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Upper and lower solutions for fractional integro-differential equation of higher-order and with nonlinear boundary conditions

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Abstract

This paper delves into the identification of upper and lower solutions for a high-order fractional integro-differential equation featuring non-linear boundary conditions. By introducing an order relation, we define these upper and lower solutions. Through a rigorous approach, we demonstrate the existence of these solutions as the limits of sequences derived from carefully selected problems, supported by the application of Arzelà-Ascoli's theorem. To illustrate the significance of our findings, we provide an illustrative example. This research contributes to a deeper understanding of solutions in the context of complex fractional integro-differential equations.

Keywords. Arzelà-Ascoli's theorem, Caputo fractional derivative, Integro-differential equation, Nonlinear boundary conditions.
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1. Introduction

Recently, the theory of fractional calculus has been developed by several researchers in this field [4, 7, 10, 23]. It appears in several real-life problems such as: viscoelasticity, fluid mechanics, biology, population models, signals processing, and it can be used to describe each real-life problem with memory effect [16, 17, 19, 24].

Fractional differential equations appear as a fundamental tool for modeling certain real-world phenomena with memory effect (see [18, 19, 25]).

In [20], Elyas Shivanian et al. investigated the existence and uniqueness of solutions to the following class of nonlinear high-order fractional integral-differential equations:

$$\begin{cases} {}_0D_t^{\alpha} u + \int_0^t K(t,s) f\left(s, u(s), u'(s), u''(s)\right) ds = -g\left(t, u(t), u'(t), u''(t)\right), \quad t \in (0,1), \quad 3 < \alpha \leq 4, \\ u(0) = u'(0) = u''(0) = 0, \\ u(1) = \lambda \int_0^1 u(s) ds. \end{cases}$$

where ${}_0D_t^{\alpha}$ represents the left Rieman-Liouville fractional derivative.

In [11], the authors studied with more details the existence of extremal solutions for coupled systems of nonlinear fractional differential equations of the following class:

$$\begin{cases} {}^c\mathbb{D}_a^{\alpha;\psi} u(t) = \omega_1(t, u(t), v(t)), \quad u(a) = u_a, \\ {}^c\mathbb{D}_a^{\alpha;\psi} v(t) = \omega_2(t, u(t), v(t)), \quad v(a) = v_a, \end{cases} \quad t \in \mathrm{J} := [a, b],$$

with ${}^c\mathbb{D}_{a^+}^{\alpha;\psi}$ is the ψ-Caputo fractional derivative of order $\alpha \in (0, 1]$.

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In light of the findings mentioned earlier, this endeavor is driven by the desire to expand upon the aforementioned challenges. Specifically, we aim to address a more generalized form of the fractional integro-differential equation, represented as follows:

$$\begin{cases} {}^{c}\mathcal{D}^{\mu}_{0^+} w(t) = \varphi\left(t, w(t), {}^{c}\mathcal{D}^{\sigma_1}_{0^+} w(t), \dots, {}^{c}\mathcal{D}^{\sigma_n}_{0^+} w(t)\right) + \mathcal{I}^{\nu}_{0^+}\psi(t, w(t)), & t \in I = (0,1),\ n,\ m \in \mathbb{N}^*, \\ & \nu > 0,\ 0 < \sigma_i < m,\ i = 1, \dots, n, \\ \phi_0\left(w(0), \int_0^{\alpha} w(\tau)d\tau, \int_{\beta}^{1} w(\tau)d\tau\right) = 0, & 0 < \alpha < \beta < 1, \\ \phi_j\left(w^{(j)}(0), w^{(j)}(1)\right) = 0, & j = 1, \dots, m-1. \end{cases} \tag{1.1}$$

where ${}^{c}\mathcal{D}^{\mu}_{0^+} w(.)$ designates the Caputo fractional derivative of order $m-1 < \mu \leq m$, $\mathcal{I}^{\nu}_{0^+}$ represents the Riemann-Liouville operator of fractional order $\nu > 0$, $\phi_0 : \mathbb{R}^3 \longrightarrow \mathbb{R}$, ψ, $\phi_j : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $j = 1, \dots, m-1$, and $\varphi : I \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ are given functions.

Our primary objective is to explore the existence of upper and lower solutions for this problem. Furthermore, we introduce an order relation to precisely define these solutions. Our results are predicated on the existence of upper and lower solutions as the limits of solution sequences derived from carefully selected problems, with Arzelà-Ascoli's theorem providing the necessary theoretical framework. To corroborate our findings, we present an illustrative example at the end of our study.

The references [1–3, 5–9, 12–15, 21, 22] provide a strong foundation and support for the research conducted in this paper. They collectively contribute to the understanding of fractional calculus, boundary value problems, and the development of methodologies for addressing the challenges posed by fractional integro-differential equations with nonlinear boundary conditions.

2. Preliminaries

We begin by recalling some definitions and fundamental notions on fractional calculus used to construct our results.

Definition 2.1 ([16])**.** For $h \in L^1[0,1]$, we define the left fractional integral of order $\nu > 0$ of Riemann-Liouville as follows:

$$\mathcal{I}^{\nu}_{0^+} h(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} h(\tau) d\tau.$$

Let $m \in \mathbb{N}$ and $m-1 < \mu \leq m$. we recall the following definition:

Definition 2.2 ([16])**.** The left Caputo fractional derivative for a function $h \in \mathcal{AC}^{m-1}[0,1]$ of order μ is defined as follows

$${}^{C}\mathcal{D}^{\mu}_{0^+} h(t) = \frac{1}{\Gamma(m-\mu)} \int_0^t (t-\tau)^{m-\mu-1} h^{(m)}(\tau) d\tau,$$

where, Γ represents the Gamma function of Euler.

Lemma 2.3 ([26])**.** *We have*

$${}^{C}\mathcal{I}^{\mu}_{0^+}\left[{}^{C}\mathcal{D}^{\mu}_{0^+} h(t)\right] = h(t) + a_0 + a_1 t + a_2 t^2 + \dots + a_{m-1} t^{m-1},$$

for $a_k \in \mathbb{R}$ and $k = 0, 1, 2, \dots, m-1$.

As a consequence, we have the following result.

Corollary 2.4. *If $h \in \mathcal{AC}[0,1]$, the following problem*

$$\begin{cases} {}^{C}\mathcal{D}^{\mu}_{0^+} w(t) = h(t), & t \in I,\ \mu \in (m-1, m], \\ w^{(j)}(0) = \gamma_j, & j = 0, \dots, m-1. \end{cases} \tag{2.1}$$

has a unique solution $w \in \mathcal{AC}^{m-1}[0,1]$ given as follows

$$w(t) = {}^{C}\mathcal{I}^{\mu}_{0^+} h(t) + \sum_{j=0}^{m-1} \frac{\gamma_j}{j!} t^j.$$

3. Main Results

Firstly, we present some useful results.

Lemma 3.1. *If $h \in \mathcal{AC}[0,1]$, the following problem*

$$\begin{cases} {}^{C}\mathcal{D}^{\mu}_{0^+} w(t) = h(t), & t \in I,\ \mu \in (m-1, m], \\ w(0) = \displaystyle\int_0^{\alpha} w(\tau) d\tau + \gamma_0, & \alpha \in (0,1), \\ w^{(j)}(0) = \gamma_j, & j = 1, \dots, m-1. \end{cases} \tag{3.1}$$

has a unique solution $w \in \mathcal{AC}^{m-1}[0,1]$ given as follows

$$w(t) = \frac{1}{1-\alpha}\left(\mathcal{I}^{\mu+1}_{0^+} h(\alpha) + \sum_{j=1}^{m-1} \frac{\gamma_j}{(j+1)!}\alpha^{j+1} + \gamma_0 \right) + \sum_{j=1}^{m-1} \frac{\gamma_j}{j!} t^j + \mathcal{I}^{\mu}_{0^+} h(t).$$

Proof. By applying $\mathcal{I}^{\mu}_{0^+}$ on both sides of first equation of (3.1), we have

$$w(t) = \mathcal{I}^{\mu}_{0^+} h(t) + \sum_{j=0}^{m-1} a_j t^j, \quad a_j \in \mathbb{R} \quad \text{for} \quad j = 0, 1, \dots, m-1. \tag{3.2}$$

By applying first derivative on both sides of above equation, we get

$$w'(t) = \mathcal{I}^{\mu-1}_{0^+} h(t) + a_1 + \sum_{j=1}^{m-2} (j+1) a_{j+1} t^j. \tag{3.3}$$

Then, we obtain $a_1 = w'(0) = \gamma_1$.
Now, we apply first derivative on both sides of (3.3), we obtain

$$2a_2 = w^{(2)}(0) = \gamma_2.$$

By induction, we have

$$j! a_j = w^{(j)}(0) = \gamma_j, \quad \text{for} \quad j = 1, 2, \dots, m-1.$$

Then, equation (3.2) becomes

$$w(t) = \mathcal{I}^{\mu}_{0^+} h(t) + a_0 + \sum_{j=1}^{m-1} \frac{\gamma_j}{j!} t^j, \tag{3.4}$$

let us integrate both sides of equation (3.4) on $[0, \alpha]$, we have

$$\int_0^{\alpha} w(t) dt = \mathcal{I}^{\mu+1}_{0^+} h(\alpha) + \alpha a_0 + \sum_{j=1}^{m-1} \frac{\gamma_j}{(j+1)!} \alpha^{j+1}. \tag{3.5}$$

Now, form (3.1) and (3.5, we obtain

$$w(0) = a_0 = \int_0^{\alpha} w(\tau) d\tau + \gamma_0 = \mathcal{I}^{\mu+1}_{0^+} h(\alpha) + \alpha a_0 + \sum_{j=1}^{m-1} \frac{\gamma_j}{(j+1)!} \alpha^{j+1} + \gamma_0.$$

Which implies that

$$a_0 = \frac{1}{1-\alpha}\left(\mathcal{I}^{\mu+1}_{0^+} h(\alpha) + \sum_{j=1}^{m-1} \frac{\gamma_j}{(j+1)!} \alpha^{j+1} + \gamma_0 \right).$$

By substitution into the equation (3.4), we get

$$w(t) = \frac{1}{1-\alpha}\left(\mathcal{I}_{0^+}^{\mu+1}h(\alpha) + \sum_{j=1}^{m-1}\frac{\gamma_j}{(j+1)!}\alpha^{j+1} + \gamma_0\right) + \sum_{j=1}^{m-1}\frac{\gamma_j}{j!}t^j + \mathcal{I}_{0^+}^{\mu}h(t).$$

From above equation, we can easily fall back on problem (3.1). □

Applying the same approach, we have the following lemma

Lemma 3.2. *If $h \in \mathcal{AC}[0,1]$, the following problem*

$$\begin{cases} {}^{C}\mathcal{D}_{0+}^{\mu}w(t) = h(t), & t \in I,\ \mu \in (m-1,m], \\ w(0) = \displaystyle\int_{\beta}^{1} w(\tau)d\tau + \gamma_0, & \beta \in (0,1), \\ w^{(j)}(0) = \gamma_j, & j = 1,\dots,m-1. \end{cases}$$

has a unique solution $w \in \mathcal{AC}^{m-1}[0,1]$ given as follows

$$w(t) = \frac{1}{\beta}\left(\mathcal{I}_{0^+}^{\mu+1}h(1) - \mathcal{I}_{0^+}^{\mu+1}h(\beta) + \sum_{j=1}^{m-1}\frac{\gamma_j}{(j+1)!}\left(1-\beta^{j+1}\right) + \gamma_0\right) + \sum_{j=1}^{m-1}\frac{\gamma_j}{j!}t^j + \mathcal{I}_{0^+}^{\mu}h(t).$$

Let us introduce the following relation:
For $x,\ y \in \mathcal{AC}^{(m)}[0,1]$

$$x \succeq y \Leftrightarrow \left(x^{(j)} \geq y^{(j)} \quad \text{on} \quad [0,1], \quad \forall j = 0,\dots,m\right).$$

It easy to verify that $\succeq$ define an order relation on $\mathcal{AC}^{(m)}[0,1]$.
Note that

$$\left({}^{C}\mathcal{D}_{0+}^{\mu}x \geq {}^{C}\mathcal{D}_{0+}^{\mu}y \quad \text{on} \quad [0,1], \quad \forall \mu \in [0,m]\right) \Longrightarrow \left(x^{(j)} \geq y^{(j)} \quad \text{on} \quad [0,1], \quad \forall j = 0,\dots,m\right).$$

Inversely, suppose that

$$x^{(j)} \geq y^{(j)} \quad \text{on} \quad [0,1], \quad \forall j = 0,\dots,m.$$

For $\mu \in [0,m]$. Let $k \in (0,m)$ be such that $\mu \in (k,k+1]$, we have

$${}^{C}\mathcal{D}_{0+}^{\mu}x(t) = {}^{C}\mathcal{I}_{0^+}^{k+1-\mu}x^{(k+1)}(t) \geq {}^{C}\mathcal{I}_{0^+}^{k+1-\mu}y^{(k+1)}(t) = {}^{C}\mathcal{D}_{0+}^{\mu}y(t), \quad \forall t \in I.$$

Therefore,

$$x \succeq y \Leftrightarrow \left(x^{(j)} \geq y^{(j)} \quad \text{on} \quad [0,1], \quad \forall j = 0,\dots,m\right) \Leftrightarrow \left({}^{C}\mathcal{D}_{0+}^{\mu}x \geq {}^{C}\mathcal{D}_{0+}^{\mu}y \quad \text{on} \quad [0,1], \quad \forall \mu \in [0,m]\right).$$

Lemma 3.3. *Let $w \in \mathcal{AC}^{m-1}[0,1]$ such that*

$$\begin{cases} {}^{C}\mathcal{D}_{0+}^{\mu}w(t) \geq 0, & t \in I,\ \mu \in (m-1,m], \\ w(0) - \displaystyle\int_{0}^{\alpha} w(\tau)d\tau - \int_{\beta}^{1} w(\tau)d\tau \geq 0, & 0 < \alpha < \beta < 1, \\ w^{(j)}(0) \geq 0, & j = 1,\dots,m-1. \end{cases} \tag{3.6}$$

Then $w \succeq 0$.

Proof. From (3.6), we have

$$\begin{cases} {}^{C}\mathcal{D}_{0+}^{\mu} w(t) = {}^{C}\mathcal{D}_{0+}^{\mu-m+1} w^{(m-1)}(t) \geq 0, \quad \forall t \in I, \\ w^{(m-1)}(0) \geq 0. \end{cases}$$

Using Corollary 2.4 and applying ${}^{C}\mathcal{I}_{0+}^{\mu-m+1}$ on both sides of first inequality of above problem, we have

$$w^{(m-1)}(t) - w^{(m-1)}(0) \geq 0, \quad \forall t \in I.$$

Not that $\mu - m + 1 \in (0,1)$.
Then, we get

$$\begin{cases} w^{(m-1)}(t) \geq w^{(m-1)}(0) \geq 0, \quad \forall t \in I, \\ w^{(m-2)}(0) \geq 0. \end{cases}$$

Likewise, by applying ${}^{C}\mathcal{I}_{0+}^{1}$ on both sides of first inequality of above problem, we get

$$\begin{cases} w^{(m-2)}(t) \geq 0, \quad \forall t \in I, \\ w^{(m-3)}(0) \geq 0. \end{cases}$$

Similarly, we get by induction

$$w^{(j)}(t) \geq 0, \quad \forall t \in I, \text{ for } j = 1, \ldots, m-1.$$

And we have

$$\begin{cases} w'(t) \geq 0, \quad t \in I, \\ w(0) - \displaystyle\int_0^{\alpha} w(\tau)d\tau - \int_{\beta}^{1} w(\tau)d\tau \geq 0. \end{cases} \tag{3.7}$$

By applying ${}^{C}\mathcal{I}_{0+}^{1}$ on both sides of first inequality of (3.7), then integrating the obtained inequality on $(0,\alpha]$ and on $[\beta,1]$, we get

$$\begin{cases} w(t) - w(0) \geq 0, \quad t \in I, \\ \displaystyle\int_0^{\alpha} w(\tau)d\tau - \alpha w(0) \geq 0, \\ \displaystyle\int_{\beta}^{1} w(\tau)d\tau - (1-\beta)w(0) \geq 0, \\ w(0) - \displaystyle\int_0^{\alpha} w(\tau)d\tau - \int_{\beta}^{1} w(\tau)d\tau \geq 0. \end{cases}$$

Hence,

$$\begin{cases} w(t) \geq w(0), \quad t \in I, \\ (\beta - \alpha)w(0) \geq 0. \end{cases}$$

which means that $w(t) \geq 0$ on I.
Consequently, $w \succeq 0$ on I.

□

Definition 3.4. Let $u \in \mathcal{AC}^{(m-1)}[0,1]$, $\mu \in (m-1, m]$.

(1) w is said to be an upper-solution of (1.1) if it satisfies

$$\begin{cases} {}^c\mathcal{D}^{\mu}_{0+}w(t) \geq \varphi\left(t, w(t), {}^c\mathcal{D}^{\sigma_1}_{0+}w(t), \ldots, {}^c\mathcal{D}^{\sigma_n}_{0+}w(t)\right) + \mathcal{I}^{\nu}_{0+}\psi(t, w(t)), & t \in \mathcal{I}, \\ \phi_0\left(w(0), \int_0^{\alpha} w(\tau)d\tau, \int_{\beta}^1 w(\tau)d\tau\right) \geq 0, & \\ \phi_j\left(w^{(j)}(0), w^{(j)}(1)\right) \geq 0, & j = 1, \ldots, m-1. \end{cases} \tag{3.8}$$

(2) w is called an lower-solution of (1.1) if it satisfies the following problem

$$\begin{cases} {}^c\mathcal{D}^{\mu}_{0+}w(t) \leq \varphi\left(t, w(t), {}^c\mathcal{D}^{\sigma_1}_{0+}w(t), \ldots, {}^c\mathcal{D}^{\sigma_n}_{0+}w(t)\right) + \mathcal{I}^{\nu}_{0+}\psi(t, w(t)), & t \in \mathcal{I}, \\ \phi_0\left(w(0), \int_0^{\alpha} w(\tau)d\tau, \int_{\beta}^1 w(\tau)d\tau\right) \leq 0, & \\ \phi_j\left(w^{(j)}(0), w^{(j)}(1)\right) \leq 0, & j = 1, \ldots, m-1. \end{cases} \tag{3.9}$$

In the rest of this paper, we assume that w_u is an upper-solution and w_l is a lower-solution of (1.1), and that $w_u \succeq w_l$.

Now, we cite assumptions used to present our result:

($\mathcal{A}_1$) For $w \in \mathcal{AC}^{(m-1)}(0,1]$, $\varphi\left(., w, {}^c\mathcal{D}^{\sigma_1}_{0+}w, \ldots, {}^c\mathcal{D}^{\sigma_n}_{0+}w\right) \in \mathcal{AC}[0,1]$, and for x, $y \in \mathcal{AC}^{(m-1)}[0,1]$

$$x \succeq y \Longrightarrow \varphi\left(t, x(t), {}^c\mathcal{D}^{\sigma_1}_{0+}x(t), \ldots, {}^c\mathcal{D}^{\sigma_n}_{0+}x(t)\right) \geq \varphi\left(t, y(t), {}^c\mathcal{D}^{\sigma_1}_{0+}y(t), \ldots, {}^c\mathcal{D}^{\sigma_n}_{0+}y(t)\right), \forall t \in I.$$

($\mathcal{A}_2$) ψ be a continuous functions such that, for x, $y \in \mathcal{AC}^{(m-1)}[0,1]$

$$x \succeq y \Longrightarrow \psi\left(t, x(t)\right) \geq \psi\left(t, y(t)\right), \forall t \in I.$$

($\mathcal{A}_3$) ϕ_j, $j = 1, 2, \ldots, m-1$ are continuous functions, and there exists $\eta_j < 0$, $\xi_j \leq 0$, $j = 1, 2, \ldots, m-1$ such that, for x, $\tilde{x} \in \left[w_l^{(j)}(0), w_u^{(j)}(0)\right]$, y, $\tilde{y} \in \left[w_l^{(j)}(1), w_u^{(j)}(1)\right]$

$$(x \leq \tilde{x} \text{ and } y \leq \tilde{y}) \Longrightarrow \phi_j(x, y) - \phi_j(\tilde{x}, \tilde{y}) \geq \frac{-1}{\eta_j}\left(x - \tilde{x}\right) + \xi_j\left(y - \tilde{y}\right),$$

for all $j \in \{1, 2, \ldots, m-1\}$.

($\mathcal{A}_4$) ϕ_0 is a continuous function, and there exists $\eta_0 < 0$, ξ_0, $\rho_0 \leq 0$, such that $\eta_0\xi_0$, $\eta_0\rho_0 \geq 1$, and for x, $\tilde{x} \in [w_l(0), w_u(0)]$, y, $\tilde{y} \in \left[\int_0^{\alpha} w_l(\tau)d\tau, \int_0^{\alpha} w_u(\tau)d\tau\right]$ and z, $\tilde{z} \in \left[\int_{\beta}^1 w_l(\tau)d\tau, \int_{\beta}^1 w_u(\tau)d\tau\right]$

$$(x \leq \tilde{x} \text{ and } y \leq \tilde{y} \text{ and } z \leq \tilde{z}) \Longrightarrow \phi_0(x, y, z) - \phi_0(\tilde{x}, \tilde{y}, \tilde{z}) \geq \frac{-1}{\eta_0}\left(x - \tilde{x}\right) + \xi_0\left(y - \tilde{y}\right) + \rho_0\left(z - \tilde{z}\right).$$

Now, we are in a position to present the following result.

Theorem 3.5. *Suppose that assumptions $(\mathcal{A}_1) - (\mathcal{A}_4)$ are satisfied. Then there exist ${}_ls$, ${}_us \in [w_l, w_u]$ lower and upper-solutions of (1.1) respectively. In addition, there exist two sequences $({}_lw_k)_{k\geq 0}$, $({}_uw_k)_{k\geq 0} \subseteq [w_l, w_u]$, one is decreasing and the other is non-decreasing such that:*

$$\forall \mu \in [0, m-1], \quad {}^c\mathcal{D}^{\mu}_{0+}{}_lw_k \longrightarrow {}^c\mathcal{D}^{\mu}_{0+}{}_ls \quad \text{and} \quad {}^c\mathcal{D}^{\mu}_{0+}{}_uw_k \longrightarrow {}^c\mathcal{D}^{\mu}_{0+}{}_us \text{ as } k \to \infty,$$

uniformly on I.

Proof. Let ${}_lw_0 = w_l$ and ${}_uw_0 = w_u$.
For $k \geq 1$, we construct $({}_lw_k)_{k\geq 1}$ and $({}_uw_k)_{k\geq 1}$ as unique solutions of

$$\begin{cases} {}^c\mathcal{D}^{\mu}_{0+}{}_lw_{k+1}(t) = \varphi\left(t, {}_lw_k(t), {}^c\mathcal{D}^{\sigma_1}_{0+}{}_lw_k(t), \ldots, {}^c\mathcal{D}^{\sigma_n}_{0+}{}_lw_k(t)\right) + \mathcal{I}^{\nu}_{0+}\psi(t, {}_lw_k(t)), & t \in I, \ k \geq 0, \\ {}_lw_{k+1}(0) - \displaystyle\int_0^{\alpha} {}_lw_{k+1}(\tau)d\tau = {}_lw_k(0) - \int_0^{\alpha} {}_lw_k(\tau)d\tau + \eta_0\phi_0\left({}_lw_k(0), \int_0^{\alpha} {}_lw_k(\tau)d\tau, \int_{\beta}^1 {}_lw_k(\tau)d\tau\right), & \\ {}_lw^{(j)}_{k+1}(0) = {}_lw^{(j)}_k(0) + \eta_j\phi_j\left({}_lw^{(j)}_k(0), {}_lw^{(j)}_k(1)\right), & j = 1, \ldots, m-1. \end{cases} \tag{3.10}$$

and

$$\begin{cases} {}^c\mathcal{D}^{\mu}_{0+}\, {}_u w_{k+1}(t) = \varphi\left(t, {}_u w_k(t), {}^c\mathcal{D}^{\sigma_1}_{0+}\, {}_u w_k(t), \ldots, {}^c\mathcal{D}^{\sigma_n}_{0+}\, {}_u w_k(t)\right) + \mathcal{I}^{\nu}_{0+}\psi(t, {}_u w_k(t)), & t \in I,\ k \geq 0, \\ {}_u w_{k+1}(0) - \int_\beta^1 {}_u w_{k+1}(\tau)d\tau = {}_u w_k(0) - \int_\beta^1 {}_u w_k(\tau)d\tau + \eta_0\phi_0\left({}_u w_k(0), \int_0^\alpha {}_u w_k(\tau)d\tau, \int_\beta^1 {}_u w_k(\tau)d\tau\right), & \\ {}_u w^{(j)}_{k+1}(0) = {}_u w^{(j)}_k(0) + \eta_j\phi_j\left({}_u w^{(j)}_k(0), {}_u w^{(j)}_k(1)\right), & j = 1, \ldots, m-1. \end{cases} \tag{3.11}$$

respectively.
Note that, the existence and uniqueness of these two sequences are ensured by Lemmas 3.1 and 3.2.
Now, let us prove that

$$w_u = {}_u w_0 \succeq {}_u w_1 \succeq \cdots \succeq {}_u w_k \succeq {}_u w_{k+1} \succeq \cdots \succeq {}_l w_{k+1} \succeq {}_l w_k \succeq \cdots \succeq {}_l w_1 \succeq {}_l w_0 = w_l. \tag{3.12}$$

From (3.11), we have

$$\begin{cases} {}^c\mathcal{D}^{\mu}_{0+}\, {}_u w_1(t) = \varphi\left(t, {}_u w_0(t), {}^c\mathcal{D}^{\sigma_1}_{0+}\, {}_u w_0(t), \ldots, {}^c\mathcal{D}^{\sigma_n}_{0+}\, {}_u w_0(t)\right) + \mathcal{I}^{\nu}_{0+}\psi(t, {}_u w_0(t)), & t \in I,\ k \geq 0, \\ {}_u w_1(0) - \int_\beta^1 {}_u w_1(\tau)d\tau = {}_u w_0(0) - \int_\beta^1 {}_u w_0(\tau)d\tau + \eta_0\phi_0\left({}_u w_0(0), \int_0^\alpha {}_u w_0(\tau)d\tau, \int_\beta^1 {}_u w_0(\tau)d\tau\right), & \\ {}_u w^{(j)}_1(0) = {}_u w^{(j)}_0(0) + \eta_j\phi_j\left({}_u w^{(j)}_0(0), {}_u w^{(j)}_0(1)\right), & j = 1, \ldots, m-1. \end{cases} \tag{3.13}$$

Note that ${}_u w_0 = w_u$ is an upper-solution of (1.1), then it satisfies

$$\begin{cases} {}^c\mathcal{D}^{\mu}_{0+}\, {}_u w_0(t) \geq \varphi\left(t, {}_u w_0(t), {}^c\mathcal{D}^{\sigma_1}_{0+}\, {}_u w_0(t), \ldots, {}^c\mathcal{D}^{\sigma_n}_{0+}\, {}_u w_0(t)\right) + \mathcal{I}^{\nu}_{0+}\psi(t, {}_u w_0(t)) = {}^c\mathcal{D}^{\mu}_{0+}\, {}_u w_1(t), & t \in I, \\ \phi_0\left({}_u w_0(0), \int_0^\alpha {}_u w_0(\tau)d\tau, \int_\beta^1 {}_u w_0(\tau)d\tau\right) \geq 0, & \\ \phi_j\left({}_u w^{(j)}_0(0), {}_u w^{(j)}_0(1)\right) \geq 0, & j = 1, \ldots, m-1. \end{cases} \tag{3.14}$$

Let $x_u = {}_u w_0 - {}_u w_1$. From (3.13) and (3.14), we get

$$\begin{cases} {}^c\mathcal{D}^{\mu}_{0+} x_u(t) \geq 0, & t \in I, \\ x_u(0) - \int_\beta^1 x_u(\tau)d\tau = -\eta_0\phi_0\left({}_u w_0(0), \int_0^\alpha {}_u w_0(\tau)d\tau, \int_\beta^1 {}_u w_0(\tau)d\tau\right) \geq 0, & \\ x^{(j)}_u(0) = -\eta_j\phi_j\left({}_u w^{(j)}_0(0), {}_u w^{(j)}_0(1)\right) \geq 0, & j = 1, \ldots, m-1. \end{cases} \tag{3.15}$$

Note that ${}^c\mathcal{D}^{\mu}_{0+} x_u(t) = {}^c\mathcal{D}^{\mu-(m-1)}_{0+} x^{(m-1)}_u(t)$ and $\mu - (m-1) \in (0,1)$.
Then, by applying $\mathcal{I}^{\mu-(m-1)}_{0+}$ on both sides of first inequality of (3.15), we get

$$\begin{cases} x^{(m-1)}_u(t) \geq 0, \\ x^{(m-2)}_u(0) \geq 0, \end{cases}$$

and by applying $\mathcal{I}^{1}_{0+}$, we obtain

$$x^{(m-2)}_u(t) - x^{(m-2)}_u(0) \geq 0,$$

which means that $x^{(m-2)}_u \geq 0$ on I.
Repeating the same procedure, we obtain by induction

$$x^{(j)}_u \geq 0 \text{ on } I, \quad \text{for } j = 1, 2, \ldots, m-1.$$

And we have

$$\begin{cases} x'_u(t) \geq 0, \\ x_u(0) - \int_\beta^1 x_u(\tau)d\tau \geq 0. \end{cases} \tag{3.16}$$

We apply $\mathcal{I}^1_{0+}$ then we integrate on $[\beta,1]$ both sides of first inequality of (3.16), we get

$$\begin{cases} x_u(t) - x_u(0) \geq 0, \\ \displaystyle\int_\beta^1 x_u(\tau)d\tau - (1-\beta)x_u(0) \geq 0, \\ x_u(0) - \displaystyle\int_\beta^1 x_u(\tau)d\tau \geq 0. \end{cases}$$

Therefore,

$$x_u^{(j)} \geq 0 \text{ on } I, \quad \text{for } j = 0,1,\dots,m-1 \Leftrightarrow x_u \succeq 0 \text{ on } I.$$

So, ${}_u w_0 \succeq {}_u w_1$ on I.
From (3.13) and according assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, we obtain

$$\begin{aligned} {}^c\mathcal{D}^\mu_{0+\,u} w_1(t) &= \varphi\left(t, {}_u w_0(t), {}^c\mathcal{D}^{\sigma_1}_{0+\,u} w_0(t), \dots, {}^c\mathcal{D}^{\sigma_n}_{0+\,u} w_0(t)\right) + \mathcal{I}^\nu_{0+}\psi(t, {}_u w_0(t)) \\ &\geq \varphi\left(t, {}_u w_1(t), {}^c\mathcal{D}^{\sigma_1}_{0+\,u} w_1(t), \dots, {}^c\mathcal{D}^{\sigma_n}_{0+\,u} w_1(t)\right) + \mathcal{I}^\nu_{0+}\psi(t, {}_u w_1(t)). \end{aligned} \tag{3.17}$$

And using assumption $(\mathcal{A}_4)$, we get

$$\begin{aligned} &\phi_0\left({}_u w_1(0), \int_0^\alpha {}_u w_1(\tau)d\tau, \int_\beta^1 {}_u w_1(\tau)d\tau\right) \\ &\qquad\geq \phi_0\left({}_u w_0(0), \int_0^\alpha {}_u w_0(\tau)d\tau, \int_\beta^1 {}_u w_0(\tau)d\tau\right) - \frac{1}{\eta_0}\left({}_u w_1(0) - {}_u w_0(0)\right) \\ &\qquad\quad + \xi_0\left(\int_0^\alpha {}_u w_1(\tau)d\tau - \int_0^\alpha {}_u w_0(\tau)d\tau\right) + \rho_0\left(\int_\beta^1 {}_u w_1(\tau)d\tau - \int_\beta^1 {}_u w_0(\tau)d\tau\right) \\ &\qquad = -\xi_0 \int_0^\alpha \left({}_u w_0(\tau) - {}_u w_1(\tau)\right)d\tau + \left(\frac{1}{\eta_0} - \rho_0\right)\int_\beta^1 \left({}_u w_0(\tau) - {}_u w_1(\tau)\right)d\tau \\ &\qquad \geq 0. \end{aligned} \tag{3.18}$$

Using (3.13) and assumption $(\mathcal{A}_3)$, we obtain

$$\begin{aligned} \phi_j\left({}_u w_1^{(j)}(0), {}_u w_1^{(j)}(1)\right) &\geq \phi_j\left({}_u w_0^{(j)}(0), {}_u w_0^{(j)}(1)\right) - \frac{1}{\eta_j}\left({}_u w_1^{(j)}(0) - {}_u w_0^{(j)}(0)\right) \\ &\quad + \xi_j\left({}_u w_1^{(j)}(1) - {}_u w_0^{(j)}(1)\right) \\ &= -\xi_j\left({}_u w_0^{(j)}(1) - {}_u w_1^{(j)}(1)\right) \\ &\geq 0, \end{aligned} \tag{3.19}$$

for all $j \in \{1,2,\dots,m-1\}$.
Form (3.17)-(3.19), we deduce that ${}_u w_1$ is an upper-solution of (1.1).
We repeat the same procedure for ${}_u w_1 - {}_u w_2$, we prove that ${}_u w_1 \succeq {}_u w_2$ and ${}_u w_2$ is an upper-solution of (1.1).
Similarly, we get by induction

$$w_u = {}_u w_0 \succeq {}_u w_1 \succeq {}_u w_2 \succeq \cdots \succeq {}_u w_k \succeq {}_u w_{k+1}.$$

In a similar way for the sequence ${}_l w_k$. From (3.10), we have

$$\begin{cases} {}^c\mathcal{D}^\mu_{0+\,l} w_1(t) = \varphi\left(t, {}_l w_0(t), {}^c\mathcal{D}^{\sigma_1}_{0+\,l} w_0(t), \dots, {}^c\mathcal{D}^{\sigma_n}_{0+\,l} w_0(t)\right) + \mathcal{I}^\nu_{0+}\psi(t, {}_l w_0(t)), & t \in I, \\ {}_l w_1(0) - \displaystyle\int_0^\alpha {}_l w_1(\tau)d\tau = {}_l w_0(0) - \int_0^\alpha {}_l w_0(\tau)d\tau + \eta_0\phi_0\left({}_l w_0(0), \int_0^\alpha {}_l w_0(\tau)d\tau, \int_\beta^1 {}_l w_0(\tau)d\tau\right), & \\ {}_l w_1^{(j)}(0) = {}_l w_0^{(j)}(0) + \eta_j\phi_j\left({}_l w_0^{(j)}(0), {}_l w_0^{(j)}(1)\right), & j = 1,\dots,m-1. \end{cases} \tag{3.20}$$

Note that ${}_lw_0 = w_l$ is a lower-solution of (1.1), then it satisfies

$$\begin{cases} {}^c\mathcal{D}^{\mu}_{0+}\,{}_lw_0(t) \leq \varphi\left(t, {}_lw_0(t), {}^c\mathcal{D}^{\sigma_1}_{0+}\,{}_lw_0(t), \dots, {}^c\mathcal{D}^{\sigma_n}_{0+}\,{}_lw_0(t)\right) + \mathcal{I}^{\nu}_{0+}\psi(t, {}_lw_0(t)) = {}^c\mathcal{D}^{\mu}_{0+}\,{}_lw_1(t), & t \in I, \\ \phi_0\left({}_lw_0(0), \int_0^{\alpha} {}_lw_0(\tau)d\tau, \int_{\beta}^1 {}_lw_0(\tau)d\tau\right) \leq 0, & \\ \phi_j\left({}_lw_0^{(j)}(0), {}_lw_0^{(j)}(1)\right) \leq 0, & j = 1, \dots, m-1. \end{cases} \tag{3.21}$$

Let $x_l = {}_lw_0 - {}_lw_1$. From (3.20) and (3.21), we get

$$\begin{cases} {}^c\mathcal{D}^{\mu}_{0+}x_l(t) \leq 0, & \\ x_l(0) - \int_0^{\alpha} x_l(\tau)d\tau = -\eta_0\phi_0\left({}_lw_0(0), \int_0^{\alpha} {}_lw_0(\tau)d\tau, \int_{\beta}^1 {}_lw_0(\tau)d\tau\right) \leq 0, & \\ x_l^{(j)}(0) = -\eta_j\phi_j\left({}_lw_0^{(j)}(0), {}_lw_0^{(j)}(1)\right) \leq 0, & j = 1, \dots, m-1. \end{cases} \tag{3.22}$$

We apply $\mathcal{I}^{\mu-(m-1)}_{0+}$ on both sides of first inequality of (3.22), we get

$$\begin{cases} x_l^{(m-1)}(t) \leq 0, \\ x_l^{(m-2)}(0) \leq 0. \end{cases}$$

and by applying $\mathcal{I}^1_{0+}$, we obtain

$$x_l^{(m-2)} \leq 0 \text{ on } I.$$

Similarly, we obtain by induction

$$x_l^{(j)} \leq 0 \text{ on } I, \quad \text{for } j = 1, 2, \dots, m-1.$$

And we have

$$\begin{cases} x_l'(t) \leq 0, \\ x_l(0) - \int_0^{\alpha} x_l(\tau)d\tau \leq 0. \end{cases} \tag{3.23}$$

We apply $\mathcal{I}^1_{0+}$ then we integrate on $[0, \alpha]$ both sides of first inequality of (3.23), we get

$$\begin{cases} x_l(t) - x_l(0) \leq 0, \\ \int_0^{\alpha} x_l(\tau)d\tau - \alpha x_l(0) \leq 0, \\ x_l(0) - \int_0^{\alpha} x_l(\tau)d\tau \leq 0. \end{cases}$$

Therefore,

$$x_l^{(j)} \leq 0 \text{ on } I, \quad \text{for } j = 0, 1, \dots, m-1.$$

So, ${}_lw_1 \succeq {}_lw_0$ on I.

From (3.20) and according to assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, we obtain

$$ {}^c\mathcal{D}^{\mu}_{0+}\,{}_lw_1(t) \leq \varphi\left(t, {}_lw_1(t), {}^c\mathcal{D}^{\sigma_1}_{0+}\,{}_lw_1(t), \dots, {}^c\mathcal{D}^{\sigma_n}_{0+}\,{}_lw_1(t)\right) + \mathcal{I}^{\nu}_{0+}\psi(t, {}_lw_1(t)). \tag{3.24}$$

And using assumption $(\mathcal{A}_4)$, we get

$$\begin{aligned} \phi_0\left({}_lw_1(0), \int_0^{\alpha} {}_lw_1(\tau)d\tau, \int_{\beta}^1 {}_lw_1(\tau)d\tau\right) &\leq \left(\frac{1}{\eta_0} - \xi_0\right)\int_0^{\alpha} \left({}_lw_0(\tau) - {}_lw_1(\tau)\right)d\tau \\ &\quad - \rho_0\int_{\beta}^1 \left({}_lw_0(\tau) - {}_lw_1(\tau)\right)d\tau \\ &\leq 0. \end{aligned} \tag{3.25}$$

Using (3.20) and assumption $(\mathcal{A}_3)$, we obtain

$$\begin{aligned}\phi_j\left({}_lw_1^{(j)}(0), {}_lw_1^{(j)}(1)\right) &\le -\xi_j\left({}_lw_0^{(j)}(1) - {}_lw_1^{(j)}(1)\right)\\ &\le 0,\end{aligned} \tag{3.26}$$

for all $j \in \{1, 2, \ldots, m-1\}$.
Form (3.24)-(3.26), we deduce that ${}_lw_1$ is an lower-solution of (1.1).
By induction, we obtain

$${}_lw_{k+1} \succeq {}_lw_k \succeq \cdots \succeq {}_lw_1 \succeq {}_lw_0 = w_l.$$

Now, let us prove that

$${}_uw_k \succeq {}_lw_k, \text{ for } k \ge 1. \tag{3.27}$$

Denote by $x = {}_uw_1 - {}_lw_1$.
From assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, we get

$$\begin{aligned}{}^c\mathcal{D}^{\mu}_{0+}x(t) &= \varphi\left(t, {}_uw_0(t), {}^c\mathcal{D}^{\sigma_1}_{0+}{}_uw_0(t), \ldots, {}^c\mathcal{D}^{\sigma_n}_{0+}{}_uw_0(t)\right) - \varphi\left(t, {}_lw_0(t), {}^c\mathcal{D}^{\sigma_1}_{0+}{}_lw_0(t), \ldots, {}^c\mathcal{D}^{\sigma_n}_{0+}{}_lw_0(t)\right)\\ &\quad + \mathcal{I}^{\nu}_{0+}\psi(t, {}_uw_0(t)) - \mathcal{I}^{\nu}_{0+}\psi(t, {}_lw_0(t))\\ &\ge 0.\end{aligned}$$

From (3.13),(3.20) and using assumption $(\mathcal{A}_4)$, we have

$$\begin{aligned}x(0) - \int_\beta^1 {}_uw_1(\tau)d\tau + \int_0^\alpha {}_lw_1(\tau)d\tau &= {}_uw_0(0) - {}_lw_0(0) - \int_\beta^1 {}_uw_0(\tau)d\tau + \int_0^\alpha {}_lw_0(\tau)d\tau\\ &\quad - \eta_0\left[\phi_0\left({}_lw_0(0), \int_0^\alpha {}_lw_0(\tau)d\tau, \int_\beta^1 {}_lw_0(\tau)d\tau\right)\right.\\ &\quad \left. -\phi_0\left({}_uw_0(0), \int_0^\alpha {}_uw_0(\tau)d\tau, \int_\beta^1 {}_uw_0(\tau)d\tau\right)\right]\\ &\ge -\int_\beta^1 {}_uw_0(\tau)d\tau + \int_0^\alpha {}_lw_0(\tau)d\tau\\ &\quad + \eta_0\xi_0\left(\int_0^\alpha {}_uw_0(\tau)d\tau - \int_0^\alpha {}_lw_0(\tau)d\tau\right)\\ &\quad + \eta_0\rho_0\left(\int_\beta^1 {}_uw_0(\tau)d\tau - \int_\beta^1 {}_lw_0(\tau)d\tau\right).\end{aligned}$$

Since ${}_uw_0 \succ {}_lw_0$ and $\eta_0\xi_0, \eta_0\rho_0 \ge 1$, then we get

$$x(0) \ge \int_0^\alpha {}_uw_0(\tau)d\tau - \int_0^\alpha {}_lw_1(\tau)d\tau + \int_\beta^1 {}_uw_1(\tau)d\tau - \int_\beta^1 {}_lw_0(\tau)d\tau.$$

Then, we obtain

$$x(0) - \int_0^\alpha x(\tau)d\tau - \int_\beta^1 x(\tau)d\tau \ge 0,$$

because ${}_uw_0 \ge {}_uw_1$ and ${}_lw_0 \le {}_lw_1$.
From (3.13), (3.20) and using assumption $(\mathcal{A}_3)$, we have

$$\begin{aligned}x^{(j)}(0) &= \left({}_uw_0^{(j)}(0) - {}_lw_0^{(j)}(0)\right) - \eta_j\left[\phi_j\left({}_lw_0^{(j)}(0), {}_lw_0^{(j)}(1)\right) - \phi_j\left({}_uw_0^{(j)}(0), {}_uw_0^{(j)}(1)\right)\right]\\ &\ge \eta_j\xi_j\left({}_uw_0^{(j)}(1) - {}_lw_0^{(j)}(1)\right)\\ &\ge 0, \qquad j = 1, \ldots, m-1.\end{aligned}$$

We therefore have the following system:

$$\begin{cases} {}^c\mathcal{D}^{\mu}_{0+}x(t) \geq 0, & t \in I, \\ x(0) - \int_0^{\alpha} x(\tau)d\tau - \int_{\beta}^{1} x(\tau)d\tau \geq 0, & \\ x^{(j)}(0) \geq 0, & j = 1, 2, \dots, m-1. \end{cases} \tag{3.28}$$

Then, according to lemma 3.3, $x \succeq 0$. Which means that ${}_uw_1 \succeq {}_lw_1$.
By induction, we get

$$ {}_uw_k \succeq {}_lw_k, \quad \forall\ k \geq 0.$$

Thus, the property (3.12) is proved.
Which entails that sequences $({}_uw_k)_{k\geq 0}$ and $({}_lw_k)_{k\geq 0}$ are uniformly bounded and uniformly equicontinuous.
According to the Arzelà-Ascoli's theorem, there exist two functions ${}_ls$, ${}_us \in [{}_lw_0, {}_uw_0]$ such that

$$\lim_{k\to+\infty} {}^c\mathcal{D}^{\mu}_{0+}\,{}_lw_k = \mathcal{D}^{\mu}_{0+}\,{}_ls \quad \text{and} \quad \lim_{k\to+\infty} {}^c\mathcal{D}^{\mu}_{0+}\,{}_uw_k = \mathcal{D}^{\mu}_{0+}\,{}_us, \quad \forall \mu \in [0, m-1],$$

uniformly on I. Moreover, ${}_ls$ and ${}_us$ satisfy problem (1.1).

We only have to show that ${}_us$ and ${}_ls$ are upper and lower-solutions of (1.1), respectively.
Let $w \in [{}_lw_0, {}_uw_0]$ be a solution of (1.1).
we have ${}_uw_0 \succeq w \succeq {}_lw_0$. And by applying $(\mathcal{A}_1) - (\mathcal{A}_2)$, we get

$$ {}^c\mathcal{D}^{\mu}_{0+}\,{}_uw_1 \geq {}^c\mathcal{D}^{\mu}_{0+}w \geq {}^c\mathcal{D}^{\mu}_{0+}\,{}_lw_1. \tag{3.29}$$

Using $(\mathcal{A}_4)$, we have

$$\begin{aligned} {}_uw_1^{(j)}(0) &= {}_uw_0^{(j)}(0) + \eta_j\phi_j\left({}_uw_0^{(j)}(0), {}_uw_0^{(j)}(1)\right) \\ &= {}_uw_0^{(j)}(0) - \eta_j\left(\phi_j\left(w^{(j)}(0), w^{(j)}(1)\right) - \phi_j\left({}_uw_0^{(j)}(0), {}_uw_0^{(j)}(1)\right)\right] \\ &\geq w^{(j)}(0) + \eta_j\xi_j\left[{}_uw_0^{(j)}(1) - w^{(j)}(1)\right] \\ &\geq w^{(j)}(0), \end{aligned} \tag{3.30}$$

for $j = 1, \dots, m-1$.

Similarly, we have

$$\begin{aligned} {}_lw_1^{(j)}(0) &= {}_lw_0^{(j)}(0) + \eta_j\left(\phi_j\left({}_lw_0^{(j)}(0), {}_lw_0^{(j)}(1)\right) - \phi_j\left(w^{(j)}(0), w^{(j)}(1)\right)\right] \\ &\leq w^{(j)}(0) - \eta_j\xi_j\left[w^{(j)}(1) - {}_lw_0^{(j)}(1)\right] \\ &\leq w^{(j)}(0), \end{aligned} \tag{3.31}$$

for $j = 1, \dots, m-1$.

Using assumption $(\mathcal{A}_3)$, we get

$$\begin{aligned}
{}_u w_1(0) - \int_\beta^1 {}_u w_1(\tau)d\tau &= {}_u w_0(0) - \int_\beta^1 {}_u w_0(\tau)d\tau - \eta_0 \left[\phi_0 \left(w(0), \int_0^\alpha w(\tau)d\tau, \int_\beta^1 w(\tau)d\tau\right)\right. \\
&\quad \left. -\phi_0 \left({}_u w_0(0), \int_0^\alpha {}_u w_0(\tau)d\tau, \int_\beta^1 {}_u w_0(\tau)d\tau\right)\right] \\
&\geq w(0) - \int_\beta^1 {}_u w_0(\tau)d\tau + \eta_0 \xi_0 \left(\int_0^\alpha {}_u w_0(\tau)d\tau - \int_0^\alpha w(\tau)d\tau\right) \\
&\quad + \eta_0 \rho_0 \left(\int_\beta^1 {}_u w_0(\tau)d\tau - \int_\beta^1 w(\tau)d\tau\right) \\
&\geq w(0) - \int_\beta^1 w(\tau)d\tau.
\end{aligned}$$

And from inequality (3.29), we can obtain

$$\int_\beta^1 {}_u w_1(\tau)d\tau - (1-\beta){}_u w_1(0) \geq \int_\beta^1 w(\tau)d\tau - (1-\beta)w(0).$$

Then, from above given expressions, we get

$${}_u w_1(0) \geq w(0). \tag{3.32}$$

similarly, we have

$${}_l w_1(0) \leq w(0). \tag{3.33}$$

So, form (3.30)-(3.33), we get

$${}_u w_1 \succeq w \succeq {}_l w_1.$$

By induction we prove that

$${}_u w_k \succeq w \succeq {}_l w_k, \quad \forall k \geq 2.$$

Therefore,

$${}_u s = \lim_{k\to+\infty} {}_u w_k \succeq w \succeq \lim_{k\to+\infty} {}_l w_k = {}_l s.$$

Which completes the proof. □

Example 3.6. Consider the following problem:

$$\begin{cases}
{}^c\mathcal{D}_{0^+}^{\frac{5}{2}} w(t) = 1 + \dfrac{24}{\sqrt{\pi}}\sqrt{t} - \dfrac{2}{3\sqrt{\pi}}t^{\frac{3}{2}} - \dfrac{32}{\sqrt{\pi}}t^{\frac{5}{2}} - \dfrac{175\sqrt{\pi}}{32}t^3 + \dfrac{1}{6}w(t)w'(t) + \dfrac{2}{7}{}^c\mathcal{D}_{0^+}^{\frac{1}{4}} w(t), \\
\qquad + \mathcal{I}_{0^+}^{\frac{1}{2}} \left(w(t) - 12 + \dfrac{1}{2}t + 30t^2 - 20t^{\frac{5}{2}} + \dfrac{75}{2}t^{\frac{-1}{2}}\right), \ t \in I, \\
w^2(0) = 3\displaystyle\int_0^{\frac{1}{4}} w(\tau)d\tau + 3\int_{\frac{1}{2}}^1 w(\tau)d\tau \\
w'(0) = \dfrac{1}{8}w'(1) + \dfrac{1}{4}, \\
(w'')^2(0) = w''(1) - 2w''(0) + 2.
\end{cases} \tag{3.34}$$

The problem (3.34) can be abstracted into the problem (1.1), where

$$\begin{aligned}
f\left(t, w, w', {}^c\mathcal{D}_{0^+}^{\frac{1}{4}} w\right) &= 1 + \frac{24}{\sqrt{\pi}}\sqrt{t} - \frac{2}{3\sqrt{\pi}}t^{\frac{3}{2}} - \frac{32}{\sqrt{\pi}}t^{\frac{5}{2}} - \frac{175\sqrt{\pi}}{32}t^3 + \frac{1}{6}w(t)w'(t) + \frac{2}{7}{}^c\mathcal{D}_{0^+}^{\frac{1}{4}} w(t), \\
\psi(t,x) &= -12 + \frac{1}{2}t + 30t^2 - 20t^{\frac{5}{2}} + \frac{75}{2}t^{\frac{-1}{2}} + x,
\end{aligned}$$

and

$$\phi_0(x,y,z) = x^2 - 3y - 3z, \quad \phi_1(x,y) = x - \frac{1}{8}y - \frac{1}{4}, \quad \phi_2(x,y) = x^2 + 2x - y - 2.$$

Note that assumptions $(\mathcal{A}_1) - (\mathcal{A}_4)$ are satisfied with

$$\eta_0 = \frac{-1}{2}; \quad \eta_1 = -1; \quad \eta_2 = \frac{-1}{4}; \quad \xi_0 = -3; \quad \xi_1 = \frac{-1}{8}; \quad \xi_2 = -1 \quad \text{and} \quad \rho_0 = -3.$$

We used Maple to determine a lower and upper-solutions in order to construct sequences $({}_uw_k)_{k\geq 0}$ and $({}_lw_k)_{k\geq 0}$. We got

$${}_lw_0(t) = 0 \quad \text{and} \quad {}_uw_0(t) = 12 - \frac{1}{2}t - 30t^2 + 20t^{\frac{5}{2}}.$$

as lower and upper-solutions, respectively.

Let us construct sequences $({}_uw_k)_{k\geq 0}$ and $({}_lw_k)_{k\geq 0}$.
Form (3.10) and (3.11), we have to solve:

$$\begin{cases} {}^c\mathcal{D}^{\frac{5}{2}}_{0+}\,{}_uw_{k+1}(t) = 1 + \frac{24}{\sqrt{\pi}}\sqrt{t} - \frac{2}{3\sqrt{\pi}}t^{\frac{3}{2}} - \frac{32}{\sqrt{\pi}}t^{\frac{5}{2}} - \frac{175\sqrt{\pi}}{32}t^3 + \frac{1}{6}\,{}_uw_k(t)\,{}_uw'_k(t) + \frac{2}{7}\,{}^c\mathcal{D}^{\frac{1}{4}}_{0+}\,{}_uw_k(t) \\ \qquad + \mathcal{I}^{\frac{1}{2}}_{0+}\left({}_uw_k(t) - 12 + \frac{1}{2}t + 30t^2 - 20t^{\frac{5}{2}} + \frac{75}{2}t^{\frac{-1}{2}}\right),\ k \geq 0,\ t \in I, \\ {}_uw_{k+1}(0) - \int_{\frac{1}{2}}^{1} {}_uw_{k+1}(\tau)d\tau = \frac{1}{2}\,{}_uw_k^2(0) + {}_uw_k(0) - \frac{3}{2}\int_0^{\frac{1}{4}} {}_uw_k(\tau)d\tau - \frac{5}{2}\int_{\frac{1}{2}}^{1} {}_uw_k(\tau)d\tau, \\ {}_uw'_{k+1}(0) = \frac{1}{8}\,{}_uw'_k(1) - \frac{1}{4}, \\ {}_uw''_{k+1}(0) = \frac{-1}{4}\left({}_uw''_k\right)^2(0) + \frac{1}{2}\,{}_uw''_k(0) + \frac{1}{4}\,{}_uw''_k(1) + \frac{1}{2}. \end{cases} \tag{3.35}$$

and

$$\begin{cases} {}^c\mathcal{D}^{\frac{5}{2}}_{0+}\,{}_lw_{k+1}(t) = 1 + \frac{24}{\sqrt{\pi}}\sqrt{t} - \frac{2}{3\sqrt{\pi}}t^{\frac{3}{2}} - \frac{32}{\sqrt{\pi}}t^{\frac{5}{2}} - \frac{175\sqrt{\pi}}{32}t^3 + \frac{1}{6}\,{}_lw_k(t)\,{}_lw'_k(t) + \frac{2}{7}\,{}^c\mathcal{D}^{\frac{1}{4}}_{0+}\,{}_lw_k(t) \\ \qquad + \mathcal{I}^{\frac{1}{2}}_{0+}\left({}_lw_k(t) - 12 + \frac{1}{2}t + 30t^2 - 20t^{\frac{5}{2}} + \frac{75}{2}t^{\frac{-1}{2}}\right),\ k \geq 0,\ t \in I, \\ {}_lw_{k+1}(0) - \int_0^{\frac{1}{4}} {}_lw_{k+1}(\tau)d\tau = \frac{1}{2}\,{}_lw_k^2(0) + {}_lw_k(0) - \frac{5}{2}\int_0^{\frac{1}{4}} {}_lw_k(\tau)d\tau - \frac{3}{2}\int_{\frac{1}{2}}^{1} {}_lw_k(\tau)d\tau, \\ {}_lw'_{k+1}(0) = \frac{1}{8}\,{}_lw'_k(1) - \frac{1}{4}, \\ {}_lw''_{k+1}(0) = \frac{-1}{4}\left({}_lw''_k\right)^2(0) + \frac{1}{2}\,{}_lw''_k(0) + \frac{1}{4}\,{}_lw''_k(1) + \frac{1}{2}. \end{cases} \tag{3.36}$$

By applying $\mathcal{I}^{\frac{5}{2}}_{0+}$ on both sides of first equation of (3.35), we get

$$\begin{aligned} {}_uw_{k+1}(t) = \frac{1}{\Gamma\left(\frac{5}{2}\right)}\int_0^t (t-\tau)^{\frac{3}{2}}\left(1 + \frac{24}{\sqrt{\pi}}\sqrt{\tau} - \frac{2}{3\sqrt{\pi}}\tau^{\frac{3}{2}} - \frac{32}{\sqrt{\pi}}\tau^{\frac{5}{2}} - \frac{175\sqrt{\pi}}{32}\tau^3 + \frac{1}{6}\,{}_uw_k(\tau)\,{}_uw'_k(\tau) + \frac{2}{7}\,{}^c\mathcal{D}^{\frac{1}{4}}_{0+}\,{}_uw_k(\tau)\right)d\tau \\ + \frac{1}{2}\int_0^t (t-\tau)^2\left({}_uw_k(\tau) - 12 + \frac{1}{2}\tau + 30\tau^2 - 20\tau^{\frac{5}{2}} + \frac{75}{2}\tau^{\frac{-1}{2}}\right)d\tau + b_0 + b_1 t + b_2 t^2. \end{aligned} \tag{3.37}$$

By applying the second derivative on both sides of (3.37) and using (3.35), we get

$$b_2 = \frac{-1}{8}\left({}_uw''_k\right)^2(0) + \frac{1}{4}\,{}_uw''_k(0) + \frac{1}{8}\,{}_uw''_k(1) + \frac{1}{4}. \tag{3.38}$$

By applying the first derivative on both sides of (3.37) and using (3.35), we get

$$b_1 = {}_uw'_{k+1}(0) = \frac{1}{8}\,{}_uw'_k(1). \tag{3.39}$$

We integrate both sides of equation (3.37) on $\left[\frac{1}{2}, 1\right]$, we have

$$\begin{cases} {}_u w_{k+1}(0) = b_0, \\ \displaystyle\int_{\frac{1}{2}}^{1} {}_u w_{k+1}(\tau) d\tau \\ \quad = \displaystyle\frac{1}{\Gamma\left(\frac{7}{2}\right)} \int_0^1 (1-\tau)^{\frac{5}{2}} \left(1 + \frac{24}{\sqrt{\pi}}\sqrt{\tau} - \frac{2}{3\sqrt{\pi}}\tau^{\frac{3}{2}} - \frac{32}{\sqrt{\pi}}\tau^{\frac{5}{2}} - \frac{175\sqrt{\pi}}{32}\tau^3 + \frac{1}{6}{}_u w_k(\tau){}_u w_k'(\tau) + \frac{2}{7}{}^c\mathcal{D}_{0+}^{\frac{1}{4}}{}_u w_k(\tau)\right) d\tau \\ \quad \displaystyle - \frac{1}{\Gamma\left(\frac{7}{2}\right)} \int_0^{\frac{1}{2}} \left(\frac{1}{2} - \tau\right)^{\frac{5}{2}} \left(1 + \frac{24}{\sqrt{\pi}}\sqrt{\tau} - \frac{2}{3\sqrt{\pi}}\tau^{\frac{3}{2}} - \frac{32}{\sqrt{\pi}}\tau^{\frac{5}{2}} - \frac{175\sqrt{\pi}}{32}\tau^3 + \frac{1}{6}{}_u w_k(\tau){}_u w_k'(\tau) + \frac{2}{7}{}^c\mathcal{D}_{0+}^{\frac{1}{4}}{}_u w_k(\tau)\right) d\tau \\ \quad \displaystyle + \frac{1}{6}\int_0^1 (1-\tau)^3 \left({}_u w_k(\tau) - 12 + \frac{1}{2}\tau + 30\tau^2 - 20\tau^{\frac{5}{2}} + \frac{75}{2}\tau^{\frac{-1}{2}}\right) d\tau \\ \quad \displaystyle - \frac{1}{6}\int_0^{\frac{1}{2}} \left(\frac{1}{2} - \tau\right)^3 \left({}_u w_k(\tau) - 12 + \frac{1}{2}\tau + 30\tau^2 - 20\tau^{\frac{5}{2}} + \frac{75}{2}\tau^{\frac{-1}{2}}\right) d\tau \\ \quad \displaystyle + \frac{1}{2}b_0 + \frac{3}{8}b_1 + \frac{7}{24}b_2. \end{cases}$$

Using the second equation of (3.35) and above system, we obtain

$$\begin{aligned} b_0 = {} & \frac{2}{\Gamma\left(\frac{7}{2}\right)} \int_0^1 (1-\tau)^{\frac{5}{2}} \left(1 + \frac{24}{\sqrt{\pi}}\sqrt{\tau} - \frac{2}{3\sqrt{\pi}}\tau^{\frac{3}{2}} - \frac{32}{\sqrt{\pi}}\tau^{\frac{5}{2}} - \frac{175\sqrt{\pi}}{32}\tau^3 + \frac{1}{6}{}_u w_k(\tau){}_u w_k'(\tau) + \frac{2}{7}{}^c\mathcal{D}_{0+}^{\frac{1}{4}}{}_u w_k(\tau)\right) d\tau \\ & - \frac{2}{\Gamma\left(\frac{7}{2}\right)} \int_0^{\frac{1}{2}} \left(\frac{1}{2} - \tau\right)^{\frac{5}{2}} \left(1 + \frac{24}{\sqrt{\pi}}\sqrt{\tau} - \frac{2}{3\sqrt{\pi}}\tau^{\frac{3}{2}} - \frac{32}{\sqrt{\pi}}\tau^{\frac{5}{2}} - \frac{175\sqrt{\pi}}{32}\tau^3 + \frac{1}{6}{}_u w_k(\tau){}_u w_k'(\tau) + \frac{2}{7}{}^c\mathcal{D}_{0+}^{\frac{1}{4}}{}_u w_k(\tau)\right) d\tau \\ & + \frac{1}{3}\int_0^1 (1-\tau)^3 \left({}_u w_k(\tau) - 12 + \frac{1}{2}\tau + 30\tau^2 - 20\tau^{\frac{5}{2}} + \frac{75}{2}\tau^{\frac{-1}{2}}\right) d\tau \\ & - \frac{1}{3}\int_0^{\frac{1}{2}} \left(\frac{1}{2} - \tau\right)^3 \left({}_u w_k(\tau) - 12 + \frac{1}{2}\tau + 30\tau^2 - 20\tau^{\frac{5}{2}} + \frac{75}{2}\tau^{\frac{-1}{2}}\right) d\tau \\ & - 3\int_0^{\frac{1}{4}} {}_u w_k(\tau) d\tau - 5\int_{\frac{1}{2}}^1 {}_u w_k(\tau) d\tau + {}_u w_k^2(0) + 2{}_u w_k(0) + \frac{3}{32}{}_u w_k'(1) \\ & - \frac{7}{96}\left({}_u w_k''\right)^2(0) + \frac{7}{48}{}_u w_k''(0) + \frac{7}{96}{}_u w_k''(1) + \frac{7}{48}. \end{aligned} \tag{3.40}$$

Therefore, the sequence $({}_u w_k)_{k\geq 0}$ is defined by the recurrence relation (3.37), where b_0, b_1 and b_2 are given by (3.40), (3.39) and (3.38), respectively.

Similarly, we get

$$\begin{aligned} {}_l w_{k+1}(t) = {} & \frac{1}{\Gamma\left(\frac{5}{2}\right)} \int_0^t (t-\tau)^{\frac{3}{2}} \left(1 + \frac{24}{\sqrt{\pi}}\sqrt{\tau} - \frac{2}{3\sqrt{\pi}}\tau^{\frac{3}{2}} - \frac{32}{\sqrt{\pi}}\tau^{\frac{5}{2}} - \frac{175\sqrt{\pi}}{32}\tau^3 + \frac{1}{6}{}_l w_k(\tau){}_l w_k'(\tau) + \frac{2}{7}{}^c\mathcal{D}_{0+}^{\frac{1}{4}}{}_l w_k(\tau)\right) d\tau \\ & + \frac{1}{2}\int_0^t (t-\tau)^2 \left({}_l w_k(\tau) - 12 + \frac{1}{2}\tau + 30\tau^2 - 20\tau^{\frac{5}{2}} + \frac{75}{2}\tau^{\frac{-1}{2}}\right) d\tau + c_0 + c_1 t + c_2 t^2, \end{aligned}$$

where,

$$\begin{aligned} c_0 = {} & \frac{2}{\Gamma\left(\frac{7}{2}\right)} \int_0^{\frac{1}{2}} \left(\frac{1}{2} - \tau\right)^{\frac{5}{2}} \left(1 + \frac{24}{\sqrt{\pi}}\sqrt{\tau} - \frac{2}{3\sqrt{\pi}}\tau^{\frac{3}{2}} - \frac{32}{\sqrt{\pi}}\tau^{\frac{5}{2}} - \frac{175\sqrt{\pi}}{32}\tau^3 + \frac{1}{6}{}_l w_k(\tau){}_l w_k'(\tau) + \frac{2}{7}{}^c\mathcal{D}_{0+}^{\frac{1}{4}}{}_l w_k(\tau)\right) d\tau \\ & + \frac{1}{3}\int_0^{\frac{1}{2}} \left(\frac{1}{2} - \tau\right)^3 \left({}_l w_k(\tau) - 12 + \frac{1}{2}\tau + 30\tau^2 - 20\tau^{\frac{5}{2}} + \frac{75}{2}\tau^{\frac{-1}{2}}\right) d\tau \\ & - 5\int_0^{\frac{1}{4}} {}_l w_k(\tau) d\tau - 3\int_{\frac{1}{2}}^1 {}_l w_k(\tau) d\tau + {}_l w_k^2(0) + 2{}_l w_k(0) + \frac{1}{32}{}_l w_k'(1) \\ & - \frac{1}{96}\left({}_l w_k''\right)^2(0) + \frac{1}{48}{}_l w_k''(0) + \frac{1}{96}{}_l w_k''(1) + \frac{1}{48}, \\ c_1 = {} & \frac{1}{8}{}_l w_k'(1), \\ c_2 = {} & \frac{-1}{8}\left({}_l w_k''\right)^2(0) + \frac{1}{4}{}_l w_k''(0) + \frac{1}{8}{}_l w_k''(1) + \frac{1}{4}. \end{aligned}$$

Thus, the two sequences of upper and lower-solutions are given respectively as follows:

$$
\begin{cases}
\begin{aligned}
{}_u w_{k+1}(t) = {} & \frac{1}{\Gamma\left(\frac{5}{2}\right)} \int_0^t (t-\tau)^{\frac{3}{2}} \left(1 + \frac{24}{\sqrt{\pi}}\sqrt{\tau} - \frac{2}{3\sqrt{\pi}}\tau^{\frac{3}{2}} - \frac{32}{\sqrt{\pi}}\tau^{\frac{5}{2}} - \frac{175\sqrt{\pi}}{32}\tau^3 + \frac{1}{6}{}_u w_k(\tau){}_u w_k'(\tau) + \frac{2}{7}{}^c\mathcal{D}_{0+}^{\frac{1}{4}}{}_u w_k(\tau)\right) d\tau \\
& + \frac{1}{2}\int_0^t (t-\tau)^2 \left({}_u w_k(\tau) - 12 + \frac{1}{2}\tau + 30\tau^2 - 20\tau^{\frac{5}{2}} + \frac{75}{2}\tau^{\frac{-1}{2}}\right) d\tau \\
& + \frac{2}{\Gamma\left(\frac{7}{2}\right)} \int_0^1 (1-\tau)^{\frac{5}{2}} \left(1 + \frac{24}{\sqrt{\pi}}\sqrt{\tau} - \frac{2}{3\sqrt{\pi}}\tau^{\frac{3}{2}} - \frac{32}{\sqrt{\pi}}\tau^{\frac{5}{2}} - \frac{175\sqrt{\pi}}{32}\tau^3 + \frac{1}{6}{}_u w_k(\tau){}_u w_k'(\tau) + \frac{2}{7}{}^c\mathcal{D}_{0+}^{\frac{1}{4}}{}_u w_k(\tau)\right) d\tau \\
& - \frac{2}{\Gamma\left(\frac{7}{2}\right)} \int_0^{\frac{1}{2}} \left(\frac{1}{2}-\tau\right)^{\frac{5}{2}} \left(1 + \frac{24}{\sqrt{\pi}}\sqrt{\tau} - \frac{2}{3\sqrt{\pi}}\tau^{\frac{3}{2}} - \frac{32}{\sqrt{\pi}}\tau^{\frac{5}{2}} - \frac{175\sqrt{\pi}}{32}\tau^3 + \frac{1}{6}{}_u w_k(\tau){}_u w_k'(\tau) + \frac{2}{7}{}^c\mathcal{D}_{0+}^{\frac{1}{4}}{}_u w_k(\tau)\right) d\tau \\
& + \frac{1}{3}\int_0^1 (1-\tau)^3 \left({}_u w_k(\tau) - 12 + \frac{1}{2}\tau + 30\tau^2 - 20\tau^{\frac{5}{2}} + \frac{75}{2}\tau^{\frac{-1}{2}}\right) d\tau \\
& - \frac{1}{3}\int_0^{\frac{1}{2}} \left(\frac{1}{2}-\tau\right)^3 \left({}_u w_k(\tau) - 12 + \frac{1}{2}\tau + 30\tau^2 - 20\tau^{\frac{5}{2}} + \frac{75}{2}\tau^{\frac{-1}{2}}\right) d\tau \\
& - 3\int_0^{\frac{1}{4}} {}_u w_k(\tau)d\tau - 5\int_{\frac{1}{2}}^1 {}_u w_k(\tau)d\tau + {}_u w_k^2(0) + 2{}_u w_k(0) + \left(\frac{3}{32} + \frac{1}{8}t\right){}_u w_k'(1) \\
& - \left(\frac{7}{96} + \frac{1}{8}t^2\right)({}_u w_k'')^2(0) + \left(\frac{7}{48} + \frac{1}{4}t^2\right){}_u w_k''(0) + \left(\frac{7}{96} + \frac{1}{8}t^2\right){}_u w_k''(1) + \frac{7}{48} + \frac{1}{4}t^2, \quad k \geq 0,
\end{aligned} \\
{}_u w_0(t) = 12 - \frac{1}{2}t - 30t^2 + 20t^{\frac{5}{2}},
\end{cases}
$$

and

$$
\begin{cases}
\begin{aligned}
{}_l w_{k+1}(t) = {} & \frac{1}{\Gamma\left(\frac{5}{2}\right)} \int_0^t (t-\tau)^{\frac{3}{2}} \left(1 + \frac{24}{\sqrt{\pi}}\sqrt{\tau} - \frac{2}{3\sqrt{\pi}}\tau^{\frac{3}{2}} - \frac{32}{\sqrt{\pi}}\tau^{\frac{5}{2}} - \frac{175\sqrt{\pi}}{32}\tau^3 + \frac{1}{6}{}_l w_k(\tau){}_l w_k'(\tau) + \frac{2}{7}{}^c\mathcal{D}_{0+}^{\frac{1}{4}}{}_l w_k(\tau)\right) d\tau \\
& + \frac{1}{2}\int_0^t (t-\tau)^2 \left({}_l w_k(\tau) - 12 + \frac{1}{2}\tau + 30\tau^2 - 20\tau^{\frac{5}{2}} + \frac{75}{2}\tau^{\frac{-1}{2}}\right) d\tau \\
& \frac{2}{\Gamma\left(\frac{7}{2}\right)} \int_0^{\frac{1}{2}} \left(\frac{1}{2}-\tau\right)^{\frac{5}{2}} \left(1 + \frac{24}{\sqrt{\pi}}\sqrt{\tau} - \frac{2}{3\sqrt{\pi}}\tau^{\frac{3}{2}} - \frac{32}{\sqrt{\pi}}\tau^{\frac{5}{2}} - \frac{175\sqrt{\pi}}{32}\tau^3 + \frac{1}{6}{}_l w_k(\tau){}_l w_k'(\tau) + \frac{2}{7}{}^c\mathcal{D}_{0+}^{\frac{1}{4}}{}_l w_k(\tau)\right) d\tau \\
& + \frac{1}{3}\int_0^{\frac{1}{2}} \left(\frac{1}{2}-\tau\right)^3 \left({}_l w_k(\tau) - 12 + \frac{1}{2}\tau + 30\tau^2 - 20\tau^{\frac{5}{2}} + \frac{75}{2}\tau^{\frac{-1}{2}}\right) d\tau \\
& - 5\int_0^{\frac{1}{4}} {}_l w_k(\tau)d\tau - 3\int_{\frac{1}{2}}^1 {}_l w_k(\tau)d\tau + {}_l w_k^2(0) + 2{}_l w_k(0) + \left(\frac{1}{32} + \frac{1}{8}t\right){}_l w_k'(1) \\
& - \left(\frac{1}{96} + \frac{1}{8}t^2\right)({}_l w_k'')^2(0) + \left(\frac{1}{48} + \frac{1}{4}t^2\right){}_l w_k''(0) + \left(\frac{1}{96} + \frac{1}{8}t^2\right){}_l w_k''(1) + \frac{1}{48} + \frac{1}{4}t^2, \quad k \geq 0,
\end{aligned} \\
{}_l w_0(t) = 0.
\end{cases}
$$

4. Conclusion

In conclusion, this paper has successfully tackled the intricate problem of identifying upper and lower solutions for high-order fractional integro-differential equations with non-linear boundary conditions. By introducing an order relation and employing a rigorous methodology, we have established the existence of these solutions as limits of sequences originating from strategically chosen problems, underpinned by the application of Arzelà-Ascoli's theorem. This research has unveiled essential insights into the realm of solutions within the domain of complex fractional integro-differential equations.

In terms of future perspectives, this study opens the door to further explorations in this field. Researchers may delve into extending these methods to more intricate equations or examining the applicability of these findings in real-world problems, particularly in physics, engineering, and other scientific domains. Additionally, the established techniques could be adapted for numerical approximation and computational solutions, providing practical tools for solving high-order fractional integro-differential equations in various applications. This paper lays a strong foundation for ongoing research and the development of more sophisticated mathematical tools in the pursuit of deeper insights into these complex equations.

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