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A numerical approach for solving the Fractal ordinary differential equations

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Abstract

In this paper, fractal differential equations are solved numerically. Here, the typical fractal equation is considered as follows:
$$\frac{du(t)}{dt^{\alpha}} = f\{t, u(t)\}, \ \alpha > 0,$$
f can be a nonlinear function and the main goal is to get $u(t)$. The continuous and discrete modes of this method have differences, so the subject must be carefully studied. How to solve fractal equations in their discrete form will be another goal of this research and also its generalization to higher dimensions than other aspects of this research.

Keywords. Fractal Differential Equations, Taylor series, Continuous, Discrete points.
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1. Introduction

The concept of distinctive non-local operators has attracted the interest of mathematicians around the world [1–8, 11, 17]. The fractal equations can be solved by different methods such as variational iterative method [14, 16], two step Adam-Bashforth numerical scheme in Laplace space [16], Fourier spectral method based on alternative approach [19, 20]. These methods are not very suitable for an engineer to model complex real-world problems. For example, an engineering problem in a discontinuous environment can be modeled with a new derivative. In fact, these physical events exhibit fractal behaviors. The fractal derivative is actually a natural generalization of Leibniz's derivative to the discontinuous fractal medium [10, 14, 16]. The fractal differential equations are introduced by Chek et al. As follows: Models with different fractal operators, despite their complexity, are of great importance in many fields of science and technology. Therefore, to predict the timely behavior of these models, it must be solved judicially or analytically. However, several methods have been proposed by researchers in this branch of science. New numerical methods seem to be more efficient and accurate.

In the context of applied mathematics hand in mathematics in general, there is a non-standard type of derivative known as a fractal derivative in which the variable is scaled according to t. This non-standard derivative for modeling is probably associated with physical problems; today the laws of classical physics are no longer appropriate. Such physical problems are believed to be based on Euclidean geometry and cannot be applied in non-integral fractal environments. On the other hand, we describe real-world problems, such as porous media, aquifers, turbulence, and others that typically reflect fractal properties[12, 18]. Many real-world phenomena exhibit limited or statistical fractal properties. Therefore, such cases face challenges that include fractal dimension measurement and are subject to high technical turbulence. The search for numerical and experimental constraints is also evident in the total data. However, this field has attracted the interest of many researchers and is growing rapidly. Because the fractal dimensions

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evaluated for similar statistical differentiation may have practical applications in several fields. For example, it can be used in electrochemical processes, physics, neuroscience, image analysis, sound, and physiology. Some of the fractal dimensions that are statistically similar to each other can be calculated by direct measurement, that is, by considering mathematical models that are probably similar to the formation of a fractal in the real world. Ordinary differential and integral operators cannot handle such a problem efficiently. Therefore, a new concept of differentiation has been introduced in [13]. On the other hand, in recent years, the study and development of numerical methods for ordinary fractal differential equations have been considered by many researchers, not all of whom can be mentioned here. To our knowledge, there is no study in which the discrete Taylor series method is applied to a typical fractal differential equation. Therefore, in this research, for the first time, we use Taylor's method to solve a specific class of the fractal ordinary differential equation in which these equations have exact solutions.
This work is organized into different sections. In section 2, definitions and theorems are expressed. Section 3, is devoted to the proposed method for solving the fractal ordinary differential equation. In section 4, convergence analysis is proved. The suggested method is applied to numerical examples and Mathematica codes in section 5. Finally, a brief conclusion is presented given in section 6.

2. Definitions and preliminaries of fractional derivatives

Definition 2.1. We assume that $u(y)$ is on the (a,b) and a discontinuous medium can be described by fractal dimensions. Chen et al. suggested a fractal derivative defined as [13]

$$\frac{du(y)}{dy^{\alpha}} = \lim_{t\to y}\frac{u(t)-u(y)}{t^{\alpha}-y^{\alpha}},\ \alpha>0. \tag{2.1}$$

The more generalized version is given as

$$\frac{du^{\beta}(y)}{dy^{\alpha}} = \lim_{t\to y}\frac{u^{\beta}(t)-u^{\beta}(y)}{t^{\alpha}-y^{\alpha}},\ \alpha>0,\ \beta>0. \tag{2.2}$$

Definition 2.2. [9] If $u(t)$ is continuous and is differentiable a closed interval $[a,b]$, then the fractal integral of u with order α is defined as

$${}_a^F I_t^{\alpha} u(t) = \alpha\int_0^t y^{\alpha-1}u(y)dy,\ \alpha>0. \tag{2.3}$$

Definition 2.3 (Taylor Polynomial-Single Variable). [21] Suppose that $I\subseteq\mathbb{R}$ is an open interval and that $f: I\to\mathbb{R}$ is a function of class C^k on I. For a point $a\in I$, the k-th order Taylor polynomial of f at a is the unique polynomial of order at most k, denoted $P_{a,k}(x)$, such that

$$\begin{aligned} f(a) &= P_{a,k}(0),\\ f'(a) &= P'_{a,k}(0),\\ &\ \vdots\\ f^{(k)}(a) &= P^{(k)}_{a,k}(0). \end{aligned} \tag{2.4}$$

Since the jth derivative of a polynomial evaluated at 0 gives the jth coefficient times $j!$, we can show that

$$\begin{aligned} P_{a,k}(x) &= f(a)+f'(a)x+f''(a)\frac{x^2}{2!}+\cdots+f^{(k)}(a)\frac{x^k}{k!}\\ &= \sum_{j=0}^{k} f^{(j)}(a)\frac{x^j}{j!}. \end{aligned} \tag{2.5}$$

Taylor's Theorem guarantees that $P_{a,k}(h)$ is a very good approximation of $f(a+h)$ for small h, and that the equality of the approximation increases as k increases.

Theorem 2.4 (Taylors Theorem-Single Variable)**.** *[21] Suppose that $I \subseteq \mathbb{R}$ is an open interval and that $f : I \to \mathbb{R}$ is a function of class C^k on I. Let $a \in I$ and $h \in \mathbb{R}$ such that $a + h \in I$, let $P_{a,k}(h)$ denote the kth-order Taylor polynomial at a, and define the remainder, $R_{a,k}(h)$, to be $f(a + h) - P_{a,k}(h)$, then*

$$\lim_{h\to 0} \frac{R_{a,k}(h)}{h^k} = 0. \tag{2.6}$$

When $k = 1$, we have $P_{a,1}(x) = f(a) + f'(a)x$, and so

$$R_{a,1}(h) = f(a + h) - f(a) - f'(a)h, \tag{2.7}$$

and for $k = n$, we can write $P_{a,n}(x) = f(a) + f'(a)x + \cdots + f^{(n)}(a)\frac{x^n}{n!}$, then

$$R_{a,n}(h) = f(a + h) - \left(f(a) + f'(a)h + \cdots + f^{(n)}(a)\frac{h^n}{n!}\right). \tag{2.8}$$

Moreover, if f has $n + 1$ continuous derivatives at $x = a$, the Taylor series of degree n about a is

$$\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n, \tag{2.9}$$

Theorem 2.5. *[21] Suppose f has $n + 1$ continuous derivatives on an open interval containing a, then for each x in the interval,*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k + \frac{f^{(n+1)}(c)}{(n + 1)!}(x - a)^{n+1}, \ \exists c \in (a, x), \tag{2.10}$$

where $R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1}$ is the error term of the Taylor series and is called the Lagrange formula for the remainder.

The above formula approximates $f(x)$ near a. Taylor Theorem gives bounds for the error in this approximation:

3. Main Idea

Models with fractal differential operators are very complex and also very important in many fields of science and technology Therefore, in order to predict the timely behavior of these models, they must be solved analytically or analytically numerically; however, due to the complexity of these models, we rely on the numerical scheme. We will pay attention to this point the existing numerical scheme can have limitations in the management of the mathematical equation as indicated by natural events, therefore, a new numerical scheme, which must be more accurate and efficient, is needed. In this section, we extract a new numerical scheme for solving ordinary fractal differential equations. Consider the following typical fractal differential equation:

$$\frac{du(t)}{dt^\alpha} = f\{t, u(t)\}, \ u(t_0) = u_0, \ t_0 \in [0, T], \tag{3.1}$$

we have a relationship (3.1) with the integration by the parties:

$$u(T) = u(t_0) + \alpha \int_0^T t^{\alpha-1} f(t, u(t))dt, \tag{3.2}$$

by placing $T = t_{n-1}$ and $T = t_n$ in relation (3.2), the following relations are obtained

$$u(t_{n-1}) = u(t_0) + \alpha \int_0^{t_{n-1}} t^{\alpha-1} f(t, u(t))dt, \tag{3.3}$$

$$u(t_n) = u(t_0) + \alpha \int_0^{t_n} t^{\alpha-1} f(t, u(t))dt. \tag{3.4}$$

Subtracting the ratio from (3.3) to (3.4), we have

$$u(t_n) - u(t_{n-1}) = \alpha \int_{t_{n-1}}^{t_n} t^{\alpha-1} f(t, u(t))dt, \tag{3.5}$$

or

$$u(t_n) = u(t_{n-1}) + \alpha \int_{t_{n-1}}^{t_n} t^{\alpha-1} f(t, u(t))dt. \tag{3.6}$$

The Taylor expansion of order nth of function $f(t, u(t))$ at point t_n around t_{n-1} is obtained as follows

$$\begin{aligned} f(t_n, u(t_n)) =& f(t_{n-1}, u(t_{n-1})) + \frac{\partial}{\partial t}\frac{f(t,u(t))}{1!}\Big/_{t=t_{n-1}} (t_n - t_{n-1}) + \frac{\partial^2}{\partial t^2}\frac{f(t,u(t))}{2!}\Big/_{t=t_{n-1}} (t_n - t_{n-1})^2 \\ &+ \cdots + \frac{\partial^n}{\partial t^n}\frac{f(t_{n-1},u(t_{n-1}))}{n!}\Big/_{t=t_{n-1}} (t_n - t_{n-1})^n + \frac{\partial^{n+1}}{\partial t^{n+1}}\frac{f(t,u(t))}{(n+1)!}\Big/_{t=\xi_{n-1}} (t_n - t_{n-1})^{n+1}. \end{aligned} \tag{3.7}$$

On the other hand, the above Taylor expansion can be written by the following form

$$f(t_n, u(t_n)) = P_n(t_n) + R_n(t_n), \tag{3.8}$$

and we can approximated (3.6)

$$u(t_n) = u(t_{n-1}) + \alpha \int_{t_{n-1}}^{t_n} t^{\alpha-1} P_n(t_n)dt, \tag{3.9}$$

4. Error Analysis

In this section, the truncated error of the Taylor method is presented. The general conditions are considered, in which case it will converge to the current method.

Theorem 4.1. *If $\frac{du(t)}{dt^\alpha} = f(t, u(t))$ for $\alpha > 0$ is a fractal ordinary differential equation that $n + 1$th derivative of function f for $t_{n-1} \in \mathbb{R}$ exists and f has $n+1$th continuous derivative at $\xi \in I \subseteq \mathbb{R}$ around t_{n-1},then the approximated method for the fractal ordinary differential equation is stable and*

$$u(t_n) = u(t_{n-1}) + \alpha \int_{t_{n-1}}^{t_n} t^{\alpha-1} P_n(t_n)dt + R_n^\alpha, \tag{4.1}$$

where

$$\begin{aligned} P_n(t_n) &= \sum_{m=0}^{n} \frac{\partial^m}{\partial t^m} f(t, u(t)) \Big/_{t=t_{n-1}} \frac{(t_n - t_{n-1})^m}{m!}, \\ R_n^\alpha &= \alpha \int_{t_{n-1}}^{t_n} t^{\alpha-1} r_n(\xi)dt, \ \xi \in (t_{n-1}, t_n), \end{aligned} \tag{4.2}$$

and

$$r_n(\xi) = \frac{\partial^{n+1}}{\partial t^{n+1}} f(t, u(t)) \Big/_{t=\xi} \frac{(\xi - t_{n-1})^{(n+1)}}{(n+1)!}. \tag{4.3}$$

On the other hand, by considering $M = \|r_n(\xi)\|_\infty = \sup_{t\in[a,b]} |r_n(t)|$, we have

$$\|R_n^\alpha\|_\infty \le \alpha M h^\alpha \left(n^\alpha - (n-1)^\alpha\right). \tag{4.4}$$

Proof. Since the function f has $n+1$th continuous derivative at t_{n+1} around t_{n-1}, so according to Theorem 2.4, we can approximate $f(t,u(t))$

$$\begin{aligned} f(t,u(t)) =& \sum_{m=0}^{n} \frac{\partial^m}{\partial t^m} f(t,u(t)) \Big/_{t=t_{n-1}} \frac{(t_n - t_{n-1})^m}{m!} + \frac{\partial^{n+1}}{\partial t^{n+1}} f(t,u(t)) \Big/_{t=\xi} \frac{(\xi - t_{n-1})^{(n+1)}}{(n+1)!} \\ \simeq& P_n(t_n) + r_n(\xi), \end{aligned} \tag{4.5}$$

by applying the presented method, we conclude

$$u(t_n) - u(t_{n-1}) = \alpha \int_{t_{n-1}}^{t_n} t^{\alpha-1} P_n(t_n) dt + \alpha \int_{t_{n-1}}^{t_n} t^{\alpha-1} r_n(\xi) dt, \tag{4.6}$$

and by considering $R_n^\alpha(\xi)$

$$\begin{aligned} \|R_n^\alpha(\xi)\|_\infty =& \alpha \left\| \int_{t_{n-1}}^{t_n} t^{\alpha-1} r_n(\xi) dt \right\|_\infty \\ \leq& \alpha \|r_n(\xi)\|_\infty \int_{t_{n-1}}^{t_n} t^{\alpha-1} dt \\ \leq& \alpha M((t_n)^\alpha - (t_{n-1})^\alpha) \end{aligned} \tag{4.7}$$

Since $t_i = t_0 + ih$, then

$$\begin{aligned} \|R_n^\alpha(\xi)\|_\infty \leq& \alpha M \left(n^\alpha h^\alpha - (n-1)^\alpha h^\alpha\right) \\ \leq& \alpha M h^\alpha \left(n^\alpha - (n-1)^\alpha\right). \end{aligned} \tag{4.8}$$

□

5. Numerical experiments of the Taylor Method

In order to validate the efficiency of the Taylor technique, we test several numerical experiments of fractal differential equations. The computation was done in the Mathematica programming. Moreover, the numerical results are demonstrated by some tables and figures. To show the approximate solutions, we use interpolation of obtained points by Taylor method and then compare the exact solution and the approximate solution in graphs and tables.

Example 5.1. *Let us consider the fractal differential equation given by*

$${}_0^F D_t^\alpha u(t) = -bu(t), \ \alpha > 0, \tag{5.1}$$

with the function u differentiable. Equation (5.1) can be converted to

$$u'(t) = -\alpha b u(t) t^{\alpha-1}, \tag{5.2}$$

so, we can write

$$\frac{u'(t)}{u(t)} = -\alpha b t^{\alpha-1}, \tag{5.3}$$

and

$$\ln u(t) = -bt^\alpha. \tag{5.4}$$

Now, the exact solution is

$$u(t) = u(0)e^{-bt^\alpha}, \ \alpha > 0. \tag{5.5}$$

In order to solve Equation (5.1), numerically and by considering $\alpha = 1$, it is necessary to divide the interval $[0,1]$ as follows to the n subinterval $[t_i, t_{i+1}]$, for $i = 0, 1, \ldots, n$. Using the iterative method with the Taylor expansion technique, the numerical results calculated in Figures 1, 2, and 3 for discrete points are obtained for $n = 10$, $n = 100$ and $n = 1000$, respectively. By applying the interpolation of these computed points, we can draw the absolute errors. These figures show the absolute error of the presented method. In Figure 4, the exact and approximate solutions are

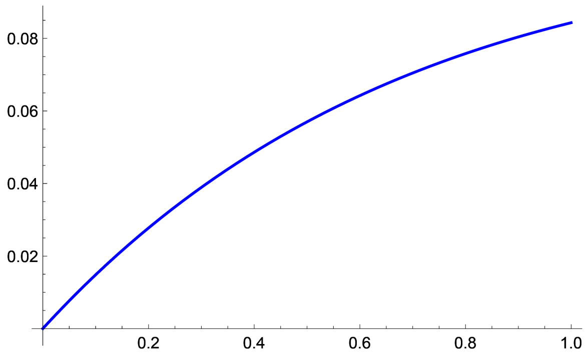

FIGURE 1. $|u_{10}(t) - u(t)|$ of Example 5.1 with $\alpha = 1$.

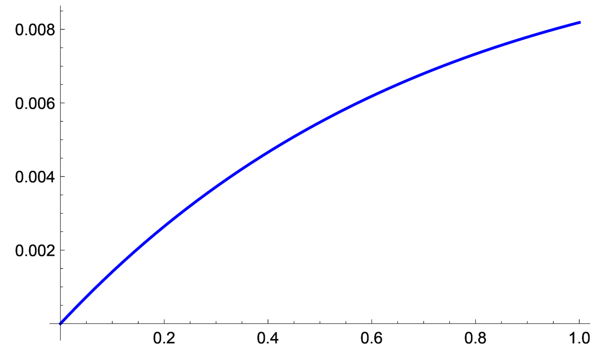

FIGURE 2. $|u_{100}(t) - u(t)|$ of Example 5.1 with $\alpha = 1$.

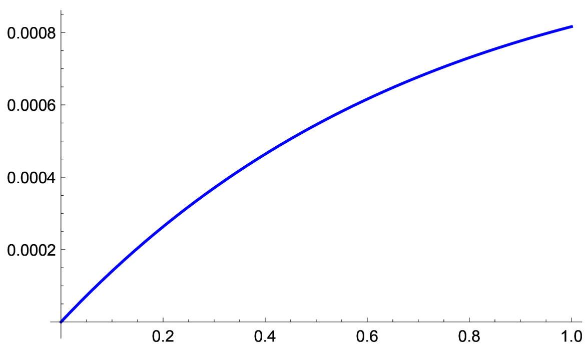

FIGURE 3. $|u_{1000}(t) - u(t)|$ of Example 5.1 with $\alpha = 1$.

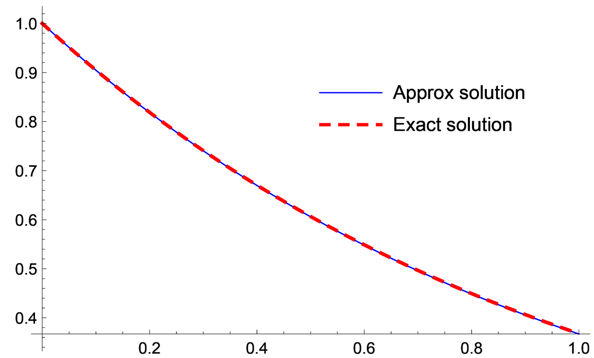

FIGURE 4. The exact solution and the approximate solution $u_{1000}(t)$ of Example 5.1 with $\alpha = 1$.

TABLE 1. Error comparison of Example 5.1.

	$\alpha = 1$			
Node	*Exact solution*	$\lvert u_{10}(t) - u(t)\rvert$	$\lvert u_{100}(t) - u(t)\rvert$	$\lvert u_{1000}(t) - u(t)\rvert$
0.0	1.000000	0.00×10^{-0}	0.00×10^{-0}	0.00×10^{-0}
0.1	0.904837	1.48×10^{-2}	1.41×10^{-3}	1.40×10^{-4}
0.2	0.818731	2.77×10^{-2}	2.64×10^{-3}	2.63×10^{-4}
0.3	0.740818	3.89×10^{-2}	3.72×10^{-3}	3.70×10^{-4}
0.4	0.670320	4.86×10^{-2}	4.66×10^{-3}	4.64×10^{-4}
0.5	0.606531	5.70×10^{-2}	5.47×10^{-3}	5.45×10^{-4}
0.6	0.548812	6.42×10^{-2}	6.18×10^{-3}	6.16×10^{-4}
0.7	0.496585	7.05×10^{-2}	6.80×10^{-3}	6.77×10^{-4}
0.8	0.449329	7.58×10^{-2}	7.33×10^{-3}	7.31×10^{-4}
0.9	0.406570	8.04×10^{-2}	7.79×10^{-3}	7.77×10^{-4}
1.0	0.367879	8.43×10^{-2}	8.19×10^{-3}	8.16×10^{-4}

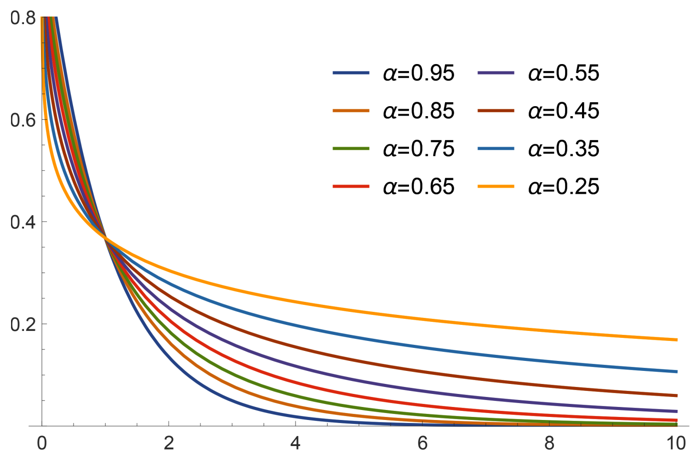

FIGURE 5. Exact solutions of Example 5.1 with different values of fractal order α.

compared for $\alpha = 1$ and $n = 1000$ which clearly shows the coordination and compatibility of the exact and approximate numerical solution. Assuming $\alpha = 1$, the numerical results of the absolute errors are checked in Table 1. In Figure 5, the numerical exact solutions are demonstrated for various values of fractal order.

Example 5.2. *We consider the following equation*

$$ {}_0^F D_t^{\alpha} u(t) = f(t), \ \alpha > 0, \tag{5.6}$$

where $f(t) = e^t$. The exact solution is given as

$$u(t) = {}_0^F I_t^{\alpha} f(t), \ \alpha > 0, \tag{5.7}$$

so, we have

$$u(t) = \alpha \left[(-t)^{-\alpha} t^{\alpha} \left(\Gamma(\alpha) - \Gamma(\alpha, -t)\right)\right], \ \alpha > 0. \tag{5.8}$$

According to Eq. (5.6), and by considering $\alpha = 0.95$, we should divide the interval $[0, 1]$ to the n subinterval $[t_i, t_{i+1}]$, for $i = 0, 1, \ldots, n$. We use the iterative method with the Taylor expansion technique and calculate the numerical results in Figures 6, 7, and 8. In Figure 9, the exact and approximate solutions are compared for $\alpha = 0.95$ which clearly shows the coordination and compatibility of the exact and approximate numerical solutions. In Table 2, the absolute errors are illustrated for $\alpha = 0.95$. In Figure 10, the numerical exact solutions are demonstrated for various values of α.

Example 5.3. *Consider the following fractal differential equation*

$$ {}_0^F D_t^{\alpha} u(t) = f(t), \ \alpha > 0, \tag{5.9}$$

where $f(t) = \cos t$. The exact solution is given as

$$u(t) = {}_0^F I_t^{\alpha} f(t), \ \alpha > 0, \tag{5.10}$$

so, we have

$$u(t) = t^{\alpha} \left({}^{p}F_q \left[\{\alpha/2\}, \{\frac{1}{2}, 1 + \frac{\alpha}{2}\}; -\frac{t}{2} \right] \right), \ \alpha > 0. \tag{5.11}$$

Consider Equation (5.9) and $\alpha = 0.95$, so, by using the presented method for $n = 10$, $n = 100$, and $n = 1000$, we can calculate the numerical results in Figures 11, 12, and 13. In Figure 14, the exact and approximate solutions are compared for $\alpha = 0.95$. In Table 3, the absolute errors are illustrated for $\alpha = 0.95$. In Figure 15, the numerical exact solutions are demonstrated for various values of fractal order α.

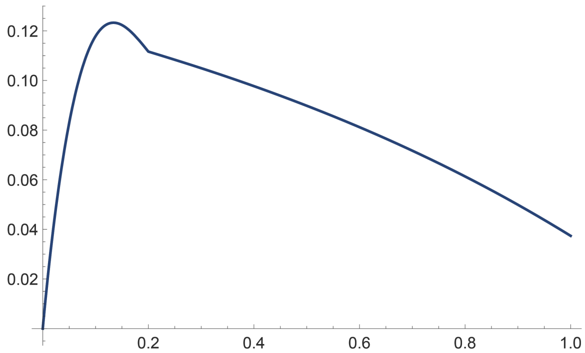

FIGURE 6. $|u_{10}(t) - u(t)|$ of Example 5.2 with $\alpha = 0.95$.

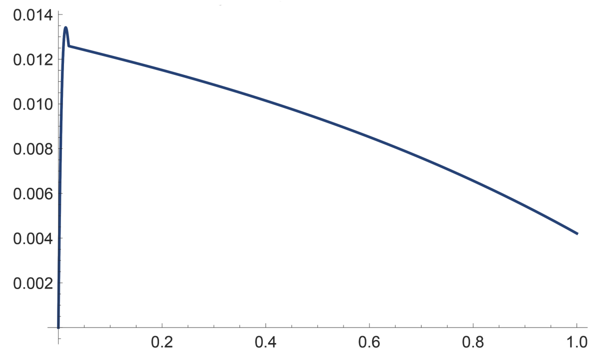

FIGURE 7. $|u_{100}(t) - u(t)|$ of Example 5.2 with $\alpha = 0.95$.

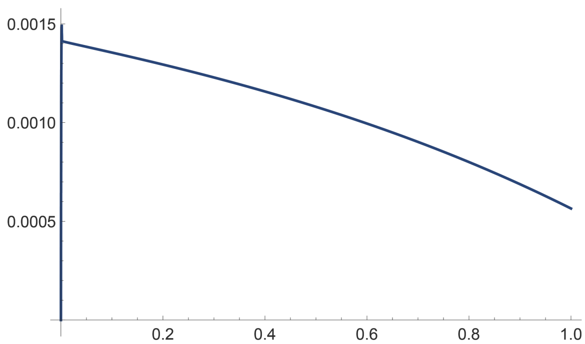

FIGURE 8. $|u_{1000}(t) - u(t)|$ of Example 5.2 with $\alpha = 0.95$.

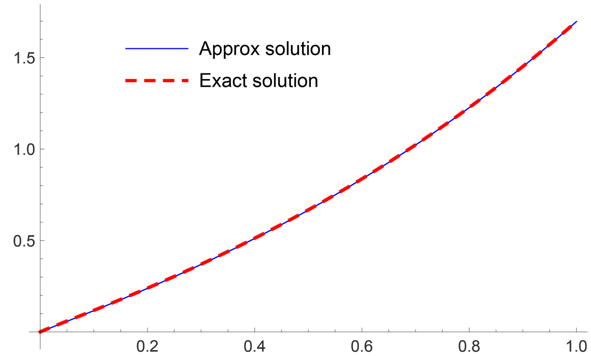

FIGURE 9. The exact solution and the approximate solution $u_{1000}(t)$ of Example 5.2 with $\alpha = 0.95$.

TABLE 2. Error comparison of Example 5.2.

	$\alpha = 0.95$			
Node	*Exact solution*	$\lvert u_{10}(t) - u(t)\rvert$	$\lvert u_{100}(t) - u(t)\rvert$	$\lvert u_{1000}(t) - u(t)\rvert$
0.1	0.117853 - 3.46945×10^{-18}**i**	1.18×10^{-1}	1.21×10^{-2}	1.35×10^{-3}
0.2	0.239348 - 1.38777×10^{-17}**i**	1.12×10^{-1}	1.15×10^{-2}	1.29×10^{-3}
0.3	0.370165 + 2.77555×10^{-17}**i**	1.04×10^{-1}	1.09×10^{-2}	1.23×10^{-3}
0.4	0.512309 - 0.00000×10^{-00}**i**	9.77×10^{-2}	1.01×10^{-2}	1.16×10^{-3}
0.5	0.667435 - 0.00000×10^{-00}**i**	8.98×10^{-2}	9.37×10^{-3}	1.08×10^{-3}
0.6	0.837160 - 2.77555×10^{-17}**i**	8.12×10^{-2}	8.52×10^{-3}	9.95×10^{-4}
0.7	1.023170 - 2.77555×10^{-17}**i**	7.17×10^{-2}	7.59×10^{-3}	9.02×10^{-4}
0.8	1.227280 + 2.77555×10^{-17}**i**	6.13×10^{-2}	6.57×10^{-3}	8.00×10^{-4}
0.9	1.451450 - 2.77555×10^{-17}**i**	4.99×10^{-2}	5.44×10^{-3}	6.88×10^{-4}
1.0	0.117853 - 0.00000×10^{-00}**i**	3.74×10^{-2}	4.21×10^{-3}	5.65×10^{-4}

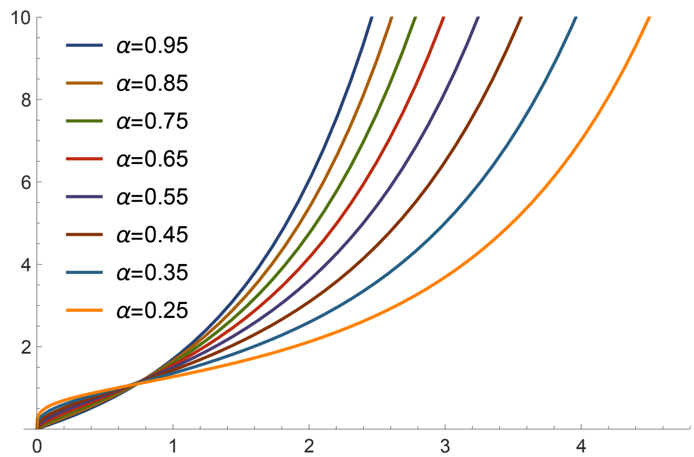

FIGURE 10. Real part of exact solutions of Example 5.2 with different values of fractal order α.

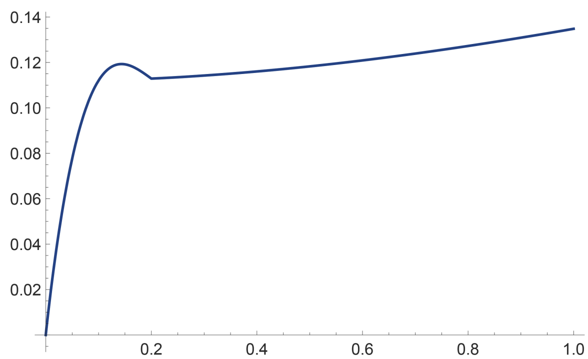

FIGURE 11. $|u_{10}(t) - u(t)|$ of Example 5.3 with $\alpha = 0.95$.

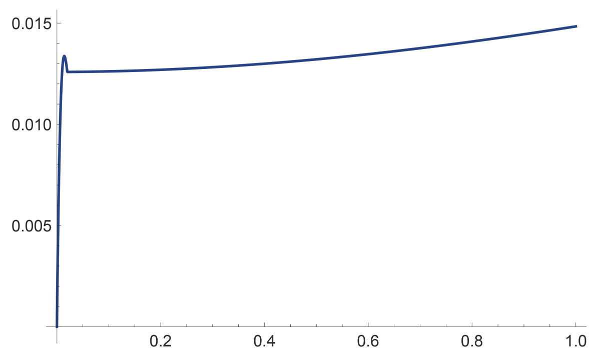

FIGURE 12. $|u_{100}(t) - u(t)|$ of Example 5.3 with $\alpha = 0.95$.

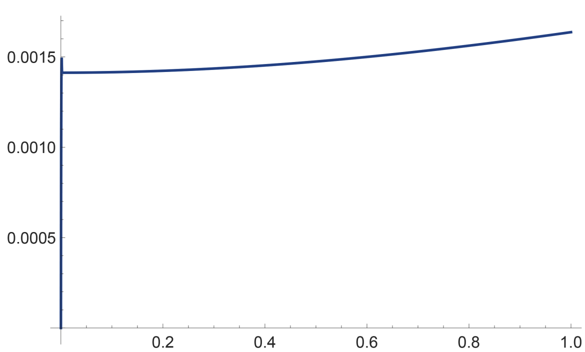

FIGURE 13. $|u_{1000}(t) - u(t)|$ of Example 5.3 with $\alpha = 0.95$.

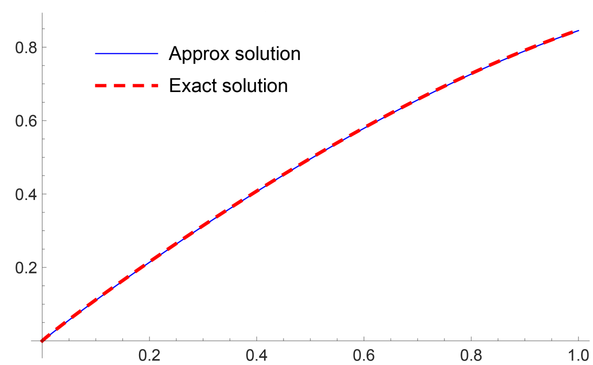

FIGURE 14. The exact solution and the approximate solution $u_{1000}(t)$ of Example 5.3 with $\alpha = 0.95$.

TABLE 3. Error comparison of Example 5.3.

Node	$\alpha = 0.95$ *Exact solution*	$\lvert u_{10}(t) - u(t)\rvert$	$\lvert u_{100}(t) - u(t)\rvert$	$\lvert u_{1000}(t) - u(t)\rvert$
0.0	*0.000000*	0.00×10^{-0}	0.00×10^{-0}	0.00×10^{-0}
0.1	*0.112021*	1.12×10^{-1}	1.26×10^{-2}	1.41×10^{-3}
0.2	*0.215366*	1.13×10^{-1}	1.27×10^{-2}	1.42×10^{-3}
0.3	*0.314018*	1.14×10^{-1}	1.28×10^{-2}	1.44×10^{-3}
0.4	*0.408049*	1.16×10^{-1}	1.30×10^{-2}	1.45×10^{-3}
0.5	*0.497053*	1.18×10^{-1}	1.32×10^{-2}	1.47×10^{-3}
0.6	*0.580475*	1.21×10^{-1}	1.35×10^{-2}	1.50×10^{-3}
0.7	*0.657725*	1.24×10^{-1}	1.38×10^{-2}	1.53×10^{-3}
0.8	*0.728220*	1.27×10^{-1}	1.41×10^{-2}	1.56×10^{-3}
0.9	*0.791409*	1.31×10^{-1}	1.45×10^{-2}	1.60×10^{-3}
1.0	*0.846792*	1.35×10^{-1}	1.48×10^{-2}	1.64×10^{-3}

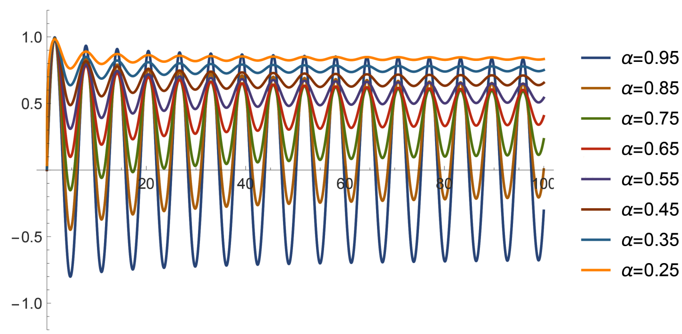

FIGURE 15. Exact solutions of Example 5.3 with different values of fractal order α.

6. Conclusion

The fractal differential equations numerically are solved. The continuous and discrete modes of the presented method are considered. Also , solving the fractal equations in the discrete form and its generalization to higher dimensions are studied.

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