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Wong-Zakai approximation of stochastic Volterra integral equations

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Abstract

This study aims to investigate a stochastic Volterra integral equation driven by fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. We employ the Wong-Zakai approximation to simplify this intricate problem, transforming the stochastic integral equation into an ordinary integral equation. Moreover, we consider the convergence and the rate of convergence of the Wong-Zakai approximation for this kind of equation.

Keywords. Wong-Zakai approximation, Fractional Brownian motion, Volterra integral equation.

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1. Introduction

Stochastic Volterra equations stand at the intersection of complex systems modeling and stochastic calculus, providing a robust framework for understanding real-world phenomena influenced by random processes. These equations, incorporating both Volterra integral terms and stochastic elements, find applications in diverse fields such as finance, biology, physics, and engineering.

In stochastic Volterra equations, fractional Brownian motion introduces a layer of intricacy. Fractional Brownian motion, characterized by its self-similar properties and long-range dependence, poses significant challenges regarding analytical solutions. Consequently, the numerical approximation of solutions emerges as a pivotal avenue for researchers and practitioners seeking to unravel the behavior of these equations.

In the realm of infinite-dimensional spaces, the exploration of stochastic Volterra equations driven by Brownian motion was studied in [1]. Subsequently, the study extended to stochastic Volterra equations driven by general semimartingales, as discussed in [12]. However, despite the progress made in recent years, the numerical solution of these equations remains a vibrant and evolving research area.

In [7], the authors applied the stochastic θ-method to address these equations, demonstrating its mean-square convergence of order $\frac{1}{2}$. Additionally, investigations into stochastic equations featuring weakly singular kernels were conducted in [11].

Exploring alternative approaches, [18] proposed a numerical scheme for nonlinear stochastic Itô Volterra integral equations, leveraging Haar wavelets. Simultaneously, [13] delved into the shifted Jacobi operational matrix method as an innovative avenue of research in this domain.

In this study, our focus centers on investigating the numerical solution for a specific stochastic integral equation driven by fractional Brownian motion, represented as:

$$X_t = X_0 + \int_0^t K_H(t,s)b(X_s)\mathrm{d}s + \int_0^t K_H(t,s)\sigma(X_s)\mathrm{d}W_s,\ t \in [0,1], \tag{1.1}$$

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where W_t denotes Brownian motion and $K_H(t,s)$ is a deterministic kernel. Notably, the process

$$B^H(t) = \int_0^t K_H(t,s)\mathrm{d}W_s,$$

is a fractional Brownian motion (fBm) characterized by its Hurst parameter $H \in (\frac{1}{2}, 1)$. For this fBm $\mathbb{E}(B^H(t)) = 0$, and its covariance function is given by

$$\mathbb{E}(B^H(t)B^H(s)) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

It's worth mentioning that when $H = \frac{1}{2}$, $B^H(t)$ reduces to the standard Brownian motion. This process finds practical applications in diverse fields such as finance, and economics [10].

In previous studies, notable progress has been made in understanding stochastic Volterra integral equations driven by fBm. In [2], researchers established the existence and uniqueness of solutions for these equations when driven by fBms with a Hurst parameter $H > \frac{1}{2}$. Additionally, in Hilbert space, researchers in [6] proved the existence and uniqueness of mild solutions for similar equations.

The Wong-Zakai (WZ) approximation is a crucial technique for solving stochastic differential equations (SDEs). This method offers a promising way to simplify the complexity inherent in stochastic equations by transforming stochastic integrals into ordinary ones. Researchers have demonstrated that by replacing the Brownian motion in an SDE with a suitably chosen absolutely continuous process (referred to as the WZ process), the solution of the approximating equation converges almost surely to the Stratonovich form of the original equation [15, 17]. In the WZ approximation, the Brownian motion is first approximated with an absolutely continuous process before applying proper time-discretisation schemes. This process serves as an intermediate step in the approximation. The key advantage of employing the WZ approximation lies in transforming the SDE into an ordinary differential equation (ODE). In simpler terms, instead of dealing with the complexities of an SDE, researchers can focus on solving a more straightforward ODE, making the approach notably simpler and more manageable.

In prior research, the WZ approximation has been investigated in various numerical schemes for solving stochastic delay differential equations, as demonstrated in [3]. Additionally, [8] explored an implicit Milstein method for SDEs employing the WZ approximation. To approximate Brownian motion within the WZ method, Fourier approximation techniques were utilized in studies such as [15, 16] and also in [3]. In a different domain, [14] applied the classical piecewise linear interpolation in combination with WZ approximation to solve stochastic Volterra equations driven by fBm. However, the rate of convergence was not determined in their study.

In this paper, we employ the Wong-Zakai approximation to simplify Equation (1.1), transforming the stochastic integral equation into an ordinary integral equation. Furthermore, we investigate the convergence and rate of convergence of the Wong-Zakai approximation for this specific type of equation.

Moving forward, the structure of the paper unfolds as follows: the subsequent section provides necessary preliminary information. Section 3 offers a concise overview of the WZ approximation technique and outlines the method's specifics. Finally, in section 4, we delve into establishing the rate of convergence for the WZ approximation method applied to stochastic Volterra equations.

2. Setting of the problem

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{W_t : 0 \le t \le 1\}$ be a Brownian motion defined on this space. Consider the following stochastic Volterra integral which is the Stratonovich form of Equation (1.1):

$$X_t = X_0 + \int_0^t K_H(t,s)\left[b(X_s) + a_H s^{H-\frac{1}{2}}\sigma\sigma'(X_s)\right]\mathrm{d}s + \int_0^t K_H(t,s)\sigma(X_s)\mathrm{d}W_s, \quad t \in [0,1], \tag{2.1}$$

where $X_0 \in \mathbb{R}$

$$a_H = \frac{2H-1}{2H+1}\left(\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}\right)^{\frac{1}{2}}\Gamma(\frac{3}{2}-H)\Gamma(H-\frac{1}{2}), \tag{2.2}$$

and b and σ satisfy in the following assumption:
A. Let $b \in C_b^1(\mathbb{R}), \sigma \in C_b^2(\mathbb{R})$ be functions such that for all $X_1, X_2 \in \mathbb{R}$ we have

$$|\sigma\sigma'(X_1) - \sigma\sigma'(X_2)|^2 + |\sigma(X_1) - \sigma(X_2)|^2 + |b(X_1) - b(X_2)|^2 \leq L|X_1 - X_2|^2, \tag{2.3}$$

where L is a positive constant.
From now on, fix $H \in (\frac{1}{2}, 1)$ and consider the deterministic kernel $K_H : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ by

$$K_H(t,s) = \begin{cases} C_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} \mathrm{d}u, & 0 < s \leq t \leq 1, \\ 0, & otherwise \end{cases} \tag{2.4}$$

where

$$C_H = (H - \frac{1}{2})\sqrt{\frac{2H\Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2})\Gamma(2 - 2H)}}.$$

In the next proposition, some elementary properties of K_H are introduced [4, 5] .

Proposition 2.1. *The mapping $s \longrightarrow K_H(t,s)$ is continuous on the $\{0 < s \leq t\}$ and*

- *There is a positive constant θ_H which satisfies this condition*
$$K_H(t,s) \leq \theta_H s^{\frac{1}{2}-H}, \ 0 < s \leq t \leq 1. \tag{2.5}$$
- *For every $0 < s \leq t$*
$$\int_0^t |K_H(t,r) - K_H(s,r)|^2 \mathrm{d}r = (t-s)^{2H}. \tag{2.6}$$
- *The mapping $t \longrightarrow K_H(t,s)$ is differentiable and*
$$\frac{\partial}{\partial t} K_H(t,s) = C_H (\frac{s}{t})^{\frac{1}{2}-H} (t-s)^{H-\frac{3}{2}}. \tag{2.7}$$
- $$\int_0^t \frac{\partial K_H(t,s)}{\partial t} \mathrm{d}s = \frac{2a_H}{2H+1} t^{H-\frac{1}{2}}. \tag{2.8}$$
- *Let $1 \leq p < \frac{2}{2H-1}$, then*
$$\sup_{0 \leq t \leq 1} \|K_H(t,.)\|_{L^p([0,1])} < \infty. \tag{2.9}$$
- $$\int_0^s K_H(s,r)\mathrm{d}r = a_H s^{H+\frac{1}{2}}, \tag{2.10}$$
where a_H is defined in Equation (2.2).

The following proposition shows the existence of the solution of the Equation (2.1).

Proposition 2.2. *[4] Let assumption A holds, then Equation (2.1) has a pathwise unique continuous solution X_t, which satisfies the following condition*

$$\sup_{0 \leq t \leq 1} \mathbb{E}\big(|X_t|^2\big) < \infty. \tag{2.11}$$

3. Wong-Zakai approximation

Kloeden and Platen [9] introduced the Karhuen-Loève expansion of Brownian motion as follows:

$$W(t) = \sum_{j=0}^{\infty} \int_0^t m_j(s) \mathrm{d}s \xi_j \;\; 0 \le t \le T = 1, \tag{3.1}$$

where $m_j(t), j = 1, 2, 3, ...,$ are a set of complete orthonormal bases in the Hilbert space $L^2[0, 1]$ and $\xi_j, j = 1, 2, ...$ are defined by

$$\xi_j = \int_0^1 m_j(s) \mathrm{d}W(s). \tag{3.2}$$

Clearly, $\xi_j, j = 1, 2, ...$ are Gaussian random variables such that $\mathbb{E}(\xi_i) = 0$. The isometry property of the Itô integrals ensures that

$$\mathbb{E}[\xi_i \xi_j] = \int_0^1 m_i(s) m_j(s) \mathrm{d}s = \delta_{i,j}, \tag{3.3}$$

where $\delta_{i,j}$ represents the Kronecker delta function. Consequently, $\xi_i, i = 1, 2, ...$ are independent standard Gaussian random variables.
The WZ approximation stands as a semi-discretization technique, involving the truncation of the spectral expansion of Brownian motion as depicted in Equation (3.1) prior to any discretization in time. It's important to note that various forms of the WZ approximation exist, and in this study, we employ a specific orthogonal expansion of Brownian motion, as described below:

$$W^N(t) = \sum_{j=0}^{N-1} \int_0^t m_j(s) \mathrm{d}s \xi_j \quad t \in [0, 1]. \tag{3.4}$$

Now by taking a partition $0 = t_0 < t_1 < ... < t_{N-1} < t_N = T = 1$ and choosing a truncated complete orthonormal bases $m_i^j(t), \; i = 1, 2, 3, ..., N_h$ in $L^2[t_j, t_{j+1}]$ for $j = 0, 1, 2, ..., N$ we derive the following piecewise spectral expansion from Equation (3.4):

$$W^N(t) = \sum_{j=0}^{N-1} \sum_{k=1}^{N_h} \int_{t_j}^t m_k^j(s) \mathrm{d}s \xi_k^j, \tag{3.5}$$

where $\xi_k^j = \int_{t_j}^{t_{j+1}} m_k^j(s) \mathrm{d}W_s$. Let the orthonormal bases $m_k^j(t)$ in $L^2[t_j, t_{j+1}]$ be selected as trigonometric functions defined as follows:

$$m_1^j(t) = \frac{1}{\sqrt{t_{j+1} - t_j}}, \; m_k^j(t) = \sqrt{\frac{2}{t_{j+1} - t_j}} \cos\Big(\frac{(k-1)\pi(t - t_j)}{t_{j+1} - t_j}\Big), \; k = 2, 3, ..., \; t \in [t_j, t_{j+1}]. \tag{3.6}$$

Wuan in [19] showed that by choosing $m_j(t)$ as trigonometric functions the expansion (3.4) converges in a mean square sense to (3.1). Consider a WZ approximation of $X(t)$ in (2.1) as follows

$$X_t^N = X_0^N + \int_0^t K_H(t, s) b(X_s^N) \mathrm{d}s + \int_0^t K_H(t, s) \sigma(X_s^N) \mathrm{d}W_s^N, \quad t \in [0, 1], \tag{3.7}$$

where $X_0^N = X_0$. The given equation is, in fact, an ordinary integral equation, offering a considerably simpler solving process compared to Equation (2.1). In the subsequent section, we establish the mean square convergence of X_t^N to X_t and ascertain its convergence rate.

4. Main results

Let $N \in \mathbb{N}$ and $t_{j+1} - t_j = h,\ j = 1, 2, ..., N-1$, with $h = \frac{1}{N}$. Define $N_h = \lceil h^{-x} \rceil$, where $x \in \mathbb{R}^+$ satisfies the following inequalities,

$$\begin{aligned} &x < \min\left\{H - \frac{1}{2}, \frac{3}{2} - 2H\right\} \text{ for } \frac{1}{2} < H < \frac{3}{4}, \\ &x < \min\left\{H - \frac{1}{2}, 2 - 2H\right\} \text{ for } \frac{3}{4} < H < 1. \end{aligned} \tag{4.1}$$

For ease of presentation, it is important to note that throughout the remainder of this paper, the symbol C represents a random constant. Importantly, this constant remains independent of h and may vary from line to line, simplifying our notation. The subsequent theorem serves to demonstrate the regularity of both the solution (2.1) and the WZ approximation (3.7).

Lemma 4.1. *Let X_t and $X_t^N, t \in [0,1]$ be the solution of (2.1) and (3.7) respectively. For $s \in (t_j, t_{j+1}), j = 0, 1, ..., N$, we have the following inequalities*

$$\begin{aligned} \mathbb{E}[|X_s^N - X_{t_j}^N|^2] &\le C_1 h^{\frac{1}{2}-x}, \\ \mathbb{E}[|X_s - X_{t_j}|^2] &\le C_2 h^{2-2H}, \end{aligned} \tag{4.2}$$

where C_1 and C_2 are some positive constants.

Proof. By Equation (3.7) we can write

$$\begin{aligned} X_s^N - X_{t_j}^N = &\int_0^{t_j} \big[K_H(s,r) - K_H(t_j,r)\big] b(X_r^N) \mathrm{d}r + \int_0^{t_j} \big[K_H(s,r) - K_H(t_j,r)\big] \sigma(X_r^N) \mathrm{d}W_r^n \\ &+ \int_{t_j}^s K_H(s,r) b(X_r^N) \mathrm{d}r + \int_{t_j}^s K_H(s,r) \sigma(X_r^N) \mathrm{d}W_r^N, \end{aligned}$$

therefore, by substituting the WZ approximation (3.5) we get

$$\begin{aligned} \mathbb{E}\left(|X_s^N - X_{t_j}^N|^2\right) \le C\Bigg(&\mathbb{E}\left|\int_0^{t_j} \left[K_H(s,r) - K_H(t_j,r)\right] b(X_r^N) \mathrm{d}r\right|^2 \\ &+ \mathbb{E}\left|\sum_{n=0}^{j-1} \sum_{k=1}^{N_h} \int_{t_n}^{t_{n+1}} \left[K_H(s,r) - K_H(t_j,r)\right] m_k^n(r) \sigma(X_r^N) \mathrm{d}r \xi_k^j\right|^2 \\ &+ \mathbb{E}\left|\int_{t_j}^s K_H(s,r) b(X_r^N) \mathrm{d}r\right|^2 + \mathbb{E}\left|\sum_{k=1}^{N_h} \int_{t_j}^s K_H(s,r) m_k^j(r) \sigma(X_r^N) \mathrm{d}r \xi_k^j\right|^2\Bigg). \end{aligned}$$

By applying Cauchy Schwarz's inequality, Itô's isometry, and relying on assumption A, we derive

$$\begin{aligned} \mathbb{E}\big(|X_s^N - X_{t_j}^N|^2\big) \le C\Bigg(&\int_0^s \big[K_H(s,r) - K_H(t_j,r)\big]^2 \mathrm{d}r \\ &+ \sum_{n=0}^{j-1} \sum_{k=1}^{N_h} \Big(\int_{t_n}^{t_{n+1}} \big[K_H(s,r) - K_H(t_j,r)\big]^2 \mathrm{d}r\Big) \Big(\int_{t_n}^{t_{n+1}} (m_k^n(r))^2 \mathrm{d}r\Big) \\ &+ h \int_{t_j}^s K_H^2(s,r) \mathrm{d}r + h \sum_{k=1}^{N_h} \Big(\int_{t_j}^s K_H^4(s,r) \mathrm{d}r\Big)^{\frac{1}{2}} \Big(\int_{t_j}^{t_{j+1}} (m_k^j(r))^4 \mathrm{d}r\Big)^{\frac{1}{2}}\Bigg). \end{aligned}$$

Based on Equations (2.6), we deduce

$$\begin{aligned}\mathbb{E}\big(|X_s^N - X_{t_j}^N|^2\big) &\leq Ch^{2H} + CN_h \int_0^s \big[K_H(s,r) - K_H(t_j,r)\big]^2 \mathrm{d}r + Ch\int_{t_j}^s K_H^2(s,r)\mathrm{d}r \\ &\quad + Ch^{\frac{1}{2}} N_h \left(\int_{t_j}^s K_H^4(s,r)\mathrm{d}r\right)^{\frac{1}{2}} \\ &\leq C(1+N_h)h^{2H} + Ch \sup_{0\leq s\leq 1}\int_0^1 K_H^2(s,r)\mathrm{d}r + Ch^{\frac{1}{2}}N_h\left(\sup_{0\leq s\leq 1}\int_0^1 K_H^4(s,r)\mathrm{d}r\right)^{\frac{1}{2}} \\ &\leq C_1 h^{\frac{1}{2}} N_h \leq C_1 h^{\frac{1}{2}-x}.\end{aligned} \tag{4.3}$$

Now, let's proceed to estimate $\mathbb{E}(|X_s - X_{t_j}|^2)$. Let $b_1(X_r) = b(X_r) + a_H s^{H-\frac{1}{2}}\sigma\sigma'(X_r)$, so we can express

$$\begin{aligned}X_s - X_{t_j} &= \int_0^{t_j} \left[K_H(s,r) - K_H(t_j,r)\right] b_1(X_r)\mathrm{d}r + \int_0^{t_j} \left[K_H(s,r) - K_H(t_j,r)\right]\sigma(X_r)\mathrm{d}W_r \\ &\quad + \int_{t_j}^s K_H(s,r) b_1(X_r)\mathrm{d}r + \int_{t_j}^s K_H(s,r)\sigma(X_r)\mathrm{d}W_r.\end{aligned}$$

Utilizing Cauchy-Schwarz's inequality, Itô's isometry, and assumption A, we conclude

$$\begin{aligned}\mathbb{E}\left(|X_s - X_{t_j}|^2\right) &\leq C\Bigg(t_j \int_0^{t_j} |K_H(s,r) - K_H(t_j,r)|^2 \mathrm{d}r + t_j \int_0^{t_j} |K_H(s,r) - K_H(t_j,r)|^2 \mathrm{d}r \\ &\quad + h\int_{t_j}^s K_H^2(s,r)\mathrm{d}r + \int_{t_j}^s K_H^2(s,r)\mathrm{d}r\Bigg) \\ &\leq C\Bigg(\int_0^s |K_H(s,r) - K_H(t_j,r)|^2 \mathrm{d}r + \int_0^s \left[K_H(s,r) - K_H(t_j,r)\right]^2 \mathrm{d}r + \int_{t_j}^s K_H^2(s,r)\mathrm{d}r\Bigg).\end{aligned}$$

By using (2.5), (2.6) and the mean value theorem we get

$$\mathbb{E}\left(|X_s - X_{t_j}|^2\right) \leq Ch^{2H} + C(h+1)\int_{t_j}^s r^{1-2H}\mathrm{d}r \leq C_2 h^{2-2H}.$$

Thus (4.2) is satisfied. □

In the following theorem, we show that the WZ approximation (3.7) is convergent to the solution of the stochastic Volterra equation (3.4), and we will obtain the rate of convergence.

Theorem 4.2. *Let X_t represent the exact solution of the stochastic Volterra equation (2.1) and X_t^N be its WZ approximation as defined in (3.7). Assuming that assumption A is satisfied, we can establish that*

$$\mathbb{E}\left(|X_t^N - X_t|^2\right) = O(h^\gamma), \tag{4.4}$$

where

$$\gamma = \min\left\{(2H-1-2x), (3-4H-2x)\right\}, \ \textit{for}\ \frac{1}{2} < H < \frac{3}{4},$$

and

$$\gamma = \min\left\{(2-2H-x), (2H-1-2x),\right\}, \ \textit{for}\ \frac{3}{4} < H < 1.$$

Proof. By subtracting (3.7) from (2.1) we deduce

$$X_t^N - X_t = \alpha_N + \beta_N,$$

where

$$\alpha_N(t) = \int_0^t K_H(t,s)\big[b(X_s^N) - b(X_s)\big]\mathrm{d}s,$$
$$\beta_N(t) = \int_0^t K_H(t,s)\sigma(X_s^N)\mathrm{d}W_s^N - \int_0^t K_H(t,s)\sigma(X_s)\mathrm{d}W_s - a_H\int_0^t K_H(t,s)s^{H-\frac{1}{2}}\sigma\sigma'(X_s)\mathrm{d}s.$$

Assumption A, CauchySchwarz's inequality and (2.5) yield

$$\mathbb{E}\big[|\alpha_N(t)|^2\big] \le C\int_0^t K_H^2(t,s)\mathbb{E}\big(|X_s^N - X_s|^2\big)\mathrm{d}s \le C\int_0^t s^{1-2H}\mathbb{E}\big(|X_s^N - X_s|^2\big)\mathrm{d}s.$$

For the second term, i.e, β_N, we can write

$$\beta_N(t) = \beta_{N,1}(t) + \beta_{N,2}(t) + \beta_{N,3}(t),$$

where

$$\beta_{N,1}(t) = \sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}} K_H(t,s)\sigma(X_{t_j}^N)\mathrm{d}W_s^N - \int_0^t K_H(t,s)\sigma(X_s)\mathrm{d}W_s,$$
$$\beta_{N,2}(t) = \sum_{j=0}^{N-1}\sum_{k=1}^{N_h}\int_{t_j}^{t_{j+1}} K_H(t,s)m_k^j(s)\int_{t_j}^{s}\sigma'(X_u^N)\Big(\int_0^u \frac{\partial K_H(u,r)}{\partial u}\sigma(X_r^N)\mathrm{d}W_r^N\Big)\mathrm{d}u\mathrm{d}s\xi_k^j$$
$$- a_H\int_0^t K_H(t,s)s^{H-\frac{1}{2}}\sigma\sigma'(X_s)\mathrm{d}s,$$
$$\beta_{N,3}(t) = \sum_{j=0}^{N-1}\sum_{k=1}^{N_h}\int_{t_j}^{t_{j+1}} K_H(t,s)m_k^j(s)\int_{t_j}^{s}\sigma'(X_u^N)\Big(\int_0^u \frac{\partial K_H(u,r)}{\partial u}b(X_r^N)\mathrm{d}r\Big)\mathrm{d}u\mathrm{d}s\xi_k^j.$$

To estimate $\beta_{N,1}(t)$, we break it down into two components

$$\beta_{N,1}(t) = \sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}} K_H(t,s)\sigma(X_{t_j}^N)[\mathrm{d}W_s^N - \mathrm{d}W_s] + \sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}} K_H(t,s)[\sigma(X_{t_j}^N) - \sigma(X_s)]\mathrm{d}W_s$$
$$= b_{N,1}(t) + b_{N,2}(t).$$

Breaking down $b_{N,1}(t)$, we can express it as:

$$b_{N,1}(t) = \int_0^{t_1} K_H(t,s)\sigma(X_{t_j}^N)[\mathrm{d}W_s^N - \mathrm{d}W_s] + \sum_{j=1}^{N-1}\int_{t_j}^{t_{j+1}}[K_H(t,s) - K_H(t,t_j)]\sigma(X_{t_j}^N)\mathrm{d}W_s^N$$
$$+ \sum_{j=1}^{N-1}\int_{t_j}^{t_{j+1}} K_H(t,t_j)\sigma(X_{t_j}^N)[\mathrm{d}W_s^N - \mathrm{d}W_s] + \sum_{j=1}^{N-1}\int_{t_j}^{t_{j+1}}[K_H(t,t_j) - K_H(t,s)]\sigma(X_{t_j}^N)\mathrm{d}W_s$$
$$:= \sum_{i=1}^{4} b_{N,1}^i(t).$$

Assumption A and Cauchy-Schwarz's inequality yield

$$\begin{aligned}
\mathbb{E}[|b^1_{N,1}(t)|^2] &= \mathbb{E}\Big[\Big|\int_0^{t_1} K_H(t,s)\sigma(X^N_{t_j})\mathrm{d}W^N_s - \int_0^{t_1} K_H(t,s)\sigma(X^N_{t_j})\mathrm{d}W_s\Big|^2\Big] \\
&\le 2C\mathbb{E}\Big[\Big|\int_0^{t_1} K_H(t,s)\sigma(X^N_{t_j})\mathrm{d}W^N_s\Big|^2\Big] + 2C\mathbb{E}\Big[\Big|\int_0^{t_1} K_H(t,s)\sigma(X^N_{t_j})\mathrm{d}W_s\Big|^2\Big] \\
&\le 2C\sum_{k=1}^{N_h}\Big|\int_0^{t_1} K_H(t,s)m^0_k(s)\mathrm{d}s\Big|^2 + 2C\int_0^{t_1} K^2_H(t,s)\mathrm{d}s \\
&\le 2C\sum_{k=1}^{N_h}\Big(\int_0^{t_1} K^2_H(t,s)\mathrm{d}s\Big)\Big(\int_0^{t_1}(m^0_k(s))^2\mathrm{d}s\Big) + 2C\int_0^{t_1} s^{1-2H}\mathrm{d}s \\
&\le 2C(N_h+1)\int_0^{t_1} s^{1-2H}\mathrm{d}s = 2C(N_h+1)h^{2-2H},
\end{aligned}$$

therefore

$$\mathbb{E}[|b^1_{N,1}(t)|^2] \le CN_h h^{2-2H}.$$

To estimate $\mathbb{E}[|b^2_{N,1}(t)|^2]$, by assumption A we have

$$\begin{aligned}
\mathbb{E}|b^2_{N,1}(t)|^2 &= \mathbb{E}\left|\sum_{j=1}^{N-1}\sum_{k=1}^{N_h}\int_{t_j}^{t_{j+1}}[K_H(t,s)-K_H(t,t_j)]\sigma(X^N_{t_j})m^j_k(s)\mathrm{d}s\xi^j_k\right|^2 \\
&\le C\sum_{j=1}^{N-1}\sum_{k=1}^{N_h}\mathbb{E}\left|\int_{t_j}^{t_{j+1}}[K_H(t,s)-K_H(t,t_j)]\sigma(X^N_{t_j})m^j_k(s)\mathrm{d}s\right|^2 \\
&\le C\sum_{j=1}^{N-1}\sum_{k=1}^{N_h}\left(\int_{t_j}^{t_{j+1}}|K_H(t,s)-K_H(t,t_j)|^2\mathrm{d}s\right)\left(\int_{t_j}^{t_{j+1}}(m^j_k(s))^2\mathrm{d}s\right).
\end{aligned}$$

Since $s \le t \le 1$ and $|s-t_j| \le h$, by simple calculation we get

$$|K_H(t,s)-K_H(t,t_j)| \le C|s^{\frac{1}{2}-H} - t_j^{\frac{1}{2}-H}| + C(s-t_j)^{H-\frac{1}{2}}.$$

Hence, employing the mean value theorem, we can deduce:

$$\begin{aligned}
\mathbb{E}|b^2_{N,1}(t)|^2 &\le C\left(2\sum_{j=1}^{N-1}\sum_{k=1}^{N_h}\int_{t_j}^{t_{j+1}}(s^{\frac{1}{2}-H}-t_j^{\frac{1}{2}-H})^2\mathrm{d}s + 2\sum_{j=1}^{N-1}\sum_{k=1}^{N_h}\int_{t_j}^{t_{j+1}}(s-t_j)^{2H-1}\mathrm{d}s\right) \\
&\le C\left(2N_h h^2 + 2N_h h^{2H-1}\right) \le 2CN_h\left[h^2+h^{2H-1}\right].
\end{aligned}$$

For the estimation of $b^3_{N,1}(t)$, by assumption A and the fact that $\int_{t_j}^{t_{j+1}}\mathrm{d}W^n_s = \Delta W_j = W(t_{j+1}) - W(t_j)$, we have

$$\mathbb{E}[|b^3_{N,1}(t)|^2] \le \sum_{j=1}^{N-1}|K_H(t,t_j)|^2\sigma^2(X^N_{t_j})\mathbb{E}\left|\int_{t_j}^{t_{j+1}}\mathrm{d}W^N_s - \int_{t_j}^{t_{j+1}}\mathrm{d}W_s\right|^2 = 0.$$

Finally, for the estimation of $b^4_{N,1}(t)$, we employ a similar calculation as for $b^2_{N,1}(t)$, and find

$$\begin{aligned}\mathbb{E}|b^4_{N,1}(t)|^2 &\leq C\sum_{j=1}^{N-1}\int_{t_j}^{t_{j+1}}[K_H(t,t_j)-K_H(t,s)]^2\mathrm{d}s\\ &\leq 2C\sum_{j=1}^{N-1}\int_{t_j}^{t_{j+1}}(s^{\frac{1}{2}-H}-t_j^{\frac{1}{2}-H})^2\mathrm{d}s+2C\sum_{j=1}^{N-1}\int_{t_j}^{t_{j+1}}(s-t_j)^{2H-1}\mathrm{d}s\\ &\leq 2C[h^2+h^{2H-1}].\end{aligned}$$

Therefore, we conclude

$$\mathbb{E}[|b_{N,1}(t)|^2]\leq CN_h[h^{2H-1}+h^{2-2H}]. \tag{4.5}$$

Now, to estimate $b_{N,2}(t)$, we can express it as follows:

$$\begin{aligned}b_{N,2}(t)&=\sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}}K_H(t,s)\left[\sigma(X^N_{t_j})-\sigma(X_s)\right]\mathrm{d}W_s\\ &=\sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}}K_H(t,s)\left[\sigma(X^N_{t_j})-\sigma(X^N_s)\right]\mathrm{d}W_s+\sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}}K_H(t,s)\left[\sigma(X^N_s)-\sigma(X_s)\right]\mathrm{d}W_s\\ &:=b^1_{N,2}(t)+b^2_{N,2}(t).\end{aligned}$$

For $b^1_{N,2}(t)$, we can use Lemma 4.1 and Equation (2.5) to obtain the following estimate:

$$\begin{aligned}\mathbb{E}|b^1_{N,2}(t)|^2&=\sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}}K^2_H(t,s)\mathbb{E}\big(|X^N_{t_j}-X^N_s|^2\big)\mathrm{d}s\leq Ch^{\frac{1}{2}}N_h\sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}}K^2_H(t,s)\mathrm{d}s\\ &\leq Ch^{\frac{1}{2}}N_h\sup_{0\leq s\leq 1}\int_0^1K^2_H(t,s)\mathrm{d}s\leq Ch^{\frac{1}{2}}N_h,\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\mathbb{E}|b^2_{N,2}(t)|^2&\leq C\sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}}K^2_H(t,s)\mathbb{E}\big(|X^N_s-X_s|^2\big)\mathrm{d}s\\ &\leq C\int_0^t s^{1-2H}\mathbb{E}\big(|X^N_s-X_s|^2\big)\mathrm{d}s.\end{aligned} \tag{4.7}$$

From (4.5)-(4.7) we conclude

$$\mathbb{E}\big[|\beta_{N,1}(t)|^2\big]\leq Ch^{y-x}+C\int_0^t s^{1-2H}\mathbb{E}\big(|X^N_s-X_s|^2\big)\mathrm{d}s, \tag{4.8}$$

where $y=\min\left\{2H-1,2-2H,\frac{1}{2}\right\}$.
Now, we will estimate $\mathbb{E}[|\beta_{N,2}(t)|^2]$, for this aim we can decompose $\beta_{N,2}(t)$ into different terms as follows:

$$\beta_{N,2}(t)=\sum_{i=1}^{7}\beta^i_{N,2}(t),$$

where

$$\beta^1_{N,2}(t) = \sum_{j=0}^{N-1} \sigma'(X^N_{t_j}) \sum_{k=1}^{N_h} \int_{t_j}^{t_{j+1}} K_H(t,s) m^j_k(s) \int_{t_j}^{s} \Big(\int_0^{t_j} \frac{\partial K_H(u,r)}{\partial u} \sigma(X^N_r) \mathrm{d}W^N_r \Big) \mathrm{d}u\mathrm{d}s \xi^j_k,$$

$$\beta^2_{N,2}(t) = \sum_{j=0}^{N-1} \sum_{k=1}^{N_h} \int_{t_j}^{t_{j+1}} K_H(t,s) m^j_k(s) \int_{t_j}^{s} \big[\sigma'(X^N_u) - \sigma'(X^N_{t_j})\big] \Big(\int_0^{t_j} \frac{\partial K_H(u,r)}{\partial u} \sigma(X^N_r) \mathrm{d}W^N_r \Big) \mathrm{d}u\mathrm{d}s \xi^j_k,$$

$$\beta^3_{N,2}(t) = \sum_{j=0}^{N-1} \sum_{k=1}^{N_h} \sum_{l=1}^{N_h} \int_{t_j}^{t_{j+1}} K_H(t,s) m^j_k(s) \int_{t_j}^{s} \sigma'(X^N_u) \Big(\int_{t_j}^{u} \frac{\partial K_H(u,r)}{\partial u} m^j_l(r) \big[\sigma(X^N_r) - \sigma(X_r)\big] \mathrm{d}r \Big) \mathrm{d}u\mathrm{d}s \xi^j_k \xi^j_l,$$

$$\beta^4_{N,2}(t) = \sum_{j=0}^{N-1} \sum_{k=1}^{N_h} \sum_{l=1}^{N_h} \int_{t_j}^{t_{j+1}} K_H(t,s) m^j_k(s) \int_{t_j}^{s} \big[\sigma'(X^N_u) - \sigma'(X^N_{t_j})\big]$$
$$\Big(\int_{t_j}^{u} \frac{\partial K_H(u,r)}{\partial u} \sigma(X_r) m^j_l(r) \mathrm{d}r \Big) \mathrm{d}u\mathrm{d}s \xi^j_k \xi^j_l,$$

$$\beta^5_{N,2}(t) = \sum_{j=0}^{N-1} \sum_{k=1}^{N_h} \sum_{l=1}^{N_h} \sigma\sigma'(X_{t_j}) \int_{t_j}^{t_{j+1}} K_H(t,s) m^j_k(s) \int_{t_j}^{s} \int_{t_j}^{u} \frac{\partial K_H(u,r)}{\partial u} m^j_l(r) \mathrm{d}r\mathrm{d}u\mathrm{d}s \xi^j_k \xi^j_l$$
$$- a_H \int_0^t K_H(t,s) s^{H-\frac{1}{2}} \sigma\sigma'(X_s) \mathrm{d}s,$$

$$\beta^6_{N,2}(t) = \sum_{j=0}^{N-1} \sum_{k=1}^{N_h} \sum_{l=1}^{N_h} \sigma'(X^N_{t_j}) \int_{t_j}^{t_{j+1}} K_H(t,s) m^j_k(s) \int_{t_j}^{s} \int_{t_j}^{u} \frac{\partial K_H(u,r)}{\partial u} m^j_l(r) \big[\sigma(X^N_r) - \sigma(X_r)\big] \mathrm{d}r\mathrm{d}u\mathrm{d}s \xi^j_k \xi^j_l,$$

$$\beta^7_{N,2}(t) = \sum_{j=0}^{N-1} \sum_{k=1}^{N_h} \sum_{l=1}^{N_h} \int_{t_j}^{t_{j+1}} K_H(t,s) m^j_k(s) \int_{t_j}^{s} \big[\sigma'(X^N_u) - \sigma'(X_{t_j})\big] \int_{t_j}^{u} \frac{\partial K_H(u,r)}{\partial u} m^j_l(r) \sigma(X_r) \mathrm{d}r\mathrm{d}u\mathrm{d}s \xi^j_k \xi^j_l.$$

For estimating $\beta^1_{N,2}(t)$, we can proceed as follows:

$$\beta^1_{N,2}(t) = \sum_{k=1}^{N_h} \sum_{l=1}^{N_h} \sigma'(X^N_{t_1}) \int_{t_1}^{t_2} K_H(t,s) m^1_k(s) \int_{t_1}^{s} \Big(\int_0^{t_1} \frac{\partial K_H(u,r)}{\partial u} \sigma(X^N_r) m^0_l(r) \mathrm{d}r \Big) \mathrm{d}u\mathrm{d}s \xi^1_k \xi^0_l$$
$$+ \sum_{j=2}^{N-1} \sum_{k=1}^{N_h} \sum_{l=1}^{N_h} \sigma'(X^N_{t_j}) \int_{t_j}^{t_{j+1}} K_H(t,s) m^j_k(s) \int_{t_j}^{s} \Big(\int_0^{t_1} \frac{\partial K_H(u,r)}{\partial u} \sigma(X^N_r) m^0_l(r) \mathrm{d}r \Big) \mathrm{d}u\mathrm{d}s \xi^j_k \xi^0_l$$
$$+ \sum_{j=1}^{N-1} \sum_{k=1}^{N_h} \sum_{n=1}^{j-1} \sum_{l=1}^{N_h} \sigma'(X^N_{t_j}) \int_{t_j}^{t_{j+1}} K_H(t,s) m^j_k(s) \int_{t_j}^{s} \Big(\int_{t_n}^{t_{n+1}} \frac{\partial K_H(u,r)}{\partial u} \sigma(X^N_r) m^n_l(r) \mathrm{d}r \Big) \mathrm{d}u\mathrm{d}s \xi^j_k \xi^n_l$$
$$= \beta^{1,1}_{N,2}(t) + \beta^{1,2}_{N,2}(t) + \beta^{1,3}_{N,2}(t).$$

For estimating $\mathbb{E}\left|\beta^{1,1}_{N,2}(t)\right|^2$ by assumption A, we obtain

$$\mathbb{E}|\beta^{1,1}_{N,2}(t)|^2 \le C \sum_{k=1}^{N_h} \sum_{l=1}^{N_h} \left[\int_{t_1}^{t_2} |K_H(t,s)||m^1_k(s)| \int_{t_1}^{s} \left(\int_0^{t_1} \frac{\partial K_H(u,r)}{\partial u} |m^0_l(r)| \mathrm{d}r \right) \mathrm{d}u\mathrm{d}s \right]^2 \mathbb{E}[(\xi^1_k \xi^0_l)^2].$$

By using Cauchy Schwarz's inequality, mean value theorem and (2.8) we conclude

$$\begin{aligned}\mathbb{E}\big[|\beta^{1,1}_{N,2}(t)|^2\big] &\le \frac{C}{h^2}N_h^2\Big(\int_{t_1}^{t_2} K_H^2(t,s)\mathrm{d}s\Big)\Big(\int_{t_1}^{t_2}(s-t_1)\int_{t_1}^{s}\Big(\int_0^u \frac{\partial K_H(u,r)}{\partial u}\mathrm{d}r\Big)^2\mathrm{d}u\mathrm{d}s\Big)\\ &\le Ca_H^2\frac{N_h^2}{h^2}\Big(\int_{t_1}^{t_2} s^{1-2H}\mathrm{d}s\Big)\Big(\int_{t_1}^{t_2}(s-t_1)\int_{t_1}^{s}u^{2H-1}\mathrm{d}u\mathrm{d}s\Big)\\ &\le CN_h^2h^2.\end{aligned}$$

Again by Cauchy Schwarz's inequality and assumption A we get

$$\begin{aligned}\mathbb{E}\big[|\beta^{1,2}_{N,2}(t)|^2\big] &\le \sum_{j=2}^{N-1}\sum_{k=1}^{N_h}\sum_{l=1}^{N_h}\Big[\int_{t_j}^{t_{j+1}}|K_H(t,s)||m_k^j(s)|\int_{t_j}^{s}\Big(\int_0^{t_1}\frac{\partial K_H(u,r)}{\partial u}|m_l^0(r)|\mathrm{d}r\Big)\mathrm{d}u\mathrm{d}s\Big]^2\\ &\le \frac{C}{h^2}N_h^2\sum_{j=2}^{N-1}\Big(\int_{t_j}^{t_{j+1}}K_H^2(t,s)\mathrm{d}s\Big)\Big(\int_{t_j}^{t_{j+1}}(s-t_j)\int_{t_j}^{s}h\int_0^{t_1}(\frac{\partial K_H(u,r)}{\partial u})^2\mathrm{d}r\mathrm{d}u\mathrm{d}s\Big)\\ &\le \frac{C}{h}N_h^2\sum_{j=2}^{N-1}\Big(\int_{t_j}^{t_{j+1}}s^{1-2H}\mathrm{d}s\Big)\Big(\int_{t_j}^{t_{j+1}}(s-t_j)\int_{t_j}^{s}\int_0^{t_1}(\frac{r}{u})^{1-2H}(u-r)^{2H-3}\mathrm{d}r\mathrm{d}u\mathrm{d}s\Big),\end{aligned}$$

mean value theorem implies that

$$\mathbb{E}\big[|\beta^{1,2}_{N,2}(t)|^2\big] \le CN_h^2\sum_{j=2}^{N-1}\frac{1}{j}h^2 \le CN_h^2h.$$

By applying similar arguments to $\mathbb{E}\big[|\beta^{1,3}_{N,2}(t)|^2\big]$, we have

$$\mathbb{E}\big[|\beta^{1,3}_{N,2}(t)|^2\big] \le CN_h^2h^2\sum_{j=1}^{N-1}j^{2-2H} \le CN_h^2h^2h^{2H-3} = CN_h^2h^{2H-1}.$$

Consequently, we can deduce that

$$\mathbb{E}\big[|\beta^{1}_{N,2}(t)|^2\big] \le CN_h^2h^{2H-1} \le Ch^{2H-1-2x}. \tag{4.9}$$

In the same manner we can conclude

$$\mathbb{E}\big[|\beta^{2}_{N,2}(t)|^2\big] \le Ch^{2H-1-2x}. \tag{4.10}$$

Our next goal is to estimate $\mathbb{E}\big[|\beta^{3}_{N,2}(t)|^2\big]$, where

$$\begin{aligned}\beta^3_{N,2}(t) &= \sum_{k=1}^{N_h}\sum_{l=1}^{N_h}\int_0^{t_1}K_H(t,s)m_k^0(s)\int_0^s\sigma'(X_u^N)\Big(\int_0^u\frac{\partial K_H(u,r)}{\partial u}m_l^0(r)\big[\sigma(X_r^N)-\sigma(X_r)\big]\mathrm{d}r\Big)\mathrm{d}u\mathrm{d}s\,\xi_k^0\xi_l^0\\ &+\sum_{j=1}^{N-1}\sum_{k=1}^{N_h}\sum_{l=1}^{N_h}\int_{t_j}^{t_{j+1}}K_H(t,s)m_k^j(s)\int_{t_j}^s\sigma'(X_u^N)\Big(\int_{t_j}^u\frac{\partial K_H(u,r)}{\partial u}m_l^j(r)\big[\sigma(X_r^N)-\sigma(X_r)\big]\mathrm{d}r\Big)\mathrm{d}u\mathrm{d}s\,\xi_k^j\xi_l^j\\ &:= \beta^{3,1}_{N,2}(t)+\beta^{3,2}_{N,2}(t),\end{aligned}$$

by the mutual independence of all Gaussian random variables ξ_k^j, $k=1,2,...,N_h$, $j=0,1,...,N$ and by assumption A we get

$$\begin{aligned}
&\mathbb{E}\big[|\beta_{N,2}^{3,1}(t)|^2\big]\\
&\leq 2\mathbb{E}\Big[\Big|\sum_{k=1}^{N_h}\int_0^{t_1}K_H(t,s)m_k^0(s)\int_0^s\sigma'(X_u^N)\Big(\int_0^u\frac{\partial K_H(u,r)}{\partial u}m_k^0(r)\big[\sigma(X_r^N)-\sigma(X_r)\big]\mathrm{d}r\Big)\mathrm{d}u\mathrm{d}s(\xi_k^0)^2\Big|^2\Big]\\
&+2\mathbb{E}\Big[\Big|\sum_{k\neq l=1}^{N_h}\int_0^{t_1}K_H(t,s)m_k^0(s)\int_0^s\sigma'(X_u^N)\Big(\int_0^u\frac{\partial K_H(u,r)}{\partial u}m_l^0(r)\big[\sigma(X_r^N)-\sigma(X_r)\big]\mathrm{d}r\Big)\mathrm{d}u\mathrm{d}s\xi_k^0\xi_l^0\Big|^2\Big]\\
&\leq 2C\sum_{k=1}^{N_h}\Big[\int_0^{t_1}|K_H(t,s)||m_k^0(s)|\int_0^s\int_0^u\frac{\partial K_H(u,r)}{\partial u}|m_k^0(r)|\mathrm{d}r\mathrm{d}u\mathrm{d}s\Big]^2\mathbb{E}[(\xi_k^0)^4]\\
&+2C\sum_{k,l=1,k\neq l}^{N_h}\Big[\int_0^{t_1}|K_H(t,s)||m_k^0(s)|\int_0^s\int_0^u\frac{\partial K_H(u,r)}{\partial u}|m_l^0(r)|\mathrm{d}r\mathrm{d}u\mathrm{d}s\Big]^2\mathbb{E}[(\xi_k^0)^2(\xi_l^0)^2].
\end{aligned}$$

Applying Cauchy-Schwartz's inequality, (2.10) and Fubini's theorem we conclude that

$$\begin{aligned}
\mathbb{E}\big[|\beta_{N,2}^{3,1}(t)|^2\big]&\leq\frac{C}{h^2}(N_h^2+N_h)\Big(\int_0^{t_1}K_H(t,s)\int_0^sK_H(s,r)\mathrm{d}r\mathrm{d}s\Big)^2\\
&\leq\frac{C}{h^2}(N_h^2+N_h)\Big(\int_0^{t_1}K_H^2(t,s)\mathrm{d}s\Big)\Big(\int_0^{t_1}\Big(\int_0^sK_H(s,r)\mathrm{d}r\Big)^2\mathrm{d}s\Big)\\
&\leq\frac{C}{h^2}(N_h^2+N_h)\Big(\int_0^{t_1}s^{1-2H}\mathrm{d}s\Big)\int_0^{t_1}\Big(\int_0^sK_H(s,r)\mathrm{d}r\Big)^2\mathrm{d}s\\
&\leq a_H\frac{C}{h^2}(N_h^2+N_h)\Big(\int_0^{t_1}s^{1-2H}\mathrm{d}s\Big)\Big(\int_0^{t_1}s^{1+2H}\mathrm{d}s\Big)\\
&\leq C(N_h^2+N_h)h^2=Ch^{2-2x}.
\end{aligned}$$

For estimating $\mathbb{E}[|\beta_{N,2}^{3,2}(t)|^2]$, by (2.5), Cauchy Schwartz's inequality, Fubini's theorem, mean value theorem and by the same arguments as $\mathbb{E}[|\beta_{N,2}^{3,1}(t)|^2]$, we get

$$\mathbb{E}\big[|\beta_{N,2}^{3,2}(t)|^2\big]\leq C\frac{N_h^2}{h^2}\sum_{j=1}^{N-1}\Big(\int_{t_j}^{t_{j+1}}s^{1-2H}\mathrm{d}s\Big)\Big(\int_{t_j}^{t_{j+1}}(s-t_j)\int_{t_j}^sr^{1-2H}\mathrm{d}r\mathrm{d}s\Big).$$

Consequently,

$$\mathbb{E}\big[|\beta_{N,2}^{3,2}(t)|^2\big]\leq CN_h^2h^{4-4H}\sum_{j=1}^{N-1}\frac{1}{j^{4H-2}}.$$

As a result of these calculations, we deduce

$$\begin{aligned}
&\mathbb{E}\big[|\beta_{N,2}^{3}(t)|^2\big]\leq Ch^{3-4H-2x}\text{ for }\frac{1}{2}<H<\frac{3}{4},\\
&\mathbb{E}\big[|\beta_{N,2}^{3}(t)|^2\big]\leq Ch^{4-4H-2x}\text{ for }\frac{3}{4}<H<1.
\end{aligned}\tag{4.11}$$

By the same calculations for $\mathbb{E}\big[|\beta_{N,2}^{i}(t)|^2\big]$ $i=4,6,7$ we get

$$\begin{aligned}
&\mathbb{E}\big[|\beta_{N,2}^{i}(t)|^2\big]\leq Ch^{3-4H-2x}\text{ for }\frac{1}{2}<H<\frac{3}{4},\\
&\mathbb{E}\big[|\beta_{N,2}^{i}(t)|^2\big]\leq Ch^{4-4H-2x}\text{ for }\frac{3}{4}<H<1.
\end{aligned}\tag{4.12}$$

We express the term $\beta^5_{N,2}(t)$ in the following manner

$$\beta^5_{N,2}(t)=\sum_{j=1}^{4}a^j_N(t),$$

where

$$\begin{aligned}
a^1_N(t)&=a_H\sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}}K_H(t,s)s^{H-\frac12}\left[\sigma\sigma'(X_{t_j})-\sigma\sigma'(X_s)\right]\mathrm{d}s,\\
a^2_N(t)&=\sum_{j=0}^{N-1}\sum_{k=1}^{N_h}\sum_{l=1}^{N_h}\sigma\sigma'(X_{t_j})\Big[\int_{t_j}^{t_{j+1}}K_H(t,s)m^j_k(s)\Big(\int_0^s K_H(s,r)m^j_l(r)\mathrm{d}r-\int_0^{t_j}K_H(t_j,r)m^j_l(r)\mathrm{d}r\Big)\mathrm{d}s\\
&\quad-a_H\int_{t_j}^{t_{j+1}}K_H(t,s)s^{H-\frac12}\mathrm{d}s\Big]\xi^j_k\xi^j_l,\\
a^3_N(t)&=a_H\sum_{j=0}^{N-1}\sum_{k=1}^{N_h}\sum_{l=1}^{N_h}\sigma\sigma'(X_{t_j})\Big(\xi^j_k\xi^j_l-1\Big)\int_{t_j}^{t_{j+1}}K_H(t,s)s^{H-\frac12}\mathrm{d}s,\\
a^4_N(t)&=\sum_{j=0}^{N-1}\sum_{k=1}^{N_h}\sum_{l=1}^{N_h}\sigma\sigma'(X_{t_j})\Big[\int_{t_j}^{t_{j+1}}K_H(t,s)m^j_k(s)\int_0^{t_j}m^j_l(r)\Big(K_H(t_j,r)-K_H(s,r)\Big)\mathrm{d}r\mathrm{d}s\Big]\xi^j_k\xi^j_l.
\end{aligned}$$

From (2.5) and assumption A we conclude that

$$\mathbb{E}\big[|a^1_N(t)|^2\big]\le C\sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}}\mathbb{E}(|X_s-X_{t_j}|^2)\mathrm{d}s,$$

so, from Lemma 4.1, it follows that

$$\mathbb{E}\big[|a^1_N(t)|^2\big]\le Ch^{2-2H}. \tag{4.13}$$

For the next term, the same reasoning as in $\mathbb{E}[|\beta^{3,1}_{N,2}(t)|^2]$ yields

$$\begin{aligned}
\mathbb{E}\big[|a^2_N(t)|^2\big]&\le C\sum_{j=0}^{N-1}\sum_{k=1}^{N_h}\sum_{l=1}^{N_h}\Big|\int_{t_j}^{t_{j+1}}K_H(t,s)m^j_k(s)\Big(\int_0^s K_H(s,r)m^j_l(r)\mathrm{d}r-\int_0^{t_j}K_H(t_j,r)m^j_l(r)\mathrm{d}r\Big)\mathrm{d}s\\
&\quad-a_H\int_{t_j}^{t_{j+1}}K_H(t,s)s^{H-\frac12}\mathrm{d}s\Big|^2\\
&\le C\sum_{j=0}^{N-1}\sum_{k=1}^{N_h}\sum_{l=1}^{N_h}\Big|\int_{t_j}^{t_{j+1}}K_H(t,s)\Big[\frac1h\Big(\int_0^s K_H(s,r)\mathrm{d}r-\int_0^{t_j}K_H(t_j,r)\mathrm{d}r\Big)\mathrm{d}s-a_Hs^{H-\frac12}\Big]\mathrm{d}s\Big|^2.
\end{aligned}$$

Cauchy Schwarz's inequality, together with Equation (2.10) implies

$$\begin{aligned}
\mathbb{E}\big[|a^2_N(t)|^2\big]&\le Ca_H^2\sum_{j=0}^{N-1}\sum_{k=1}^{N_h}\sum_{l=1}^{N_h}\Big|\int_{t_j}^{t_{j+1}}K_H(t,s)\Big[\frac1h\Big(s^{H+\frac12}-t_j^{H+\frac12}\Big)-s^{H-\frac12}\Big]\mathrm{d}s\Big|^2\\
&\le\frac{C}{h^2}\sum_{j=0}^{N-1}\sum_{k=1}^{N_h}\sum_{l=1}^{N_h}\Big|\int_{t_j}^{t_{j+1}}K_H(t,s)\Big(s^{H-\frac12}(s-t_{j+1})\Big)\mathrm{d}s\Big|^2\\
&\le\frac{C}{h^2}\sum_{j=0}^{N-1}\sum_{k=1}^{N_h}\sum_{l=1}^{N_h}\Big(\int_{t_j}^{t_{j+1}}(s-t_{j+1})^2\mathrm{d}s\Big)\Big(\int_{t_j}^{t_{j+1}}K_H^2(t,s)s^{2H-1}\mathrm{d}s\Big)\\
&\le CN_h^2Nh^2\le Ch^{1-2x}.
\end{aligned} \tag{4.14}$$

Similarly, the independence of ξ_k^j and ξ_l^j, (2.5) and Cauchy Schwarz's inequality assert that

$$\begin{aligned}\mathbb{E}\big[|a_N^3(t)|^2\big] &\leq C\sum_{j=0}^{N-1}\sum_{k=1}^{N_h}\sum_{l=1}^{N_h}\mathbb{E}(\xi_k^j\xi_l^j-1)^2\Big|\int_{t_j}^{t_{j+1}}K_H(t,s)s^{H-\frac{1}{2}}\mathrm{d}s\Big|^2\\ &\leq Ch\sum_{j=0}^{N-1}\sum_{k,l=1,k\neq l}^{N_h}\Big(\int_{t_j}^{t_{j+1}}K_H^2(t,s)s^{2H-1}\mathrm{d}s\Big)\\ &\leq CN_h^2Nh^2\leq Ch^{1-2x}.\end{aligned} \tag{4.15}$$

Furthermore,

$$\begin{aligned}\mathbb{E}\big[|a_N^4(t)|^2\big] &\leq C\sum_{j=0}^{N-1}\sum_{k=1}^{N_h}\sum_{l=1}^{N_h}\Big|\int_{t_j}^{t_{j+1}}K_H(t,s)m_k^j(s)\int_0^{t_j}m_l^j(r)\Big[K_H(t_j,r)-K_H(s,r)\Big]\mathrm{d}r\mathrm{d}s\Big|^2\\ &\leq \frac{C}{h^2}\sum_{j=0}^{N-1}\sum_{k=1}^{N_h}\sum_{l=1}^{N_h}\Big(\int_{t_j}^{t_{j+1}}K_H^2(t,s)\mathrm{d}s\Big)\Big(\int_{t_j}^{t_{j+1}}s\Big(\int_0^s\Big[K_H(t_j,r)-K_H(s,r)\Big]\mathrm{d}r\Big)^2\mathrm{d}s\\ &\leq \frac{C}{h^2}N_h^2\sum_{j=0}^{N-1}\Big(\int_{t_j}^{t_{j+1}}K_H^2(t,s)\mathrm{d}s\Big)\Big(\int_{t_j}^{t_{j+1}}s(s-t_j)^{2H}\mathrm{d}s\Big)\\ &\leq CN_h^2h^2\sum_{j=0}^{N-1}j^{2-2H}\leq CN_h^2N^{3-2H}h^2\leq Ch^{2H-1-2x}.\end{aligned} \tag{4.16}$$

From (4.13)-(4.16) we conclude that

$$\mathbb{E}\big[|\beta_{N,2}^5(t)|^2\big]\leq CN_h^2h^{2H-1}\leq C\left(h^{2H-1-2x}+h^{2-2H}\right). \tag{4.17}$$

Consequently, according to (4.9) – (4.12), and (4.17) we get

$$\mathbb{E}\big[|\beta_{N,2}(t)|^2\big]\leq Ch^{\alpha}, \tag{4.18}$$

where

$$\alpha=\min\Big\{(2-2H),(2H-1-2x),(3-4H-2x)\Big\}\text{ for }\frac{1}{2}<H<\frac{3}{4},$$

and

$$\alpha=\min\Big\{(2-2H),(2H-1-2x),(4-4H-2x)\Big\}\text{ for }\frac{3}{4}<H<1.$$

Finally, the term $\mathbb{E}\big[|\beta_{N,3}(t)|^2\big]$ can be handled in the same way as follows

$$\begin{aligned}\beta_{N,3}(t) &= \sum_{k=1}^{N_h}\int_0^{t_1}K_H(t,s)m_k^0(s)\int_0^s\sigma'(X_u^N)\Big(\int_0^u\frac{\partial K_H(u,r)}{\partial u}b(X_r^N)\mathrm{d}r\Big)\mathrm{d}u\mathrm{d}s\xi_k^j\\ &\quad+\sum_{j=1}^{N-1}\sum_{k=1}^{N_h}\int_{t_j}^{t_j+1}K_H(t,s)m_k^j(s)\int_{t_j}^s\sigma'(X_u^N)\Big(\int_0^u\frac{\partial K_H(u,r)}{\partial u}b(X_r^N)\mathrm{d}r\Big)\mathrm{d}u\mathrm{d}s\xi_k^j\\ &= \beta_{N,3}^1(t)+\beta_{N,3}^2(t).\end{aligned}$$

So we have

$$\begin{aligned}\mathbb{E}\big[|\beta^1_{N,3}(t)|^2\big] &\leq C\sum_{k=1}^{N_h}\Big[\int_0^{t_1}|K_H(t,s)||m_k^0(s)|\int_0^s\Big(\int_0^u\frac{\partial K_H(u,r)}{\partial u}\mathrm{d}r\Big)\mathrm{d}u\mathrm{d}s\Big]^2\\&\leq C\frac{N_h}{h}\Big[\int_0^{t_1}|K_H(t,s)|\int_0^s\int_0^u\frac{\partial K_H(u,r)}{\partial u}\mathrm{d}r\mathrm{d}u\mathrm{d}s\Big]^2\\&\leq Ca_H^2\frac{N_h}{h}\Big(\int_0^{t_1}K_H^2(t,s)\mathrm{d}s\Big)\Big(\int_0^{t_1}(\int_0^s u^{H-\frac{1}{2}}\mathrm{d}u)^2\mathrm{d}s\Big)\\&\leq C\frac{N_h}{h}h^{2-2H}\Big(\int_0^{t_1}s^{2H+1}\mathrm{d}s\Big)=CN_hh^3,\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}|\big[\beta^2_{N,3}(t)|^2\big] &\leq \sum_{j=1}^{N-1}\sum_{k=1}^{N_h}\Big[\int_{t_j}^{t_j+1}|K_H(t,s)||m_k^j(s)|\int_{t_j}^s\Big(\int_0^u\frac{\partial K_H(u,r)}{\partial u}\mathrm{d}r\Big)\mathrm{d}u\mathrm{d}s\Big]^2\\&\leq C\frac{N_h}{h}\sum_{j=1}^{N-1}\Big[\int_{t_j}^{t_{j+1}}|K_H(t,s)|\int_{t_j}^s\int_0^u\frac{\partial K_H(u,r)}{\partial u}\mathrm{d}r\mathrm{d}u\mathrm{d}s\Big]^2\\&\leq Ca_H^2\frac{N_h}{h}\sum_{j=1}^{N-1}\Big(\int_{t_j}^{t_{j+1}}K_H^2(t,s)\mathrm{d}s\Big)\Big(\int_{t_j}^{t_{j+1}}(\int_{t_j}^s u^{H-\frac{1}{2}}\mathrm{d}u)^2\mathrm{d}s\Big)\\&\leq C\frac{N_h}{h}\sum_{j=1}^{N-1}\Big(\int_{t_j}^{t_{j+1}}K_H^2(t,s)\mathrm{d}s\Big)\Big(\int_{t_j}^{t_{j+1}}(s^{H+\frac{1}{2}}-t_j^{H+\frac{1}{2}})^2\mathrm{d}s\Big).\end{aligned}$$

By the mean value theorem we deduce

$$\mathbb{E}\big[|\beta^2_{N,3}(t)|^2\big]\leq CN_hh^3\sum_{j=1}^{N-1}j^{1-2H}(j+1)^{2H-1}\leq CN_hh^3N^{3-2H}\leq CN_hh^{2H}=Ch^{2H-x},$$

consequently, we obtain

$$\mathbb{E}\big[|\beta_{N,3}(t)|^2\big]\leq CN_hh^{2H}=Ch^{2H-x}. \tag{4.19}$$

Finally, by applying Equations (4.8), (4.18), and (4.19) we conclude

$$\mathbb{E}\left(|X_t^N-X_t|^2\right)\leq Ch^\gamma+C\int_0^t s^{1-2H}\mathbb{E}|X_s^N-X_s|^2\mathrm{d}s,$$

where

$$\gamma=\min\Big\{(2-2H-x),(\frac{1}{2}-x),(2H-1-2x),(3-4H-2x)\Big\},\ \text{for } \frac{1}{2}<H<\frac{3}{4},$$

and

$$\gamma=\min\Big\{(2-2H-x),(\frac{1}{2}-x),(2H-1-2x),(4-4H-2x)\Big\},\ \text{for } \frac{3}{4}<H<1.$$

By using Gronwall's lemma we get

$$\mathbb{E}\left(|X_t^N-X_t|^2\right)\leq Ch^\gamma,$$

where

$$\gamma=\min\left\{(2H-1-2x),(3-4H-2x)\right\},\ \text{for } \frac{1}{2}<H<\frac{3}{4},$$

and

$$\gamma = \min\left\{(2-2H-x),(2H-1-2x),\right\}, \text{ for } \frac{3}{4} < H < 1,$$

which is the desired conclusion. □

5. Numerical experiments

In this section, we discuss the implementation of the numerical method (3.7) by providing an example, and we present the theoretically obtained results in numerical form.

Example 5.1. Consider the following integral equation

$$X_t = \int_0^t K_H(t,s)\sin(X_s)\mathrm{d}s + \int_0^t K_H(t,s)\cos(X_s)\mathrm{d}W_s,\ t \in [0,1], \tag{5.1}$$

where W_t denotes Brownian motion and $K_H(t,s)$ is a deterministic kernel given by (2.4).

In Figures 1 and 2, we have compared the results obtained from the numerical method with the theoretical results. To achieve this, we obtained the mean square errors by averaging over 1000 independent Brownian paths over $[0,1]$ and as a reference solution, we used numerical approximation with the step-size 2^{-13}.

In 1 we obtained the results with $H = 0.6$, $N_h = \lceil h^{-x}\rceil$ with $x = 0.05$ and apply the numerical method (3.7) with 64, 128, 258, and 512 steps. From Theorem 4.2 we conclude

$$\mathbb{E}\left(|X_t^N - X_t|^2\right) = O(h^{0.1}). \tag{5.2}$$

We have plotted the best linear approximation (least squares) of the mean square errors from the numerical data and compared it with the line obtained from the theoretical outcome, whose slope is 0.1. Clearly, Figure 1 confirms that the numerical results from the Wong-Zakai method closely match the theoretical results. Moreover, Figure 2 shows the numerical results with the Hurst parameter $H = 0.8$ and $x = 0.1$. In this case, from Theorem 4.2 we conclude

$$\mathbb{E}\left(|X_t^N - X_t|^2\right) = O(h^{0.3}). \tag{5.3}$$

In Figure 2, we have compared the best linear approximation (least squares) of the mean square error from the numerical data with a linear line having a slope of 0.3. Clearly, this figure confirms the obtained theoretical results. It should be mentioned that we have used the Trapezoidal rule for the numerical integration.

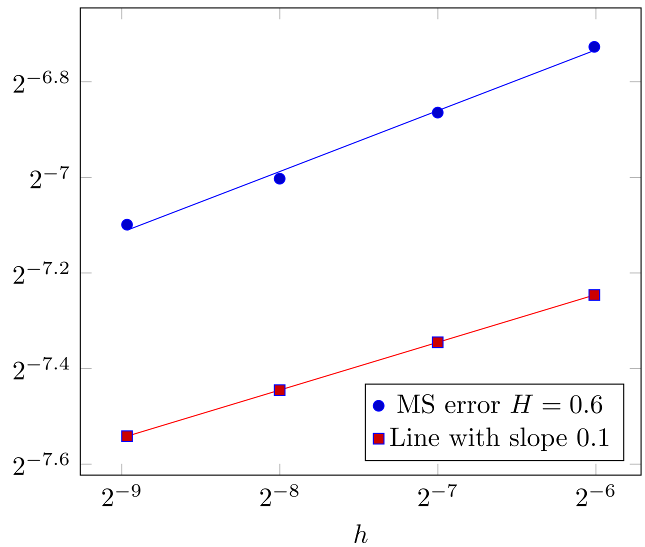

FIGURE 1. Mean square(MS) errors of the Wong-Zakai approximation for Hurst parameter $H = 0.6$ for Example 5.1.

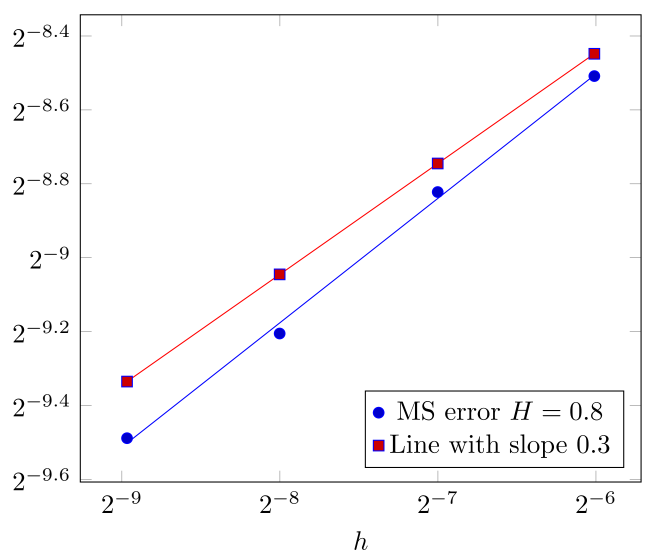

FIGURE 2. Mean square(MS) errors of the Wong-Zakai approximation for Hurst parameter $H = 0.8$ for Example 5.1.

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