Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 12, No. 3, 2024, pp. 511-522
DOI:10.22034/cmde.2023.58018.2444

On fractional linear multi-step methods for fractional order multi-delay nonlinear pantograph equation

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Abstract

This paper presents the development of a series of fractional multi-step linear finite difference methods (FLMMs) designed to address fractional multi-delay pantograph differential equations of order $0 < \alpha \leq 1$. These p-FLMMs are constructed using fractional backward differentiation formulas of first and second orders, thereby facilitating the numerical solution of fractional differential equations. Notably, we employ accurate approximations for the delayed components of the equation, guaranteeing the retention of stability and convergence characteristics in the proposed p-FLMMs. To substantiate our theoretical findings, we offer numerical examples that corroborate the efficacy and reliability of our approach.

Keywords. Fractional derivative, Fractional integration, Fractional linear multi-step methods.

2010 Mathematics Subject Classification. 34A08, 44A10.

1. Introduction

In recent years, there has been a great desire to model problems with fractional differential equations instead of using classical integer order differential equations. Because some problems are explained more accurately by using fractional derivatives. Fractional calculus plays a pivotal role in modeling complex phenomena using fractional-order differential and integral equations [2, 16, 17]. Recent research has expanded its practical utility by addressing intricate fractional systems, including those involving impulsive control, nonlinear dynamics, and proportional delay. For example, Moniri et al. have devised an efficient solver for managing impulsive control in fractional chaotic systems [16], while Moghaddam et al. present a method to handle fractional dynamical systems with impulsive effects [20]. Additionally, Mostaghim et al. have contributed a computational technique for simulating variable-order fractional Heston models in the US stock market [17]. These studies also exemplify the diverse applications of fractional calculus in various fields [13].

The pantograph delay differential equation (DDE) with proportional delays is a well-known problem that arises in various models, including mechanics and electrodynamics. Its mathematical modeling dates back to 1971 [19], sparking considerable research interest and leading to various numerical solution methods. Recent contributions include Mokhtary et al.'s computational approach for non-linear weakly singular Volterra integral equations with proportional delay [15]. Authors in [18] apply the collocation method, Shadia [22] utilizes the spline method for delay and neutral differential equations, and Moghaddam et al. address nonlinear fractional stochastic differential equations with constant time delay using an integro quadratic spline-based scheme [14]. The Runge-Kutta method is employed by authors in [11] for multi-pantograph DDEs, while the homotopy perturbation method [25] and the Adomian decomposition method [5] are applied for the generalized pantograph DDE and delay differential equations, respectively. Davarifar et al. approximate the system of delay differential equations of pantograph type using the Boubaker polynomials basis method in [3]. Isah applies the Collocation method based on the Genocchi operational matrix in [10], and more

Received: 16 August 2023 ; Accepted: 20 November 2023.
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recently, the Natural transform method is employed by authors in [24] for fractional pantograph DDEs. Other works deal with the use of Montez-Legendre polynomials and its applications to solve fractional pantograph equations [1, 23].

An efficient method for solving fractional differential equations is the fractional linear multi-step method (FLMM), which is based on fractional backward difference formulas (FBDFs). The development and analysis of FLMMs, along with their corresponding stability regions, are well-established for solving fractional differential equations. Lubich was the first to introduce the FLMM for fractional orders in the interval $(0, 1)$ [12]. Subsequently, Garrappa extended FLMMs and introduced p-FLMMs for first and second orders [7, 8]. In a recent development, authors in [6] introduced explicit first, second, third, and fourth-order p-FLMMs for fractional orders in the interval $(0, 1)$. This methodology was further extended to the fractional order interval $(1, 2)$ by Irandoust et al. [9].

In this study, we introduce two FLMMs of order one and two for solving nonlinear fractional pantograph-type multi-delay differential equations of the form:

$$\begin{cases} {}^C D_0^\alpha y(t) = \sum_{i=1}^{m} F_i(y(p_i t)) + g(t), \ 0 < p_i \leq 1, \\ \\ y'(0) = y_0, \end{cases} \tag{1.1}$$

here, $g(t)$ and $F_i(y)$, $(i = 1, 2, \dots, m)$, are known analytic functions, and ${}^C D_0^\alpha$ represents the Caputo fractional differential operator of order $0 < \alpha \leq 1$.

The remainder of this paper is organized as follows: In section 2, we provide a brief overview of fractional calculus and FLMMs. In section 3, we introduce the new FLMMs for solving the problem (1.1). Some examples are illustrated in section 4. Finally, the conclusion is stated in section 5.

2. Preliminaries

In this section, we introduce briefly some preliminary results on fractional calculus and fractional linear multi step methods which will be used in the next sections.

2.1. **Fractional Calculus.** We start this part by recalling some definitions which are quoted from [21] as follows.

Definition 2.1. Consider the real number ν and the integer n. We define the space C_ν as the space of all functions $f : R^+ \to R$ if there exists a real number $p > \nu$ and a function $f_1 \in C[0, \infty)$ such that $f(t) = t^p f_1(t)$, $\forall t\hat{A} \in R^+$. Moreover, the space C_ν^n denotes the space of all functions f such that $f^{(n)} \in C_\nu$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f(t) \in C_\nu$, $\nu \geq -1$ is defined as [21]

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \alpha > 0, \tag{2.1}$$

where $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$, $z \in C$ and a is called the index of integration. It is obvious that for $\alpha = 0$ we have $I_a^0 f(t) = f(t)$.

The operator I_a^α satisfies the following properties

- $I_a^\alpha I_a^\beta f(t) = I_a^\beta I_a^\alpha f(t) = I_a^{\alpha+\beta} f(t)$.
- $I_a^\alpha (t-a)^\gamma = \dfrac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)} (t-a)^{\alpha+\gamma}$.

where $f \in C_\nu$, $\nu \geq -1$, $\alpha,\ \beta \geq 0$ and $\gamma \geq -1$.

Definition 2.3. The Caputo fractional differential operator ${}^C D_a^\alpha$ is defined as

$${}^C D_a^\alpha f(t) = I_a^{n-\alpha} \frac{d^n}{dt^n} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \quad n-1 < \alpha < n. \tag{2.2}$$

For $\alpha = n$ the fractional differential operator ${}^C D_a^\alpha$ is equal to the ordinary differential operator D^n.

The operator ${}^CD_a^\alpha$ satisfies the following properties

- ${}^CD_a^\alpha I_a^\alpha f(t) = f(t)$.
- $I_a^\alpha[{}^CD_a^\alpha f(t)] = f(t) - \sum_{k=0}^{n-1} f^{(k)}(a)\frac{(t-a)^k}{k!},\ t > a.$
- ${}^CD_a^\alpha (t-a)^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}(t-a)^{\gamma-\alpha}$.
- ${}^CD_a^\alpha c = 0, \quad c$ is a constant,

where $n-1 < \alpha \leq n$, $n \in N$ and $f \in C_\nu^n$, $\nu \geq -1$, $\gamma > \alpha - 1$.

2.2. **Fractional Linear Multi-step Methods (FLMMs).** To introduce FLMMs we consider the common test problem

$$ {}^CD_0^\alpha y(t) = f(t, y(t)), \qquad y(0) = y_0. \tag{2.3}$$

In this equation, $f : [0, T] \times R \to R$ is considered a smooth function and $0 < \alpha < 1$ is the fractional order. The p-FLMMs are considered as

$$\sum_{j=0}^{n-1} w_j y_{n-j} = b_n y_0 + h^\alpha \sum_{j=0}^{p} \gamma_j f(t_{n-j}, y_{n-j}), \tag{2.4}$$

where p is an integer number. The coefficients w_j and γ_j, $(j = 0, 1, ...)$ are chosen properly to fulfill some appropriate consistency and convergency properties [7]. The first consistency property requires that $b_n = \sum_{j=0}^{n-1} w_j$. In the following, we list some p-FLMMs.

- **The fractional 1-FLMM Euler method of order one** [21]

$$\sum_{j=0}^{n-1} w_j y_{n-j} = b_n y_0 + h^\alpha f(t_{n-1}, y_{n-1}), \tag{2.5}$$

where

$$\begin{cases} w_j = (-1)^j \binom{\alpha}{j} = \dfrac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}, & j = 0, 1, \ldots, n-1, \\ b_n = \displaystyle\sum_{j=0}^{n-1} w_j = \dfrac{-\alpha\Gamma(n-\alpha)}{\Gamma(2-\alpha)\Gamma(n-1)}. \end{cases} \tag{2.6}$$

- **The fractional 2-FLMM method of order 2** [7]

$$\sum_{j=0}^{n-1} w_j y_{n-j} = b_n y_0 + h^\alpha \left[(2 - \frac{\alpha}{2}) f(t_{n-1}, y_{n-1}) + (\frac{\alpha}{2} - 1) f(t_{n-2}, y_{n-2})\right], \tag{2.7}$$

where the coefficients are the same as (2.6).

- **The fractional 1-FLMM Diethelm method of order $2-\alpha$** [4]

$$\sum_{j=0}^{n-1} w_j y_{n-j} = b_n y_0 + h^\alpha f(t_{n-1}, y_{n-1}), \tag{2.8}$$

where

$$\begin{cases} w_0 = 1, \\ w_j = (j+1)^{1-\alpha} - 2j^{1-\alpha} + (j-1)^{1-\alpha}, \qquad j = 1, 2, \dots, n-1, \\ b_n = \sum_{j=0}^{n-1} w_j = n^{1-\alpha} - (n-1)^{1-\alpha}. \end{cases} \tag{2.9}$$

Although there exist higher-order explicit and implicit FLMM methods, in this we apply the explicit lower-order FLMM methods in the presence of the delayed factors of the fractional pantograph equation (1.1).

3. Implementation of the method

To apply FLMM to the fractional multi-delay pantograph equation (1.1), we first discretize the proposed domain $[0, T]$ as follows

$$t_i = ih, \qquad i = 0, 1, 2, \dots, N, \tag{3.1}$$

where h is the uniform grid size ($h = t_{i+1} - t_i$). To apply FLMMs (2.5), (2.7), and (2.8) we need to evaluate $y(qt_i)$, where $0 < q < 1$ is the proportional delay factor. To maintain the order of accuracy of FLMMs, we must approximate $y(qt_i)$ with an accuracy equal to or higher than the accuracy of FLMMs. The point qt_i may not match the mesh points, so we have to approximate $y(qt_i)$ by using some adjacent nodes near qt_i. We have different choices for this.

- **Two points Lagrange interpolation of degree one:** Suppose that $t_{k-1} < qt_i \leq t_k$. Then we can approximate $y(qt_i)$ by linear two-point Lagrange interpolation at the nodes t_{k-1} and t_k as follows

$$y(qt_i) \cong \bar{y}(qt_i) = -\frac{1}{h}(qt_i - t_k)y_{k-1} + \frac{1}{h}(qt_i - t_{k-1})y_k. \tag{3.2}$$

The error approximation of interpolation polynomial gives

$$|y(qt_i) - \bar{y}(qt_i)| = |(qt_i - t_{k-1})(qt_i - t_k)|\frac{|y''(\xi)|}{2!} = \theta(1-\theta)h^2|\frac{|y''(\xi)|}{2!} = O(h^2), \tag{3.3}$$

where $qt_i - t_{k-1} = \theta h$ and $t_{k-1}, < \xi < t_k$.

- **Three points Lagrange interpolation of degree two:** Let $t_{k-1} < qt_i \leq t_k$, in this case we interpolate $y(t)$ at the points t_{k-2}, t_{k-1} and t_k by three-point Lagrange interpolation and then

$$\begin{aligned} y(qt_i) \cong \bar{y}(qt_i) \quad = \quad & \frac{1}{2h^2}(qt_i - t_{k-1})(qt_i - t_k)y_{k-2} - \frac{1}{h^2}(qt_i - t_{k-2})(qt_i - t_k)y_{k-1} \\ & + \frac{1}{2h^2}(qt_i - t_{k-2})(qt_i - t_{k-1})y_k. \end{aligned} \tag{3.4}$$

Again, the error term can be evaluated as follows

$$|y(qt_i) - \bar{y}(qt_i)| = |(qt_i - t_{k-2})(qt_i - t_{k-1})(qt_i - t_k)|\frac{|y'''(\xi)|}{3!} = \theta(1-\theta)(2-\theta)h^3|\frac{|y'''(\xi)|}{3!} = O(h^3), \tag{3.5}$$

where we let $qt_i - t_{k-1} = \theta h$ and $t_{k-1}, < \xi < t_k$.

Now we are ready to apply 1-FLMM (2.5) and (2.6) to the fractional multi-delay pantograph equation (1.1) respectively as follows.

$$\sum_{j=0}^{n-1} w_j y_{n-j} = b_n y_0 + h^\alpha \left(\sum_{i=1}^{m} F_i(y(p_i t_{n-1})) + g_{n-1} \right), \qquad n = 1, 2, \dots, N, \tag{3.6}$$

and

$$\sum_{j=0}^{n-1} w_j y_{n-j} = b_n y_0 + h^{\alpha}\left[(2-\frac{\alpha}{2})\left(\sum_{i=1}^{m} F_i(y(p_i t_{n-1})) + g_{n-1}\right)\right.$$
$$\left. +(\frac{\alpha}{2}-1)\left(\sum_{i=1}^{m} F_i(y(p_i t_{n-2})) + g_{n-2}\right)\right], \quad n = 2,3,\dots,N, \tag{3.7}$$

where $g_{n-2} = g(t_{n-2})$ and $g_{n-1} = g(t_{n-1})$. The delayed factors $y(p_i t_{n-1})$ and $y(p_i t_{n-2})$ are substituted by using (3.2) or (3.4).

The stability and convergence of the FLMMs which were studied in the literature (see [6–9, 12]) is preserved for solving delayed pantograph equations. In this work, we study the convergency of the proposed methods numerically. Although the methods (3.6) and (3.7) are explicit by nature, but due to the delay factor, the first few iterations may be calculated implicitly.

4. Illustrations

In this section, we illustrate some examples to present the accuracy and simplicity of the proposed FLMMs. We illustrate both linear and nonlinear pantograph differential equations with multiple proportional delays. To show the error and the numerical accuracy for the difference schemes (3.6) and (3.7), we denote

$$E_2(h) = \sqrt{h\sum_{i=0}^{N}(y(t_i) - y_i)^2},$$

where y_i is the approximate value for $y(x_i)$. Then, the computational convergence order can be calculated by

$$Order_2 = \log_2\left(\frac{E_2(h)}{E_2(h/2)}\right).$$

Finding the error and computational order is possible when the analytical solution of the equation is available. Otherwise, other strategies should be used to calculate the computational convergence order. In the examples provided, we have limited ourselves to drawing a graph to compare the numerical results when the analytical solution is not available.

Example 4.1. We consider the initial value problem [24]

$$^{C}D_t^{\alpha} y(t) = 1 + 2y\left(\frac{t}{2}\right) - y(t), \qquad y(0) = 0. \tag{4.1}$$

The exact solution for $\alpha = 1$ is $y(t) = t$. While for other values of α, the analytical solution is not known. This equation contains one delay with $p = \frac{1}{2}$. Applying (2.5) and (3.6) in (4.1) we get

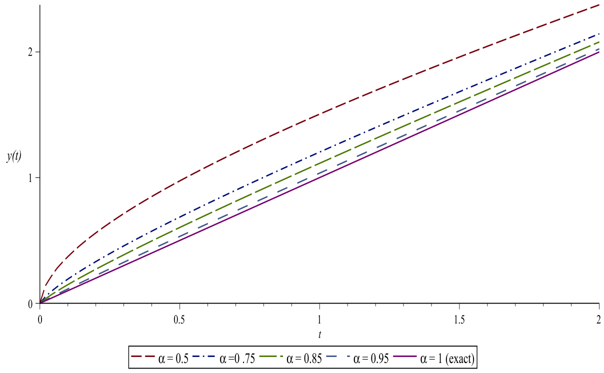
$$\sum_{j=0}^{n-1} w_j y_{n-j} = b_n y_0 + h^{\alpha}\left(1 + 2y_{\frac{n-1}{2}} - y_{n-1}\right), \qquad n = 1,2,\dots,N, \tag{4.2}$$

where y_0 is given by the initial condition and $y_{\frac{n-1}{2}} = y\left(\frac{t_{n-1}}{2}\right)$. Suppose that $t_{k-1} \le \frac{t_{n-1}}{2} < t_k$ then by (3.2) we write

$$y_{\frac{n-1}{2}} = y\left(\frac{t_{n-1}}{2}\right) \simeq -\frac{1}{h}\left(\frac{t_{n-1}}{2} - t_k\right) y_{k-1} + \frac{1}{h}\left(\frac{t_{n-1}}{2} - t_{k-1}\right) y_k = \frac{1}{2}(y_{k-1} + y_k). \tag{4.3}$$

By a similar substitution the two order 2-FLMM (2.7) can be written as follows

$$\sum_{j=0}^{n-1} w_j y_{n-j} = b_n y_0 + h^{\alpha}\left[(2-\frac{\alpha}{2})\left(1 + 2y_{\frac{n-1}{2}} - y_{n-1}\right) +(\frac{\alpha}{2}-1)\left(1 + 2y_{\frac{n-2}{2}} - y_{n-2}\right)\right], \quad n = 2,3,\dots,N, \tag{4.4}$$



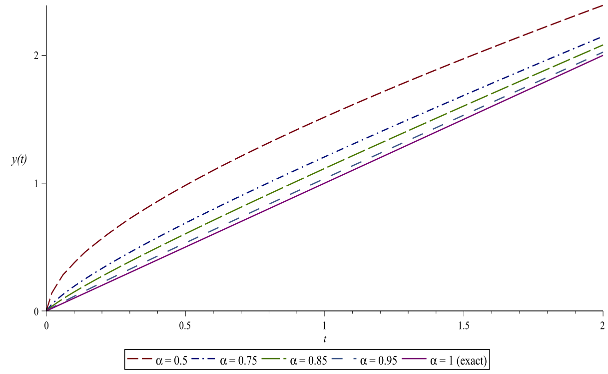


FIGURE 1. The graph of numerical solutions for Example 4.1, 1-FLMM (left), 2-FLMM (right).

where $y_{\frac{n-2}{2}}$ is considered similar to (4.3). It should be noted that step $n = 1$ should be calculated using the 1-FLMM (4.2). The numerical results for this example are plotted in Figure 1 for different values of α.

Example 4.2. Suppose the following fractional multi-delay pantograph equation [24]

$$ {}^{C}D_t^{\alpha} y(t) = t^2 - 1 - \frac{5}{6} y(t) + 4y\left(\frac{t}{2}\right) + 9y\left(\frac{t}{3}\right), \qquad y(0) = 1, \tag{4.5} $$

where the exact solution for $\alpha = 1$ is $y(t) = 1 + (67/7)t + (1675/72)t^2 + (12175/1296)t^3$. The solution is not available for $0 < \alpha < 1$. This equation contains two delays $p_1 = \dfrac{1}{2}$ and $p_2 = \dfrac{1}{3}$. Applying (2.5) and (3.6) in (4.5) we get the 1-FLMM and 2-FLMM as follows

$$ \sum_{j=0}^{n-1} w_j y_{n-j} = b_n y_0 + h^{\alpha} \left(t_{n-1}^2 - 1 - \frac{5}{6} y_{n-1} + 4y_{\frac{n-1}{2}} + 9y_{\frac{n-1}{3}} \right), \qquad n = 1, 2, \ldots, N, \tag{4.6} $$

$$ \begin{aligned} \sum_{j=0}^{n-1} w_j y_{n-j} = b_n y_0 + h^{\alpha} \Bigg[&(2 - \frac{\alpha}{2}) \left(t_{n-1}^2 - 1 - \frac{5}{6} y_{n-1} + 4y_{\frac{n-1}{2}} + 9y_{\frac{n-1}{3}} \right) \\ &+ (\frac{\alpha}{2} - 1) \left(t_{n-2}^2 - 1 - \frac{5}{6} y_{n-2} + 4y_{\frac{n-2}{2}} + 9y_{\frac{n-2}{3}} \right) \Bigg], \qquad n = 2, 3, \ldots, N. \end{aligned} \tag{4.7} $$

The numerical results for this example are plotted in Figure 2.

Example 4.3. In this example we solve the following fractional initial value problem of pantograph type [24]

$$ {}^{C}D_t^{\alpha} y(t) = \frac{1}{2} \exp\left(\frac{t}{2}\right) y\left(\frac{t}{2}\right) + \frac{1}{2} y(t), \qquad y(0) = 1. \tag{4.8} $$

The exact solution for $\alpha = 1$ is $y(t) = \exp(t)$ and, the exact solution is not available for $0 < \alpha < 1$. This equation contains one delay $p = \dfrac{1}{2}$. Applying (2.5) and (3.6) in (4.5) we get the 1-FLMM and 2-FLMM as follows

$$ \sum_{j=0}^{n-1} w_j y_{n-j} = b_n y_0 + h^{\alpha} \left(\frac{1}{2} \exp\left(\frac{t_{n-1}}{2}\right) y_{\frac{n-1}{2}} + \frac{1}{2} y_{n-1} \right), \qquad n = 1, 2, \ldots, N. \tag{4.9} $$

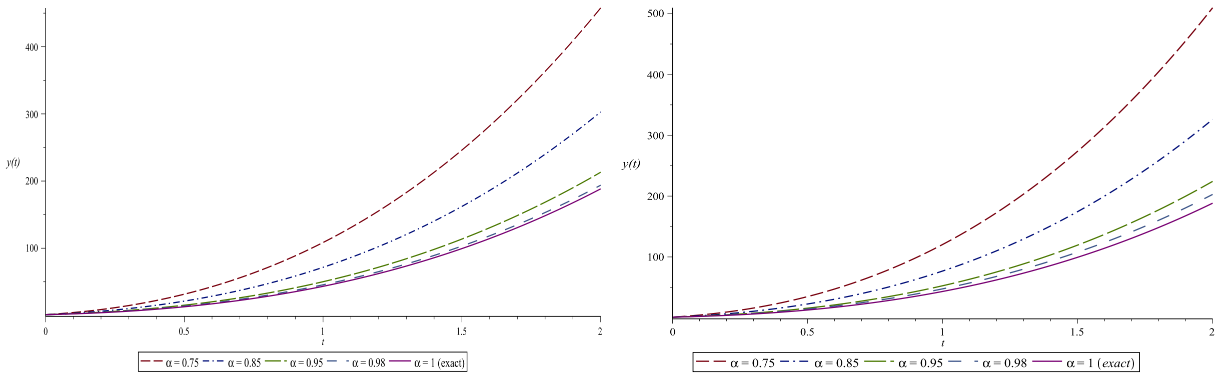


FIGURE 2. The graph of numerical solutions for Example 4.2, 1-FLMM (left), 2-FLMM (right).

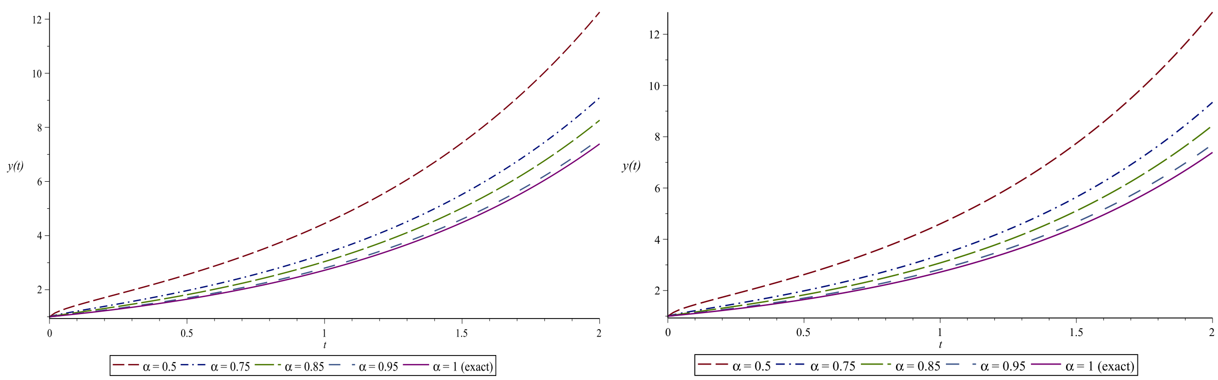


FIGURE 3. The graph of numerical solutions for Example 4.3, 1-FLMM (left), 2-FLMM (right).

$$\sum_{j=0}^{n-1} w_j y_{n-j} = b_n y_0 + h^\alpha \left[(2-\frac{\alpha}{2}) \left(\frac{1}{2} \exp\left(\frac{t_{n-1}}{2}\right) y_{\frac{n-1}{2}} + \frac{1}{2} y_{n-1} \right) \right.$$
$$\left. +(\frac{\alpha}{2}-1) \left(\frac{1}{2} \exp\left(\frac{t_{n-2}}{2}\right) y_{\frac{n-2}{2}} + \frac{1}{2} y_{n-2} \right) \right], \quad n = 2, 3, \ldots, N. \tag{4.10}$$

The numerical results for this example are plotted in Figure 3.

Example 4.4. Suppose the delayed initial value problem as

$${}^C D_t^\alpha y(t) = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} t^\alpha + \left(\frac{t}{2}\right)^{2\alpha} - t^{4\alpha} + y^2(t) - y\left(\frac{t}{2}\right), \qquad y(0) = 0. \tag{4.11}$$

The exact solution is $y(t) = t^{2\alpha}$. This equation contains one delay $p = \frac{1}{2}$. After applying (2.5) and (3.6) in (4.5) the 1-FLMM and 2-FLMM are obtained as follows

$$\sum_{j=0}^{n-1} w_j y_{n-j} = b_n y_0 + h^\alpha \left(\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} t_{n-1}^\alpha + \left(\frac{t_{n-1}}{2}\right)^{2\alpha} - t_{n-1}^{4\alpha} + y_{n-1}^2 - y_{\frac{n-1}{2}} \right), \quad n = 1, 2, \ldots, N, \tag{4.12}$$

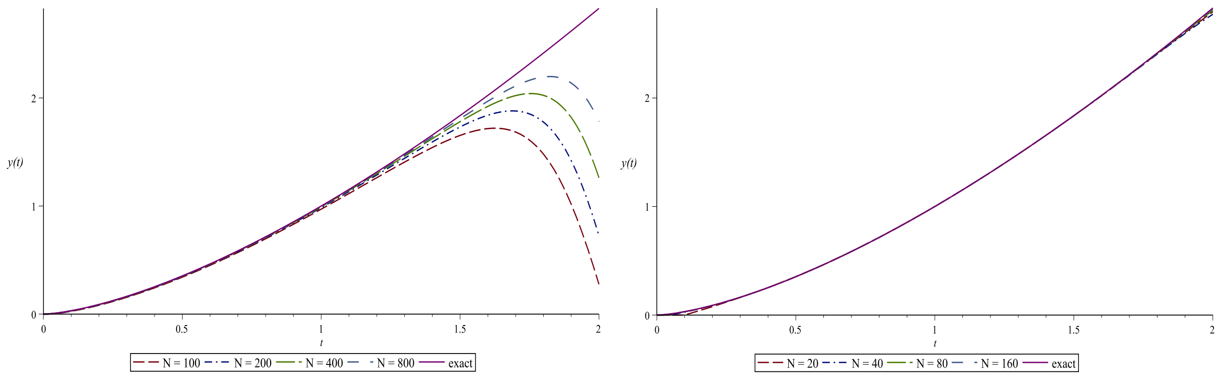


FIGURE 4. The graph of numerical solutions for Example 4.4 ($\alpha = 0.75$), 1-FLMM (left), 2-FLMM (right).

$$\begin{aligned}\sum_{j=0}^{n-1} w_j y_{n-j} \quad &= b_n y_0 + h^\alpha \left[(2-\frac{\alpha}{2}) \left(\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} t_{n-1}^{\alpha} + \left(\frac{t_{n-1}}{2}\right)^{2\alpha} - t_{n-1}^{4\alpha} + y_{n-1}^2 - y_{\frac{n-1}{2}} \right) \right. \\ &\left. +(\frac{\alpha}{2}-1) \left(\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} t_{n-2}^{\alpha} + \left(\frac{t_{n-2}}{2}\right)^{2\alpha} - t_{n-2}^{4\alpha} + y_{n-2}^2 - y_{\frac{n-2}{2}} \right) \right], n = 2,3,\dots,N. \qquad (4.13)\end{aligned}$$

The absolute errors and computational orders of convergence for Example 4.4 are illustrated in Table 1. Also, the numerical results for this example are plotted in Figure 4.

Example 4.5. We consider the initial value problem

$$ {}^{C}D_t^{\alpha} y(t) = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} t^{\alpha} - \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} + \left(\frac{t}{2}\right)^{2\alpha} - \left(\frac{t}{2}\right)^{2} - y\left(\frac{t}{2}\right), \qquad y(0) = 0. \qquad (4.14)$$

The exact solution is $y(t) = t^{2\alpha} - t^2$. This equation contains one delay $p = \frac{1}{2}$. We get the 1-FLMM and 2-FLMM as follows

$$\begin{aligned}\sum_{j=0}^{n-1} w_j y_{n-j} = b_n y_0 + h^\alpha \left(\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} t_{n-1}^{\alpha} - \frac{2}{\Gamma(3-\alpha)} t_{n-1}^{2-\alpha} + \left(\frac{t_{n-1}}{2}\right)^{2\alpha} - \left(\frac{t_{n-1}}{2}\right)^{2} - y_{\frac{n-1}{2}} \right), \\ n = 1,2,\dots,N, \qquad (4.15)\end{aligned}$$

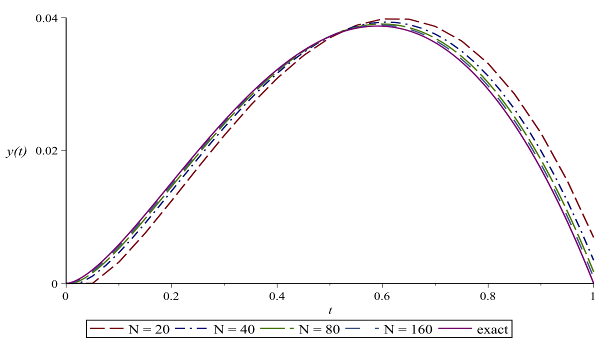
$$\begin{aligned}\sum_{j=0}^{n-1} w_j y_{n-j} = b_n y_0 + h^\alpha &\left[(2-\frac{\alpha}{2}) \left(\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} t_{n-1}^{\alpha} - \frac{2}{\Gamma(3-\alpha)} t_{n-1}^{2-\alpha} + \left(\frac{t_{n-1}}{2}\right)^{2\alpha} - \left(\frac{t_{n-1}}{2}\right)^{2} - y_{\frac{n-1}{2}} \right) \right. \\ &\left. +(\frac{\alpha}{2}-1) \left(\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} t_{n-2}^{\alpha} - \frac{2}{\Gamma(3-\alpha)} t_{n-2}^{\alpha} + \left(\frac{t_{n-2}}{2}\right)^{2\alpha} - \left(\frac{t_{n-2}}{2}\right)^{2-\alpha} - y_{\frac{n-2}{2}} \right) \right], \ n = 2,3,\dots,N. \qquad (4.16)\end{aligned}$$

Table 2 represents the absolute errors and computational convergence orders for Example 4.5. The numerical results for this example are plotted in Figure 5.

As the numerical results show, the computational order for 1-FLMM (3.6) is of order one while for 2-FLMM (3.7) the computational order for α values near one is of order two. This result is slightly worse for smaller values of α. The plotted graphs show that the numerical solutions tend to exact one.

Table 1. The absolute error and convergence orders for Example 4.4 at $t = 1$.

		$1-FLMM$		$2-FLMM$	
α	N	$E_2(h)$	$Order_2$	$E_2(h)$	$Order_2$
0.3	100	0.233e-1	–	0.175e-1	–
	200	0.115e-1	1.017	0.378e-2	1.532
	400	0.659e-2	0.805	0.169e-2	1.160
	800	0.356e-2	0.889	0.739e-3	1.195
0.5	100	0.904e-2	–	0.101e-2	–
	200	0.456e-2	0.977	0.356e-3	1.510
	400	0.232e-2	0.988	0.126e-3	1.499
	800	0.116e-2	0.994	0.448e-4	1.495
0.7	100	0.757e-2	–	0.187e-3	–
	200	0.380e-2	0.995	0.529e-4	1.820
	400	0.190e-2	0.998	0.150e-4	1.817
	800	0.952e-3	0.999	0.427e-5	1.815
0.9	100	0.674e-2	–	0.802e-4	–
	200	0.337e-2	1.001	0.203e-4	1.978
	400	0.168e-2	1.001	0.517e-5	1.975
	800	0.841e-3	1.000	0.132e-5	1.973
0.95	100	0.652e-2	–	0.764e-4	–
	200	0.326e-2	1.002	0.192e-4	1.995
	400	0.123e-2	1.001	0.482e-5	1.992
	800	0.813e-3	1.001	0.121e-5	1.991



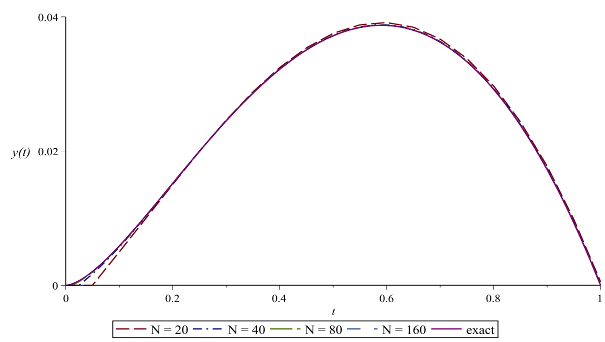


Figure 5. The graph of numerical solutions for Example 4.5, 1-FLMM (left), 2-FLMM (right).

Example 4.6. Suppose the nonlinear delayed initial value problem

$$ {}^{C}D_t^{\alpha} y(t) = -\exp(-3t) - \exp(-t) + 2\exp(-\frac{t}{3}) + y^3(t) - 2y(\frac{t}{3}), \qquad y(0) = 1. \tag{4.17}$$

The exact solution for $\alpha = 1$ is $y(t) = \exp(-t)$, while for $0 < \alpha < 1$ the exact solution is unknown. This equation contains one delay $p = \dfrac{1}{3}$. We get the 1-FLMM and 2-FLMM as follows

$$\sum_{j=0}^{n-1} w_j y_{n-j} = b_n y_0 + h^{\alpha}\left(-\exp(-3t_{n-1}) - \exp(-t_{n-1}) + 2\exp(-\frac{t_{n-1}}{3}) + y_{n-1}^3 - 2y_{\frac{n-1}{3}}\right),$$
$$n = 1, 2, \ldots, N. \tag{4.18}$$

TABLE 2. The absolute error and convergence orders for Example 4.5 at $t = 1$.

α	N	$1-FLMM$ $E_2(h)$	$1-FLMM$ $Order_2$	$2-FLMM$ $E_2(h)$	$2-FLMM$ $Order_2$
0.3	100	0.879e-2	–	0.843e-2	–
	200	0.438e-2	1.006	0.367e-2	1.201
	400	0.218e-2	1.009	0.163e-2	1.166
	800	0.109e-2	1.001	0.739e-3	1.145
0.5	100	0.446e-2	–	0.100e-2	–
	200	0.220e-2	1.020	0.354e-3	1.506
	400	0.109e-2	1.012	0.125e-3	1.498
	800	0.543e-3	1.007	0.444e-4	1.494
0.7	100	0.232e-2	–	0.165e-3	–
	200	0.115e-2	1.013	0.470e-4	1.810
	400	0.573e-3	1.007	0.134e-4	1.808
	800	0.286e-3	1.004	0.384e-5	1.807
0.9	100	0.623e-3	–	0.259e-4	–
	200	0.309e-3	1.010	0.666e-5	1.961
	400	0.154e-3	1.005	0.172e-5	1.955
	800	0.769e-4	1.003	0.444e-6	1.951
0.95	100	0.291e-3	–	0.115e-4	–
	200	0.145e-3	1.009	0.293e-5	1.971
	400	0.721e-4	1.005	0.750e-6	1.964
	800	0.360e-4	1.002	0.193e-6	1.959

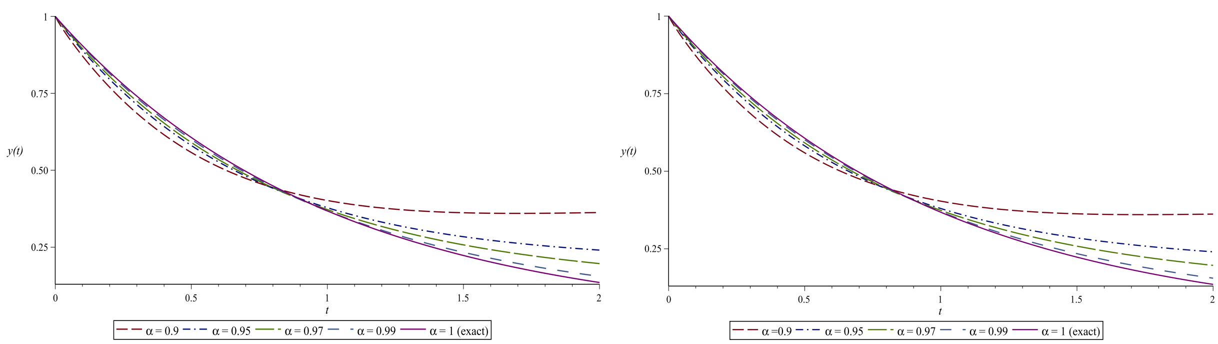


FIGURE 6. The graph of numerical solutions for Example 4.6, 1-FLMM (left), 2-FLMM (right).

$$\sum_{j=0}^{n-1} w_j y_{n-j} = b_n y_0 + h^{\alpha}\left[(2-\frac{\alpha}{2})\left(-\exp(-3t_{n-1}) - \exp(-t_{n-1}) + 2\exp(-\frac{t_{n-1}}{3}) + y_{n-1}^3 - 2y_{\frac{n-1}{3}}\right)\right.$$

$$\left.+(\frac{\alpha}{2}-1)\left(-\exp(-3t_{n-2}) - \exp(-t_{n-2}) + 2\exp(-\frac{t_{n-2}}{3}) + y_{n-2}^3 - 2y_{\frac{n-2}{3}}\right)\right], \ n = 2, 3, \ldots, N. \tag{4.19}$$

The numerical results for this example are plotted in Figure 6. The graphs show that the numerical solutions tend to the exact one as α tends 1.

5. Conclusion

In conclusion, this paper has introduced explicit fractional linear multi-step methods (FLMMs) for effectively solving fractional multi-delay pantograph differential equations. These methods denoted as 1-FMLM and 2-FLMM, maintain their convergence properties when the delayed components of the equations are appropriately approximated. Operating at first and second orders, these techniques have demonstrated their efficacy in solving both linear and non-linear equations. The numerical results have validated the theoretical findings, emphasizing the practical utility and reliability of our approach. These contributions enhance the repertoire of tools available for tackling complex fractional multi-delay pantograph differential equations, providing accurate and efficient solutions for a wide range of applications. In the continuation of this research, it is recommended to use p-FLMM methods with a higher degree of convergence and more accurate approximations for delay factors to numerically solve the pantograph equation.

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