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Efficient family of three-step with-memory methods and their dynamics

Vali Torkashvand[1,2,*], Manochehr Kazemi[3], and Masoud Azimi[2]

[1]Department of Mathematics, Shahr-e-Qods Branch, Islamic Azad University, Shahr-e-Qods, Iran.

[2]Department of Mathematics , Farhangian University , Tehran, Iran.

[3]Department of Mathematics, Ashtian Branch, Islamic Azad University, Ashtian, Iran.

Abstract

In this work, we have proposed a general manner to extend some two-parametric with-memory methods to obtain simple roots of nonlinear equations. Novel improved methods are two-step without memory and have two self-accelerator parameters that do not have additional evaluation. The methods have been compared with the nearest competitions in various numerical examples. Anyway, the theoretical order of convergence is verified. The basins of attraction of the suggested methods are presented and corresponded to explain their interpretation.

Keywords. With-memory method, Basin of attraction, Accelerator parameter, *R*-order convergence, Nonlinear equations.

2010 Mathematics Subject Classification. 65B99, 65H05.

1. Introduction

1.1. **Literature.** In recent studies, authors such as Bi et al. [2], Chun et al. [5], Cordero Barbero et al. [8], Lotfi et al. [12], Neta [16], Sharma and Arora [19], and Soleymani [23] have found the roots of nonlinear equations. They used the without-memory methods. Torkashvand [31] presented an iterative method without memory based on the family of Ostrowski's method to solve nonlinear equations. Kumar et al. [11] suggested an efficient class of fourth-order derivative-free method for multiple-roots.

In 2023, Moccari et al investigated the stability of a class two-step fourth-order methods [14]. Also, Campos et al. [7], Cordero et al. [9], Mohamadi Zadeh et al. [15], Soleymani [24], Torkashvand et al. [27–29], and Wang [34] used the with-memory method. Besides, researchers in [1] focused on the dynamics of the scheme methods. The authors in references [21] and [22] used repeat techniques for the first time to solve differential equations. Ullah et al. [32] obtained the convergence order of an adaptive method using the eigenvalue of the matrix.

1.2. **Motivation and organization.** Our objective in this essay is to study with-memory methods. In addition, we consider the dynamic behavior of the proposed method and applied to chemistry.

In section 2, we emanate a family of the with-memory methods and get a new family of three-point Steffensen-type iterative methods with memory by varying two self-accelerator parameters. Parameters are calculated using information known from the present and last iterations. The corresponding R-order of convergence is grown from 8 to 12.35. The maximal efficiency index of the with memory method is $12.35^{\frac{1}{3}} = 1.87$, which is higher than the efficiency indices of the existing without-memory methods. Numerical instances are offered in section 3 to display the convergence conduct of proposed methods for simple roots. Likewise, the application of these methods in solving chemical problems is given. In section 4, some dynamic factors associated with the proposed methods are investigated. Section 5 is a brief conclusion.

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∗ Corresponding author. Email: torkashvand1978@gmail.com.

1.3. **Definitions.**

(1) According to Kung-Traub's guess, an optimal iterative method without memory-based on $k+1$ evaluations can reach an optimal convergence order of 2^k [26]. Methods providing the Kung-Traub guess are called optimal methods.

(2) Following Traub's work [26], we propose a natural classification of iterative methods relying on the needed data from the present and prior iterations.
(a) Without-memory methods.
The category of iterative method (I.M.) is built by entering the expressions $w_1(x_k), w_2(x_k), ..., w_n(x_k)$, where x_k is the global reasoning. The I.M. φ, given as

$$x_{k+1} = \varphi(x_k, w_1(x_k), ..., w_n(x_k)), \tag{1.1}$$

is called a multi-point without-memory method. We see from (1.1) that the novel approximation x_{k+1} is gathered by the information of only previous estimate x_k, but through the n expressions w_i. (b) With-memory methods.
Let the I.M. have arguments z_j, where each such customry represents $n+1$ quantities $x_j, w_1(x_j), ...,$ $w_n(x_j)(n \geq 1)$. Then this I.M. can be reproduced in the general form as

$$x_{k+1} = \varphi(z_k; z_{k-1}, ..., z_{k-n}). \tag{1.2}$$

Such an iteration function is called a multi-point with-memory method. Namely, in each iterative step, we must preserve data of the last n approximations x_j, and for each approximation, we must determine n expressions $w_1(x_j), ..., w_n(x_j)$.

1.4. **Existing iterative method.** One of the famously grasped optimal second-order methods is Steffensen's method. This derivative-free method solves the nonlinear problems by two evaluations, as follows (SM) [25]

$$x_{m+1} = x_m - \frac{f(x_m)}{f[x_m, x_m + f(x_m)]}, m = 0, 1, 2, \cdots . \tag{1.3}$$

Ostrowski proposed the first two-point method of fourth-order as follows [17]

$$\begin{aligned} y_m &= x_m - \frac{f(x_m)}{f'(x_m)}, m = 0, 1, 2, \cdots , \\ x_{m+1} &= y_m - \frac{f(y_m)(y_m - x_m)}{2f(y_m) - f(x_m)}. \end{aligned} \tag{1.4}$$

Also, Neta [16] suggested a family of iterative method with sixth-order convergence, which is given by

$$\begin{aligned} y_m &= x_m - \frac{f(x_m)}{f'(x_m)}, m = 0, 1, \ldots , \\ z_m &= y_m - \frac{f(y_m)}{f'(x_m)} \frac{f(x_m) + \beta f(y_m)}{f(x_m) + (\beta - 2)f(y_m)}, \\ x_{m+1} &= z_m - \frac{f(z_m)}{f'(x_m)} \frac{f(x_m) - f(y_m)}{f(x_m) - 3f(y_m)}. \end{aligned} \tag{1.5}$$

Recently, Torkashvand and Araghi proposed a family of iterative with-memory methods [30] for solving nonlinear equations with convergence order four, eight, and sixteen as follows. These methods have the most efficient index.

One-step fourth-order method

$$\gamma_m = \frac{-1}{N'_{2m}(x_m)}, q_m = \frac{N''_{2m+1}(w_m)}{-2N'_{2m+1}(w_m)}, m = 1, 2, \cdots,$$
$$w_m = x_m + \gamma_m f(x_m), x_{m+1} = x_m - \frac{f(x_m)}{f[x_m, w_m] + q_m f[w_m]}, m = 0, 1, 2, \cdots. \tag{1.6}$$

Two-step eight-order method

$$\gamma_m = -\frac{1}{N'_{3m}(x_m)}, q_m = -\frac{N''_{3m+1}(w_m)}{2N'_{3m+1}(w_m)}, \lambda_m = \frac{N'''_{3m+2}(y_m)}{6}, m = 1, 2, 3, \cdots,$$
$$y_m = x_m - \frac{f(x_m)}{f[x_m, w_m] + q_m f(w_m)}, m = 0, 1, 2, \cdots,$$
$$x_{m+1} = y_m - \frac{f(y_m)}{f[w_m, y_m] + q_m f(w_m) + \lambda_m (y_m - x_m)(y_m - w_m)}(1 + \frac{f(y_m)}{f(x_m)}). \tag{1.7}$$

Three-step sixteenth-order method

$$\gamma_m = -\frac{1}{N'_{4m}(x_m)}, q_m = -\frac{N''_{4m+1}(w_m)}{2N'_{4m+1}(w_m)}, \lambda_m = \frac{N'''_{4m+2}(y_m)}{6}, \beta_m = \frac{N''''_{4m+3}(z_m)}{24}, m = 1, 2, 3, \cdots,$$
$$w_m = x_m + \gamma_m f(x_m), y_m = x_m - \frac{f(x_m)}{f[x_m, w_m] + q_m f(w_m)}, m = 0, 1, 2, \cdots,$$
$$z_m = y_m - \frac{f(y_m)}{f[y_m, x_m] + f[w_m, x_m, y_m](y_m - x_m) + \lambda_m (y_m - x_m)(y_m - w_m)},$$
$$x_{m+1} = z_m - \frac{f(z_m)}{f[x_m, z_m] + (f[w_m, x_m, y_m] - f[w_m, x_m, z_m] - f[y_m, x_m, z_m])(x_m - z_m) + \beta_m (z_m - y_m)(z_m - x_m)(z_m - w_m)}. \tag{1.8}$$

2. Description of the methods

In 2011, Soleymani proposed the three-point method following [23]

$$w_m = x_m + \gamma f(x_m), y_m = x_m - \frac{f(x_m)}{f[x_m, w_m]}, \gamma \in \mathbb{R} - \{0\}, m = 0, 1, 2, \cdots,$$
$$z_m = y_m - \frac{f(y_m)}{f[x_m, y_m] + f[y_m, w_m] - f[x_m, w_m] + \beta (y_m - x_m)(y_m - w_m)}, \beta \in \mathbb{R},$$
$$t_m = \frac{f(y_m)}{f(x_m)}, u_m = \frac{f(y_m)}{f(w_m)}, v_m = \frac{f(z_m)}{f(y_m)}, s_m = \frac{f(z_m)}{f(w_m)}, p_m = \frac{f(z_m)}{f(x_m)},$$
$$x_{m+1} = z_m - \frac{f(z_m)(G_1(t_m) + G_2(u_m) + G_3(v_m) + G_4(s_m) + G_5(p_m))}{f[x_m, z_m] + f[z_m, y_m] - f[x_m, y_m] + \beta (z_m - y_m)(z_m - x_m)}. \tag{2.1}$$

The following theorem shows beneath what conditions on the weight functions in (2.1) convergence order is eight.

Theorem 2.1. *If $I \subseteq \mathbb{R}$ is an open interval, and $f : I \to \mathbb{R}$ is a differentiable function that has a simple zero, say α. If x_0 is a primary guess to α, then method (2.1) has eight-order convergence. When the weight functions $G_1(t_m), G_2(u_m), G_3(v_m), G_4(s_m)$ and $G_5(p_m)$ satisfy the following conditions:*

$$G_1(0) = 1, G'_1(0) = G''_1(0) = G'''_1(0) = 0, |G_1^{(4)}(0)| < \infty,$$
$$G_2(0) = G'_2(0) = G''_2(0) = 0, G_2^{(3)}(0) = -(6 + 6\gamma f[x_m, w_m]), |G_2^{(4)}(0)| < \infty,$$
$$G_3(0) = G'_3(0) = 0, |G''_3(0)| < \infty,$$
$$G_4(0) = 0, G'_4(0) = 1, |G''_4(0)| < \infty,$$
$$G_5(0) = G'_5(0) = 0, |G''_5(0)| < \infty. \tag{2.2}$$

And the error equation of method (2.1) *can be presented as follows*

$$\begin{aligned} e_{m+1} &= (1+\gamma f'(\alpha))^3 c_2^2(\beta + f'(\alpha)c_2^2 - f'(\alpha)c_3)(-f'(\alpha)(3+\gamma f'(\alpha)c_2^3 \\ &+ c_2(\beta(4+3\gamma f'(\alpha)) - 2f'(\alpha)(1+\gamma f'(\alpha))c_3) \\ &+ f'(\alpha)(1+\gamma f'(\alpha))c_4)f'(\alpha)^{-2} e_m^8 + O(e_m^9). \end{aligned} \tag{2.3}$$

Proof. The proof of this theorem is given in [23]. Hence it is omitted. □

2.1. **Convergence analysis.** From error equation (2.3) can be shown that the convergence order of the family (2.1) is eight when $(1+\gamma f'(\alpha)) \neq 0$ or $(\beta + f'(\alpha)c_2^2 - f'(\alpha)c_3) \neq 0$. Consequently, it is probable to raise the convergence rate of the suggested class (2.1), if $(1+\gamma f'(\alpha)) = 0$ or $(\beta + f'(\alpha)c_2^2 - f'(\alpha)c_3) = 0$. As $f'(\alpha), f''(\alpha)$, and $f'''(\alpha)$ are not available in practice, then increasing the convergence order is not possible. Alternatively, we have used approximations $\tilde{f}'(\alpha) \approx f'(\alpha), \tilde{f}''(\alpha) \approx f''(\alpha)$, and $\tilde{f'''}(\alpha) \approx f'''(\alpha)$ determined by already available information. Consequently, by selecting $\gamma = \frac{1}{\tilde{f}'(\alpha)}$ and $\beta = \frac{\tilde{f'''}(\alpha)}{6} - \frac{\tilde{f''}(\alpha)^2}{4\tilde{f}'(\alpha)}$ convergence order increases without accepting any new functional evaluation. Thus, the principal idea in constructing with-memory methods consists of the computation of the parameters $\gamma = \gamma_m$ and $\beta = \beta_m$ as the iteration yields by the formula $\gamma_m = \frac{1}{\tilde{f}'(\alpha)}$ and $\beta_m = -\frac{\tilde{f''}(\alpha)^2}{4\tilde{f}'(\alpha)} + \frac{\tilde{f'''}(\alpha)}{6}$ for $m = 1, 2, \cdots$. Accordingly, we have approximated as follows

$$\begin{aligned} \gamma_m &= -\frac{1}{\tilde{f}'(\alpha)} = -\frac{1}{N_4'(x_m)}, \\ \beta_m &= \frac{\tilde{f'''}(\alpha)}{6} - \frac{\tilde{f''}(\alpha)^2}{4\tilde{f}'(\alpha)} = \frac{N_6'''(y_m)}{6} - \frac{(N_6''(y_m))^2}{4N_6'(y_m)}. \end{aligned} \tag{2.4}$$

where $N_4'(x_m), N_6''(y_m)$ and $N_6'''(y_m)$ are Newton's interpolation polynomials proceed via the nodes $\{x_m, x_{m-1}, w_{m-1}, y_{m-1}, z_{m-1}\}$, and $\{y_m, w_m, x_m, x_{m-1}, w_{m-1}, y_{m-1}, z_{m-1}\}$, respectively. Currently, we get the new iterative with-memory method as shadows

$$\begin{aligned} \gamma_m &= -\frac{1}{N_4'(x_m)}, \beta_m = \frac{N_6'''(y_m)}{6} - \frac{(N_6''(y_m))^2}{4N_6'(y_m)}, m = 1, 2, 3, \cdots, \\ w_m &= x_m + \gamma_m f(x_m), y_m = x_m - \frac{f(x_m)}{f[x_m, w_m]}, \gamma_m \in \mathbb{R} - \{0\}, \\ z_m &= y_m - \frac{f(y_m)}{f[x_m, y_m] + f[y_m, w_m] - f[x_m, w_m] + \beta_m (y_m - x_m)(y_m - w_m)}, \beta_m \in \mathbb{R}, , m = 0, 1, 2, \cdots, \\ x_{m+1} &= z_m - \frac{f(z_m)(1 - (1 + \gamma_m f[x_m, w_m])(\frac{f(x_m)}{f(w_m)})^3 + \frac{f(z_m)}{f(w_m)})}{f[x_m, z_m] + f[z_m, y_m] - f[x_m, y_m] + \beta_m (z_m - y_m)(z_m - x_m)}. \end{aligned} \tag{2.5}$$

Here, we prove that the new methods (2.5) have R-order of convergence 12.3. For this aim, we state the following lemma.

Lemma 2.2. *Let* $\gamma_m = -\frac{1}{N_4'(x_m)}, \beta_m = \frac{N_6'''(y_m)}{6} - \frac{(N_6''(y_m))^2}{4N_6'(y_m)}$, *then*

$$1 + \gamma_m f'(\alpha) \sim e_{m-1} e_{m-1,w} e_{m-1,y} e_{m-1,z}, (\beta_m + f'(\alpha)c_2^2 - f'(\alpha)c_3) \sim e_{m-1} e_{m-1,w} e_{m-1,y} e_{m-1,z}. \tag{2.6}$$

Proof. Assume that there are $i+1$ nodes $t_0, t_1, \cdots, t_i$ from the interval $D = [a, b]$, where a is the minimum and b is the maximum of these nodes, respectively. Then the error of Newton's interpolation polynomial $N_i(t)$ of grade i is determined by

$$f(t) - N_i(t) = \frac{f^{(i+1)}(\alpha)}{(i+1)!}\prod_{j=0}^{s}(t - t_j). \tag{2.7}$$

For $i = 4$, the equation (2.7) takes the form (holding in the sense $t_0 = x_{m-1}, t_1 = w_{m-1}, t_2 = y_{m-1}, t_3 = z_{m-1}t_4 = x_m$)

$$f(t) - N_4(t) = \frac{f^{(5)}(\alpha)}{5!}(t - x_m)(t - z_{m-1})(t - y_{m-1})(t - w_{m-1})(t - x_{m-1}). \tag{2.8}$$

Using outcome (2.8) and for t and putting $t = x_m$, we obtain

$$f'(x_m) - N_4'(x_m) = \frac{f^{(5)}(\alpha)}{5!}(x_m - x_{m-1})(x_m - w_{m-1})(x_m - y_{m-1})(x_m - z_{m-1}). \tag{2.9}$$

Now

$$x_m - x_{m-1} = (x_m - \alpha) - (x_{m-1} - \alpha) = e_m - e_{m-1}.$$

Similarly

$$\begin{aligned} x_m - w_{m-1} &= e_m - e_{m-1,w}, \\ x_m - y_{m-1} &= e_m - e_{m-1,y}, \\ x_m - z_{m-1} &= e_m - e_{m-1,z}. \end{aligned}$$

Substituting these relations in (2.9), we get

$$\begin{aligned} N_4'(x_m) &= f'(\alpha)(1 + 2c_2e_m + 3c_3e_m^2 + \ldots) - \frac{f^{(5)}(\alpha)}{5!}(e_m - e_{m-1})(e_m - x_{m-1,w})(e_m - x_{m-1,y})(e_m - x_{m-1,z}) \\ &\sim f'(\alpha)(1 + 2c_2e_m - c_5e_{m-1}e_{m-1,w}e_{m-1,y}e_{m-1,z}). \end{aligned} \tag{2.10}$$

And thus

$$1 + f'(\alpha)\gamma_m \sim 1 - \frac{1}{1 + 2c_2e_m - c_5e_{m-1}e_{m-1,w}e_{m-1,y}e_{m-1,z}} \sim e_{m-1}e_{m-1,w}e_{m-1,y}e_{m-1,z}. \tag{2.11}$$

Similarly, other result can be proved. □

Theorem 2.3. *If an initial conjecture x_0 is to the root α of $f(x)$ and the parameters γ_m and β_m in the method (2.5) is recursively-calculated unexpectedly, the convergence order of with-memory methods (2.5) is at least $\frac{1}{2}(13 + \sqrt{137}) \approx 12.35$.*

Proof. Suppose that the convergence order of the sequences w_m, y_m, z_m and x_m are at least p, q, s and r, respectively. Consequently

$$\begin{aligned} e_{m+1} &\sim e_m^r \sim e_{m-1}^{r^2}, \\ e_{m,z} &\sim e_m^s \sim e_{m-1}^{rs}, \\ e_{m,y} &\sim e_m^q \sim e_{m-1}^{rq}, \\ e_{m,w} &\sim e_m^p \sim e_{m-1}^{rp}. \end{aligned} \tag{2.12}$$

Using Lemma 1, also, the (2.3),(2.5), and (2.12) we get

$$\begin{aligned} e_{m+1} &\sim (1 + \gamma_m f'(\alpha))^3(\beta_m + f'(\alpha)c_2^2 - f'(\alpha)c_3)e_m^8 \sim (e_{m-1}e_{m-1,w}e_{m-1,y}e_{m-1,z})^4 e_m^8 \sim e_{m-1}^{4(1+p+q+s)+8r}, \\ e_{m,z} &\sim (1 + \gamma_m f'(\alpha))^2 c_2(\beta_m + f'(\alpha)c_2^2 - f'(\alpha)c_3)e_m^4 \sim (e_{m-1}e_{m-1,w}e_{m-1,y}e_{m-1,z})^3 e_m^4 \sim e_{m-1}^{3(1+p+q+s)+4r}, \\ e_{m,y} &\sim (1 + \gamma_m f'(\alpha))c_2e_m^2 \sim e_{m-1}e_{m-1,w}e_{m-1,y}e_{m-1,z}e_m^2 \sim e_{m-1}^{1+p+q+s+2r}, \\ e_{m,w} &\sim (1 + \gamma_m f'(\alpha))e_m \sim e_{m-1}e_{m-1,w}e_{m-1,y}e_{m-1,z}e_m \sim e_{m-1}^{1+p+q+s+r}. \end{aligned} \tag{2.13}$$

Comparing the exponents of e_{m-1} in four expressions (2.12) and (2.13) of $e_{m+1}, e_{m,z}, e_{m,y}, e_{m,w}$, we get four equations in the system (2.14):

$$\begin{aligned} &rp-(1+p+q+s)-r=0,\\ &rq-(1+p+q+s)-2r=0,\\ &rs-3(1+p+q+s)-4r=0,\\ &r^2-4(1+p+q+s)-8r=0. \end{aligned} \tag{2.14}$$

The retort to the overhead equations system is as follows

$$p=\frac{1}{8}(5+\sqrt{137}), q=\frac{1}{8}(13+\sqrt{137}), s=\frac{1}{8}(23+3\sqrt{137}), r=\frac{1}{2}(13+\sqrt{137}), \tag{2.15}$$

which specifies the convergence order of the scheme (2.5) is $r=\frac{1}{2}(13+\sqrt{137})\approx 12.35$ (denoted by TAKM). □

Remark 2.4. The novel method (2.5) requires four-function evaluations and has the convergence order of 12.35. Consequently, the efficiency index of the suggested methods is $12.35^{\frac{1}{4}}=1.87$, which is higher than the optimal one until four-point optimal methods without memory having efficiency indexes $EI=2^{1/2}\simeq 1.41, EI=4^{1/3}\simeq 1.59, EI=8^{1/4}\simeq 1.68, EI=16^{1/5}\simeq 1.74$, respectively.

In the next section, we have displayed the effectiveness of with-memory methods with numerical examples for solving nonlinear equations.

3. Numerical examples

Now, we also want to study the efficiency of the offered scheme and validate the academic results. For this purpose, we have utilized the following test functions [27] and displayed the approximate.

$$\begin{aligned} &h_1(t)=x\log(1+t\sin(t))+e^{-1+t^2+x\cos(t)}\sin(\pi t), \alpha=0, t_0=0.6.\\ &h_2(t)=1+\frac{1}{t^4}-\frac{1}{t}-t^2, \alpha=1, t_0=1.4.\\ &h_3(t)=\sqrt{t^4+8}\sin(\frac{\pi}{t^2+2}+\frac{t^3}{t^4+1}+\sqrt{6}+\frac{8}{17}, \alpha=-2, t_0=-2.3. \end{aligned} \tag{3.1}$$

We have selected our suggested scheme (2.5) (for $\beta=0.1$ and $\gamma=0.1$), called by TAKM for comparison with the existing robust optimal eighth-order method that was offered by Bi et al. (BRWM) [2], Kung-Traub (KTM)[10], Lotfi et al. (LSSSM) [12], Sharma-Arora(SAM) [20], and Soleymani (SM) [23], respectively. For better comparisons of our proposed methods with other existing ones, we have given of comparison tables in each test function.
The error between the two successive repeats $|x_{n+1}-x_n|$.
Furthermore, we have shown the approximation errors to the corresponding zeros of test functions in Tables 1-3, where $m(-n)$ denotes $m\times 10^{-n}$. These tables contain the values of the computational order of convergence COC calculated by the formula [18]

$$COC=\frac{Log|\frac{f(x_m)}{f(x_{m-1})}|}{Log|\frac{f(x_{m-1})}{f(x_{m-2})}|}. \tag{3.2}$$

4. Tables

Tables 1-3 show that method (2.5) costs less computing time than other methods. The main reason is that the structure of self-accelerating parameters of our method (2.5) is simple.

Table 1. Results of comparisons for different methods for $h_1(x)$.

Methods	$\lvert x_1 - \alpha \rvert$	$\lvert x_2 - \alpha \rvert$	$\lvert x_3 - \alpha \rvert$	COC
BRWM [2]	0.40(−2)	0.24(−18)	0.38(−148)	8.00
KTM[10]	0.23(−1)	0.34(−13)	0.14(−107)	8.00
LSSSM [12]	0.42(−2)	0.78(−18)	0.10(−143)	8.00
SAM[20]	0.95(−1)	0.44(−8)	0.60(−67)	8.00
SM[23]	0.22(−1)	0.15(−12)	0.12(−25)	8.00
TAKM (2.5)	0.22(−1)	0.81(−18)	0.27(−216)	12.35

Table 2. Results of comparisons for different methods for $h_2(x)$.

Methods	$\lvert x_1 - \alpha \rvert$	$\lvert x_2 - \alpha \rvert$	$\lvert x_3 - \alpha \rvert$	COC
BRWM [2]	0.54(−3)	0.13(−23)	0.14(−188)	8.00
KTM, $\gamma = 1$[10]	0.11(−1)	0.46(−12)	0.50(−95)	8.00
LSSSM [12]	0.70(−3)	0.19(−22)	0.47(−179)	8.00
SAM[20]	0.39(−3)	0.38(−26)	0.34(−210)	8.00
SM[23]	0.14(−10)	0.12(−82)	0.44(−659)	8.00
TAKM (2.5)	0.19(−3)	0.27(−41)	0.77(−497)	12.35

Table 3. Results of comparisons for different methods for $h_3(x)$.

Methods	$\lvert x_1 - \alpha \rvert$	$\lvert x_2 - \alpha \rvert$	$\lvert x_3 - \alpha \rvert$	COC
BRWM [2]	0.19(−6)	0.95(−55)	0.38(−441)	8.00
KTM, $\gamma = 1$[10]	0.98(−7)	0.47(−57)	0.14(−459)	8.00
LSSSM [12]	0.30(−6)	0.39(−52)	0.28(−419)	8.00
SAM[20]	0.33(1)	0.30(1)	0.30(1)	*div*
SM[23]	0.21(−5)	0.12(−45)	0.14(−367)	8.00
TAKM (2.5)	0.18(−6)	0.79(−83)	0.14(−999)	12.35

4.1. **Application.** In the following numerical analysis, we have proposed an example of Chemistry and have solved it. Van Der Waals' equation proffered by

$$(P + \frac{n^2 c_1}{V^2})(V + nc_2) = nRT. \tag{4.1}$$

where p is pressure, V is capacity, and the temperature T is in Kelvin units. Also, n the number of moles of the gas. R is the gas constant equals 0.0820578. c_1 and c_2 are named Van Der Waals constants that they depend on the gas kind. This equation is nonlinear in V. It can be humble to the following function of V.

$$f(V) = PV^3 - nc_2PV^2 - nRTV^2 - n^2c_1V - n^3c_1c_2. \tag{4.2}$$

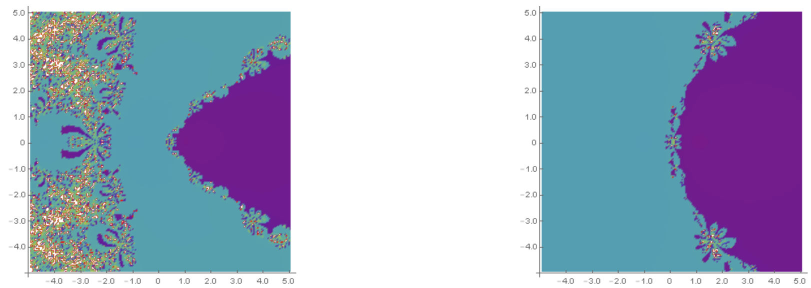
Now, if one wants to earn the volume of 1.4 moles of benzene steam under the pressure of 40 atm and temperature of $500°C$, given that Van Der Waals constants for benzene are $c_1 = 18$ and $c_2 = 0.1154$, then the puzzle that occurs is to gain roots of the polynomial

$$g(x) = -5.6998368 + 35.28x - 95.26535116x^2 + 40x^3. \tag{4.3}$$

The three roots of the equation are $x = \mp 0.173507i + 0.205425$ and $x \approx 1.97078$. As V is a volume, only the positive real roots are physically significant, and the root $x \approx 1.97078$ of the equation is acceptable. We estimated the initial conjecture 2 for this issue.

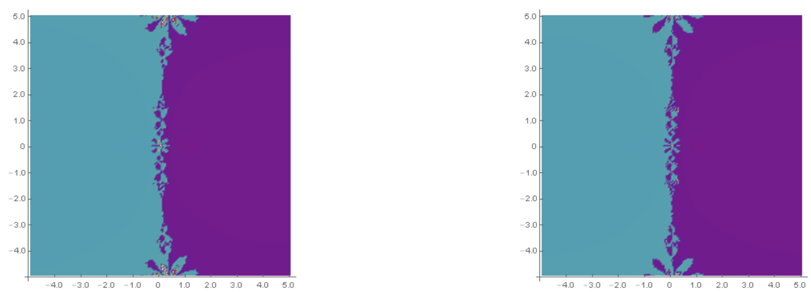
Table 4. Results of comparisons for different methods $g(x)$.

Methods	$\|x_1-\alpha\|$	$\|x_2-\alpha\|$	$\|x_3-\alpha\|$	COC
BRWM [2]	0.92(−2)	0.23(−18)	0.32(−151)	8.00
KTM[10]	0.23(−1)	0.34(−13)	0.14(−87)	8.00
LSSSM [12]	0.22(−2)	0.67(−19)	0.10(−58)	8.00
SAM[20]	0.23(−1)	0.25(−8)	0.52(−65)	8.00
SM[23]	0.23(−1)	0.45(−5)	0.42(−15)	8.00
TAKM (2.5)	0.23(−2)	0.14(−25)	0.42(−135)	12.35



(A) $\beta=\gamma=1, n=200$ (B) $\beta=\gamma=0.1, n=200$

Figure 1. Method TAKM (2.5) for detecting the polynomial roots $f(z)=z^2-1$.



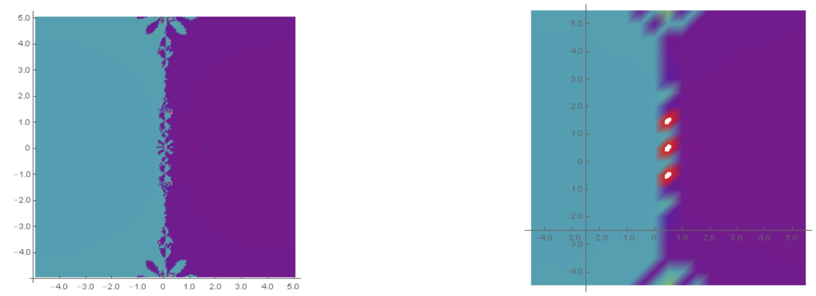
(A) $\beta=\gamma=0.01, n=200$ (B) $\beta=\gamma=0.001, n=200$

Figure 2. Method TAKM (2.5) for detecting the polynomial roots $f(z)=z^2-1$.

5. Some dynamical aspects the proposed methods

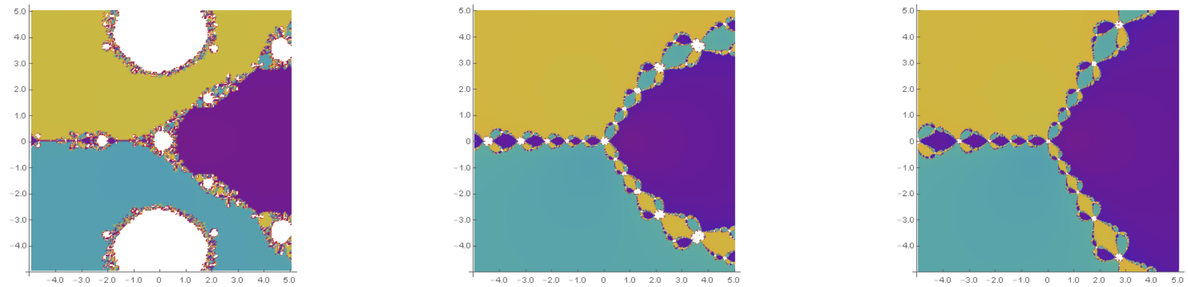
Dynamical properties of the iterative methods present us with important information about the numerical aspect of them as their stableness and validation. In what follows, we have compared the proposed methods by utilizing the attraction basins for two complex polynomials $f(z)=z^m-1, m=2,3$. Also, we have used a analogous technique in [5, 6, 13, 33] to generate the basins of attraction. We have caught a gride of 200×200 points in a rectangle $=[-5.0,5.0]\times[-5.0,5.0]\subset C$. The points as z_0 have produced the basins of attraction for the zeros of a polynomial. The sequence generated by the iterative method reaches a zero z^* of the polynomial with a tolerance $|z_m-z^*|<10^{-10}$ and a maximum of 25 iterations. We conclude that z_0 is in the basin of attraction of the zero. The attraction basin shows the iterations number needed to arrive at the solution. We have defined four values to the self-accelerator parameters, and in this section, we will select the best amount on the self-accelerator parameter.

Remark 5.1. Comparing Figures 1-4, one can conclude that the accelerator parameter plays an important role in increasing and decreasing the adsorption region, just as it plays an essential role in improving the convergence order.



(A) $\beta = \gamma = 0.001, n = 200$. (B) $\beta = \gamma = 0.001, n = 20$.

FIGURE 3. Method TAKM (2.5) for detecting the polynomial roots $f(z) = z^2 - 1$



(A) $\beta = \gamma = 0.1, n = 200$ (B) $\beta = \gamma = 0.01, n = 200$. (C) $\beta = \gamma = 0.001, n = 200$.

FIGURE 4. Method TAKM (2.5) for detecting the polynomial roots $f(z) = z^3 - 1$.

(A) Kung-Traub's method. (B) $\beta = \gamma = 0.001, n = 200$, TAKM (2.5).

FIGURE 5. Comparison of methods for finding the roots of $f(z) = z^2 - 1$.

The highest absorption region is related to the minimum parameter, which is $\gamma = 0.001$, and the lowest absorption region is when the self-accelerating parameter is equal to $\gamma = 1$. Also, the higher the number of square points, the higher the percentage of transparency of the adsorption areas. The basins of attraction of Kung-Traub optimal methods and our method are considered to find roots quadratic polynomial $f(z) = z^2 - 1$ and cubic polynomial $f(z) = z^3 - 1$ in Figures 5 and 6. The basin of attraction for TAKM is better than Kung-Traub.

6. Conclusions

In this work, we first converted a family of the eighth-order method into a with-memory method. Afterward, by entering bi self-accelerator parameters and approximating them using the available information, we have created

(A) Kung-Traub's method. (B) $\beta = \gamma = 0.001, n = 200$, TAKM (2.5).

FIGURE 6. Comparison of methods for finding the roots of $f(z) = z^3 - 1$.

methods with a convergence level of 12.35. The efficiency index of the proposed methods is equal to 1.87. The efficiency index of proposed methods is higher than the efficiency index of optimal-order methods one, two, three, four, and five. The rate of convergence improvement of the new-methods (2.5) is 54.40%, which is higher than other methods with memory and without memory mentioned in [3, 7]. Both numerical and dynamical aspects of iterative plans (2.5) support the main theorem well within any analysis equations and examples.
Further researches must be done to develop the proposed methods for system of nonlinear equations. These could be done in the next studies.

Conflict of interest

The authors declare no potential conflict of interests.

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References

[1] S. Amat, S. Busquier, and S. Plaza, *Dynamics of the King and Jarratt iterations, Aequationes mathematicae*, *69* (3) (2005), 212–223.
[2] W. Bi, Q. Wu, and H. Ren, *A new family of eighth-order iterative methods for solving nonlinear equations*, Applied Mathematics and Computation, *214* (1) (2009) 236–245.
[3] G. Candelario, A. Cordero, J.R. Torregrosa, and M.P. Vassileva, *An optimal and low computational cost fractional Newton-type method for solving nonlinear equations*, Applied Mathematics Letters, *124* (2022), 107650.
[4] B. Campos, A. Cordero, J.R. Torregrosa, and P. Vindel, *Dynamical analysis of an iterative method with memory on a family of third-degree polynomials*, AIMS Mathematics, *7*(4) 6445–6466.
[5] C. Chun, B. Neta, J. Kozdon, and M. Scott, *Choosing weight functions in iterative methods for simple roots*, Applied Mathematics and Computation, *227* (2014), 788–800.
[6] F. Chicharro, A. Cordero, J. M. Gutiérrez, and J.R. Torregrosa, *Complex dynamics of derivative-free methods for nonlinear equations*, Applied Mathematics and Computation, *219* (12) (2013), 7023–7035.
[7] B. Campos, A. Cordero Barbero, J. R, Torregrosa Sánchez, and P. Vindel Canas, *Stability of King's family of iterative methods with memory*, Journal of Computational and Applied Mathematics, *318* (2017) 504–514.
[8] A. Cordero Barbero, M. Fardi, M. Ghasemi, and J. R. Torregrosa, *Accelerated iterative methods for finding solutions of nonlinear equations and their dynamical behavior*, Calcolo, *51* (1) (2014) 17–30
[9] A. Cordero, H. Ramos, and J. R. Torregrosa, *Some variants of Halley's method with memory and their applications for solving several chemical problems*, Journal of Mathematical Chemistry, *58* (2020) 751–774.
[10] H. T. Kung and J. F. Traub, *Optimal order of one-point and multi-point iteration*, J. ACM, *21* (4) (1974) 643–651.

[11] S. Kumar, D. Kumar, J. R. Sharma, and I. K. Argyros, *An efficient class of fourth-order derivative-free method for multiple-roots*, International Journal of Nonlinear Sciences and Numerical Simulation, *24* (1) (2023) 265–275.

[12] T. Lotfi, S. Sharifi, M. Salimi and S. Siegmund, *A new class of three-point methods with optimal convergence order eight and its dynamics*, Numerical Algorithms, *68* (2) (2015) 261–288.

[13] M. Moccari and T. Lotfi, *On a two-step optimal Steffensen-type method: Relaxed local and semi-local convergence analysis and dynamical stability*, Journal of Mathematical Analysis and Applications, *468* (1) (2018) 240–269.

[14] M. Moccari, T. Lotfi, and V. Torkashvand, *On Stability of a Two-Step Method for a Fourth-degree Family by Computer Designs along with Applications*, International Journal of Nonlinear Analysis and Applications (IJNAA), *14* (4) (2023) 261–282.

[15] M. Mohamadi Zadeh, T. Lotfi, and M. Amirfakhrian, *Developing two efficient adaptive Newton-type methods with memory*, Mathematical Methods in the Applied Sciences, *42* (17) (2019) 5687–5695.

[16] B. Neta, *A sixth order family of methods for nonlinear equations*, International Journal of Computer Mathematics, *7* (1979) 157–161.

[17] A.M. Ostrowski, *Solution of Equations and Systems of Equations*, Academic Press, New York, 1960.

[18] M.S. Petković, B. Neta, L.D. Petković, and J. Džunić, *Multipoint Methods for Solving Nonlinear Equations*, Elsevier, Amsterdam, 2013.

[19] J. R. Sharma and H. Arora, *Some novel optimal eighth order derivative-free root solvers and their basins of attraction*, Applied Mathematics and Computation, *284* (2016) 149–161.

[20] J. R. Sharma and H. Arora, *A new family of optimal eighth order methods with dynamics for nonlinear equations*, Applied Mathematics and Computation, 273 (2016) 924–933.

[21] A. R. Soheili and F. Soleymani, *Iterative methods for nonlinear systems associated with finite difference approach in stochastic differential equations*, Numerical Algorithms, *71* (1) (2016) 89–102.

[22] A. R. Soheili, M. Amini, and F. Soleymani, *A family of Chaplygin-type solvers for Itô stochastic differential equations*, Applied Mathematics and Computation, *340* (2019) 296–304.

[23] F. Soleymani, *On a bi-parametric class of optimal eighth-order derivative-free methods*, International Journal of Pure and Applied Mathematics, *72* (1) (2011) 27–37.

[24] F. Soleymani, *Several iterative methods with memory using self-accelerators*, Journal of the Egyptian Mathematical Society, *21* (2) (2013) 133–141.

[25] J. Steffensen, *Remarks on iteration*, Scandinavian Actuarial Journal, *1933*(1), 64–72.

[26] J. F. Traub, *Iterative Methods for the Solution of Equations*, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.

[27] V. Torkashvand, T. Lotfi, and M. A. Fariborzi Araghi, *A new family of adaptive methods with memory for solving nonlinear equations*, Mathatical Sciences, *13* (1) (2019) 1–20.

[28] V. Torkashvand and M. Kazemi, *On an Efficient Family with Memory with High Order of Convergence for Solving Nonlinear Equations*, International Journal of Industrial Mathematics, *12* (2) (2020), 209–224.

[29] V. Torkashvand, *A two-step method adaptive with memory with eighth-order for solving nonlinear equations and its dynamic*, Computational Methods for Differential Equations, *50* (1) (2022) 1007–1026.

[30] V. Torkashvand and M. A. Fariborzi Araghi, *Construction of Iterative Adaptive Methods with Memory with* 100% *Improvement of Convergence Order*, Journal of Mathematical Extension, *15* (3) (16) (2021) 1–32.

[31] Vali Torkashvand, *Transforming Ostrowski's method into a derivative-free method and its dynamics*, Computational Mathematics and Computer Modeling with Applications, *2* (1) (2023) 1–10.

[32] M. Z. Ullah, V, Torkashvand, S. Shateyi, and M. Asma, *Using matrix eigenvalues to construct an iterative method with the highest possible efficiency index two*, Mathematics, *10* (1370) 2022 1–15.

[33] H. Veiseh, T. Lotfi, and T. Allahviranloo, *A study on the local convergence and dynamics of the two-step and derivative-free Kung-Traub's method*, Computational and Applied Mathematics, *37* (3) (2018) 2428–2444.

[34] X. Wang, *A new accelerating technique applied to a variant of Cordero-Torregrosa method*, Journal of Computational and Applied Mathematics, *330* (2018) 695–709.