Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 12, No. 3, 2024, pp. 502-510
DOI:10.22034/cmde.2023.56037.2339

Some delta $q-$fractional linear dynamic equations and a generalized delta $q-$Mittag-Leffler function

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Abstract

In this paper, we introduce a generalized delta $q-$Mittag-Leffler function. Also, we solve some Caputo delta $q-$fractional dynamic equations and these solutions are expressed by means of the newly introduced delta $q-$Mittag-Leffler function.

Keywords. Time scale calculus, $q-$fractional calculus, $q-$Mittag-Leffler function.

2010 Mathematics Subject Classification. 34-XX, 35R07, 34A08.

1. Introduction

Fractional calculus covers the subject of non-integer order derivatives and integrals [26, 43]. Due to fractional calculus becoming a potent tool with more accurate and successful results in modeling a number of complex phenomena in numerous seemingly diverse and widespread fields of science and engineering, fractional calculus and its potential applications have gained much more importance in recent years. Meanwhile, fractional differential equation models have been used in a variety of fields [6, 8, 11, 12, 14–16, 19–23, 29, 30, 34–36, 49, 50]. In these applications, fractional differentiation is often used to model phenomena exhibiting nonstandard dynamical behaviors with a long memory or hereditary effects. At the beginning of the 20th century, Jackson [18] created $q-$calculus, which is the study of calculus without limits. Al-Salam [3–5] and Agarwal [2] were the first to investigate the $q-$fractional integrals and derivatives. As $q-$fractional calculus serves as a bridge between $q-$calculus and fractional calculus, it has garnered more attention [7, 13, 31–33, 45, 46].

With the development of time scale calculus [9, 10, 17], a number of authors have recently started to concentrate on and combine the methods of time scale and $q-$fractional calculus [2, 3, 5, 13, 44]. These results relate fractional calculus on the time scale $\mathbb{T}_q := \{q^\varsigma : \varsigma \in \mathbb{Z}\} \cup \{0\}$, where $0 < q < 1$.

The Mittag-Leffler function [24, 25] plays a vital role in the solution of fractional order differential and integral equations. It has recently become a subject of rich interest in the field of fractional calculus and its applications [28, 41, 42]. Nowadays some mathematicians consider the classical Mittag-Leffler function as the queen function in fractional calculus. An enormous amount of research in the theory of Mittag-Leffler functions has been published in the literature. For a detailed account of the various generalizations, properties, and applications of the Mittag-Leffler function, readers may refer to the literature [1, 37–40, 48]. For $q-$fractional calculus, [33] proposed a new form of $q-$analogue of the Mittag-Leffler function. Recently, in [47] introduced the $q-$analogue of the generalized Mittag-Leffler function. However, there is no paper that has dealt with a generalized delta $q-$fractional Mittag-Leffler function on the time scale.

As a contribution in this direction and being motivated by what is mentioned before, in this paper, we present a generalized delta $q-$Mittag-Leffler function. Such function is obtained by solving the following linear Caputo delta

Received: 04 April 2023 ; Accepted: 18 October 2023.
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$q-$fractional dynamical equation:

$$\begin{aligned} {}^{C}D^{\alpha}_{\Delta_q,\varsigma_0}\psi(\varsigma) &= \lambda(\varsigma-\varsigma_0)_q^{\beta}\,\psi(q^{-\beta}\varsigma), \\ \psi(t_0) &= \xi, \end{aligned} \tag{1.1}$$

where $0 < \alpha < 1$, $\beta > -\alpha$, and $\lambda, \xi \in \mathbb{R}$. After that, we generalize to the higher-order case for any $\alpha > 0$, where higher-order $q-$Mittag-Leffler functions are obtained.

2. Preliminaries

This section provides the fundamental concepts and properties of $q-$calculus and $q-$fractional calculus on time scale, which are covered in [31].

Let $\mathbb{T}_q$ be the time scale for $0 < q < 1$

$$\mathbb{T}_q = \{q^{\varsigma} : \varsigma \in \mathbb{Z}\} \cup \{0\},$$

such that $\mathbb{Z}$ refers to the set of integers.

The definition of delta $q-$derivative for the function $g : \mathbb{T}_q \to \mathbb{R}$ is

$$\Delta_q g(\varsigma) = \frac{g(q\varsigma) - g(\varsigma)}{(q-1)\varsigma}, \ \ \varsigma \in \mathbb{T}_q\backslash\{0\}. \tag{2.1}$$

Now, the higher order delta $q-$derivatives are defined by:

$$\Delta_q^0 g(\varsigma) = g(\varsigma), \ \ \Delta_q^{\eta} g(\varsigma) = \Delta_q(\Delta_q^{\eta-1} g(\varsigma))\,(\eta = 1, 2, 3, \ldots). \tag{2.2}$$

The following definition is the delta $q-$integral for the function $g(\varsigma)$:

$$(I_{q,0}g)(\varsigma) = \int_0^{\varsigma} g(z)\Delta_q z = (1-q)\sum_{\ell=0}^{\infty} \varsigma q^{\ell} g(\varsigma q^{\ell}), \ \forall\, \varsigma \in \mathbb{T}_q, \tag{2.3}$$

and,

$$(I_{q,a}g)(\varsigma) = \int_a^{\varsigma} g(z)\Delta_q z = \int_0^{\varsigma} g(z)\Delta_q z - \int_0^{a} g(z)\Delta_q z, \ \forall\, a, \varsigma \in \mathbb{T}_q. \tag{2.4}$$

The basic $q-$calculus theorem provides

$$\Delta_q \int_0^{\varsigma} g(z)\Delta_q z = g(\varsigma). \tag{2.5}$$

Furthermore, if the function $g(\varsigma)$ is continuous at zero,

$$\int_0^{\varsigma} \Delta_q g(z)\Delta_q z = g(\varsigma) - g(0). \tag{2.6}$$

The following identities will also be useful.

$$\Delta_q \int_a^{\varsigma} g(\varsigma, z)\Delta_q z = \int_a^{\varsigma} \Delta_q g(\varsigma, z)\Delta_q z + g(q\varsigma, \varsigma), \tag{2.7}$$

and,

$$\Delta_q \int_{\varsigma}^{b} g(\varsigma, z)\Delta_q z = \int_{q\varsigma}^{b} \Delta_q g(\varsigma, z)\Delta_q z - g(\varsigma, \varsigma). \tag{2.8}$$

Definition 2.1. For $\vartheta \in \mathbb{N}$, the definition of delta $q-$factorial function is

$$(\varsigma - z)_q^{(0)} = 1,\ (\varsigma - z)_q^{(\vartheta)} = \prod_{r=1}^{\vartheta-1} (\varsigma - q^r z). \tag{2.9}$$

Also,

$$(r - z)_q^{(\alpha)} = \varsigma^\alpha \prod_{r=0}^{\infty} \frac{1 - \frac{z}{\varsigma} q^r}{1 - \frac{z}{\varsigma} q^{r+\alpha}}. \tag{2.10}$$

where α is positive real number but integer.

Lemma 2.2. *Suppose $\gamma, \delta \in \mathbb{R}$, we have*

(1) $(\varsigma - z)_q^{(\gamma+\delta)} = (\varsigma - z)_q^{(\gamma)} (\varsigma - q^\gamma z)_q^{(\delta)}$.
(2) $(b\varsigma - bz)_q^{(\gamma)} = b^\gamma (\varsigma - z)_q^{(\gamma)}$.
(3) $\Delta_q (\varsigma - z)_q^{(\beta)} = \frac{1-q^\gamma}{1-q} (\varsigma - z)_q^{(\gamma-1)}$.
(4) $\Delta_q (\varsigma - z)_q^{(\gamma)} = -\frac{1-q^\beta}{1-q} (\varsigma - qz)_q^{(\gamma-1)}$.

Definition 2.3. For $\gamma \in \mathbb{C} \backslash \{-m, m \in \mathbb{N}_0\}$, and the delta $q-$Gamma function is

$$\Gamma_q(\gamma) = (1-q)_q^{(\gamma-1)} (1-q)^{1-\gamma}, \tag{2.11}$$

which fulfills

$$\Gamma_q(1+\gamma) = \frac{1-q^\gamma}{1-q} \Gamma_q(\gamma),\ \ \Gamma_q(1) = 1. \tag{2.12}$$

Definition 2.4. For all continuous functions with continuous delta $q-$derivatives up to order $\eta - 1$, the space $[c, d]$ is defined as

$$\mathcal{C}_q^{(\eta)}[0, d] = \left\{ g(\varsigma) : \Delta_q^\vartheta g(\varsigma) \in \mathcal{C}[0, d],\ \forall\, \vartheta = 0, 1, \ldots, \eta \right\}. \tag{2.13}$$

Definition 2.5. Let $\varsigma, \varsigma_0 \in \mathbb{T}_q$. The fractional delta $q-$integral of order $\alpha > 0$ for the function $g : \mathbb{T}_q \to \mathbb{R}$ is

$$I^0_{\Delta_q,\varsigma_0} g(\varsigma) = g(\varsigma),$$
$$I^\alpha_{\Delta_q,\varsigma_0} g(\varsigma) = \frac{1}{\Gamma_q(\alpha)} \int_{\varsigma_0}^{\varsigma} (\varsigma - qz)_q^{(\alpha-1)} g(z) \Delta_q z. \tag{2.14}$$

For $\alpha_1, \alpha_2 > 0$, then

$$(I^{\alpha_2}_{\Delta_q,\varsigma_0} I^{\alpha_1}_{\Delta_q,\varsigma_0} g)(\varsigma) = (I^{\alpha_1+\alpha_2}_{\Delta_q,\varsigma_0} g)(\varsigma),\ \ \varsigma_0 < \varsigma. \tag{2.15}$$

Definition 2.6. Let $\varsigma, \varsigma_0 \in \mathbb{T}_q$. The Riemann-Lioville fractional delta $q-$derivative of the function $g : \mathbb{T}_q \to \mathbb{R}$ is

$$D^\alpha_{\Delta_q,\varsigma_0} g(\varsigma) = \Delta_q^\eta I^{\eta-\alpha}_{\Delta_q,\varsigma_0} g(\varsigma), \tag{2.16}$$

for α real value such that $\alpha \geq 0$ and $\eta = [\alpha] + 1$.

For $0 < \varsigma_0 < \varsigma$, $\alpha \in \mathbb{R}^+$. Then

$$D^\alpha_{\Delta_q,\varsigma_0} I^\alpha_{\Delta_q,\varsigma_0} g(\varsigma) = g(\varsigma). \tag{2.17}$$

Definition 2.7. Let $\varsigma, \varsigma_0 \in \mathbb{T}_q$. The Caputo delta $q-$fractional derivatives for the function $g : \mathbb{T}_q \to \mathbb{R}$ is

$$\begin{aligned} {}^{C}D^{\alpha}_{\Delta_q,\varsigma_0} g(\varsigma) &= I^{\eta-\alpha}_{\Delta_q,\varsigma_0} \Delta^{\eta}_q g(\varsigma) \\ &= \frac{1}{\Gamma_q(\eta-\alpha)} \int_{\varsigma_0}^{\varsigma} (\varsigma - qz)_q^{(\eta-\alpha-1)} \Delta^{\eta}_q g(z) \Delta_q z, \end{aligned} \tag{2.18}$$

for α real value such that $\alpha \geq 0$ and $\eta = [\alpha] + 1$.

Theorem 2.8. *For any $0 < \alpha < 1$, we have*

$$ {}^{C}D^{\alpha}_{\Delta_q,\varsigma_0} g(\varsigma) = D^{\alpha}_{\Delta_q,\varsigma_0} g(\varsigma) - \frac{(\varsigma-\varsigma_0)_q^{-\alpha}}{\Gamma_q(1-\alpha)} g(\varsigma_0).$$

Lemma 2.9. *Let $g : \mathbb{T}_q \to \mathbb{R}$ is defined inappropriate domains and $\alpha > 0$. Then,*

$$ I^{\alpha}_{\Delta_q,\varsigma_0} {}^{C}D^{\alpha}_{\Delta_q,\varsigma_0} g(\varsigma) = g(\varsigma) - \sum_{\ell=0}^{\eta-1} \frac{(\varsigma-\varsigma_0)_q^{(\ell)}}{\Gamma_q(\ell+1)} \Delta^{\ell}_q g(\varsigma_0), \tag{2.19}$$

and if $0 < \alpha \leq 1$, then

$$ I^{\alpha}_{\Delta_q,\varsigma_0} {}^{C}D^{\alpha}_{\Delta_q,\varsigma_0} g(\varsigma) = g(\varsigma) - g(\varsigma_0). \tag{2.20}$$

The following identity will also be utilized:

$$ I^{\alpha}_{\Delta_q,\varsigma_0} (x-\varsigma_0)_q^{\nu} = \frac{\Gamma_q(\nu+1)}{\Gamma_q(\alpha+\nu+1)} (x-\varsigma_0)_q^{\nu+\alpha}, \quad 0 < \varsigma_0 < x < \varsigma, \tag{2.21}$$

where $\nu \in (-1, \infty)$ and $\alpha \in \mathbb{R}^+$.

Definition 2.10. For $s, s_0 \in \mathbb{C}$, the delta $q-$Mittag-Leffler function is

$$ {}_{\Delta_q}E_{\alpha,\beta}(\lambda, s-s_0) = \sum_{\eta=0}^{\infty} \lambda^{\eta} \frac{(s-s_0)^{\alpha\eta}}{\Gamma_q(\alpha\eta+\beta)}, \tag{2.22}$$

when $\beta = 1$, we employ ${}_{\Delta_q}E_{\alpha}(\lambda, s-s_0) = {}_{\Delta_q}E_{\alpha,1}(\lambda, s-s_0)$.

3. Main results

In this section, we define the generalized delta $q-$Mittag-Leffler function which is analogue to the one used before in [27].

Definition 3.1. The definition of the generalized delta $q-$Mittag-Leffler function is

$$ {}_{\Delta_q}E_{\alpha,\kappa,\ell}(\lambda, \varsigma-\varsigma_0) = 1 + \sum_{\eta=1}^{\infty} \lambda^{\eta} q^{-(\eta(\eta-1)/2)\alpha(\kappa-1)(\alpha\ell+\alpha)} \Upsilon_{\eta} (\varsigma-\varsigma_0)_q^{\alpha\eta\kappa}, \tag{3.1}$$

where

$$ \Upsilon_{\eta} = \prod_{i=0}^{\eta-1} \frac{\Gamma_q\left[\alpha(i\kappa+\ell)+1\right]}{\Gamma_q\left[\alpha(i\kappa+\ell+1)+1\right]}, \forall\, \eta = 1, 2, 3, \ldots, \tag{3.2}$$

for $\ell, \lambda \in \mathbb{C}$ are complex numbers and $\kappa \in \mathbb{R}$ such that $\kappa > 0$, $\varsigma_0 \geq 0$, and $\alpha(i\kappa+\ell) \neq -1, -2, -3, \ldots$, while the generalized delta $q-$Mittag-Leffler function of order m, for all $m = 0, 1, 2, \ldots$, is

$$ {}_{\Delta_q}E^{m}_{\alpha,\kappa,\ell}(\lambda, \varsigma-\varsigma_0) = 1 + \sum_{\eta=1}^{\infty} \lambda^{\eta} q^{-\eta\alpha(\kappa-1)m} q^{-(\eta(\eta-1)/2)\alpha(\kappa-1)(\alpha\ell+\alpha)} \Upsilon_{\eta} (\varsigma-q^{m}\varsigma_0)_q^{\alpha\eta\kappa}. \tag{3.3}$$

Note that ${}_{\Delta_q}E^0_{\alpha,\kappa,\ell}(\lambda,\varsigma-\varsigma_0)={}_{\Delta_q}E_{\alpha,\kappa,\ell}(\lambda,\varsigma-\varsigma_0)$.

Remark 3.2. If $\kappa=1$, then the generalized delta $q-$Mittag-Leffler function is specifically reduced to the delta $q-$Mittag-Leffler function, with the exception of a constant factor $\Gamma_q(\alpha\ell+1)$. Namely,

$$ {}_{\Delta_q}E_{\alpha,1,\ell}(\lambda,\varsigma-\varsigma_0)=\Gamma_q(\alpha\ell+1)\,{}_{\Delta_q}E_{\alpha,\alpha\ell+1}(\lambda,\varsigma-\varsigma_0). \tag{3.4}$$

This turns up being the $q-$analogue of the identity $E_{\alpha,1,\ell}(s)=\Gamma(\alpha\ell+1)E_{\alpha,\alpha\ell+1}(s)$ [26].

Example 3.1. Consider the Caputo delta $q-$fractional dynamic equation (1.1).

Using Lemma 2.9, we get

$$\psi(\varsigma)=\psi(\varsigma_0)+\lambda\, I^{\alpha}_{\Delta_q,\varsigma_0}\left[(\varsigma-\varsigma_0)^{\beta}_q\ \psi(q^{-\beta}\varsigma)\right]. \tag{3.5}$$

According to the successive applications method,

$$\psi_\kappa(\varsigma)=\psi(\varsigma_0)+\lambda I^{\alpha}_{\Delta_q,\varsigma_0}\left[(\varsigma-\varsigma_0)^{\beta}_q\psi_{\kappa-1}(q^{-\beta}\varsigma)\right],\forall\,\kappa=1,2,3,\dots, \tag{3.6}$$

where $\psi_0(\varsigma)=\xi$.

By using Eq. (2.21), we obtain

$$\psi_1(\varsigma)=\xi+\xi\lambda\frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)}(\varsigma-\varsigma_0)^{\beta+\alpha}_q, \tag{3.7}$$

and

$$\psi_2(\varsigma)=\xi+\xi\lambda I^{\alpha}_{\Delta_q,\varsigma_0}\left[(\varsigma-\varsigma_0)^{\beta}_q\left\{1+\lambda\frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)}(q^{-\beta}\varsigma-\varsigma_0)^{\beta+\alpha}_q\right\}\right]. \tag{3.8}$$

Then by using Lemma 2.2

$$\psi_2(\varsigma)=\xi+\xi\lambda I^{\alpha}_{\Delta_q,\varsigma_0}\left[(\varsigma-\varsigma_0)^{\beta}_q+\lambda\frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)}q^{-\beta(\alpha+\beta)}(\varsigma-\varsigma_0)^{2\beta+\alpha}_q\right]. \tag{3.9}$$

Once more, based on Eq. (2.21), we obtain

$$\psi_2(\varsigma)=\xi\left[1+\lambda\frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)}(\varsigma-\varsigma_0)^{2\beta+\alpha}_q+\lambda^2\frac{\Gamma_q(2\beta+\alpha+1)}{\Gamma_q(2\beta+2\alpha+1)}q^{-\beta(\alpha+\beta)}(\varsigma-\varsigma_0)^{2\beta+2\alpha}_q\right]. \tag{3.10}$$

Using an inductive approach, for each $\kappa=1,2,\dots$, we get

$$\psi_\kappa(\varsigma)=\xi\left[1+\sum_{\eta=1}^{\kappa}\lambda^\eta q^{-\beta(\eta(\eta-1)/2)(\alpha+\beta)}\Upsilon_\eta(\varsigma-\varsigma_0)^{\eta(\alpha+\beta)}_q\right], \tag{3.11}$$

where

$$\Upsilon_\eta=\prod_{i=0}^{\eta-1}\frac{\Gamma_q\left[\alpha(i\kappa+\ell)+1\right]}{\Gamma_q\left[\alpha(i\kappa+\ell+1)+1\right]},\kappa=1+\frac{\beta}{\alpha},\ell=\frac{\beta}{\alpha},\eta=1,2,3,\dots \tag{3.12}$$

Let $\kappa\to\infty$, we arrive at the solution

$$\psi(\varsigma)=\xi\left[1+\sum_{\eta=1}^{\infty}\lambda^\eta q^{-\beta(\eta(\eta-1)/2)(\alpha+\beta)}\Upsilon_\eta(\varsigma-\varsigma_0)^{\eta(\alpha+\beta)}_q\right]. \tag{3.13}$$

Theorem 3.3. *The solution of Eq.* (1.1) *is*

$$\psi(\varsigma)=\xi\ {}_{\Delta_q}E_{\alpha,(1+(\beta/\alpha)),\beta/\alpha}(\lambda,\varsigma-\varsigma_0). \tag{3.14}$$

Remark 3.4.

(1)**:** If in Eq. (1.1) $\beta=0$, then in accordance with Eq. (3.4) and [31] we have

$${}_{\Delta_q}E_{\alpha,1,0}(\lambda,\varsigma-\varsigma_0)={}_{\Delta_q}E_{\alpha,1}(\lambda,\varsigma-\varsigma_0)={}_{\Delta_q}E_{\alpha}(\lambda,\varsigma-\varsigma_0).$$

(2): The solution of the delta $q-$Cauchy problem

$$
\begin{aligned}
&{}^{C}D^{\frac{1}{2}}_{\Delta_q,\varsigma_0}\psi(\varsigma) = \lambda(\varsigma-\varsigma_0)^{\beta}_{q}\ \psi(q^{-\beta}\varsigma), \\
&\qquad y(\varsigma_0) = \xi,
\end{aligned} \tag{3.15}
$$

where $0 < \alpha < 1$, $\beta > -\frac{1}{2}$, and $\lambda, \xi \in \mathbb{R}$, is given by

$$
\psi(\varsigma) = \xi\ {}_{\Delta_q}E_{\frac{1}{2},1+2\beta,2\beta}(\lambda, \varsigma-\varsigma_0). \tag{3.16}
$$

In order to generalize to the higher-order case, we take into account the fractional delta $q-$initial value problem:

$$
\begin{aligned}
&{}^{C}D^{\alpha}_{\Delta_q,\varsigma_0}\psi(\varsigma) = \lambda(\varsigma-\varsigma_0)^{\beta}_{q}\ \psi(q^{-\beta}\varsigma), \\
&\quad \psi^{(\eta)}(\varsigma_0) = \xi_\eta, \quad \eta = 0,1,\dots,\upsilon-1,
\end{aligned} \tag{3.17}
$$

where $\upsilon-1 < \alpha < \upsilon$, $\beta > -\alpha$, and $\lambda, \xi \in \mathbb{R}$.

Theorem 3.5. *The solution of Eq.* (3.17) *is*

$$
\psi(\varsigma) = \sum_{m=0}^{\upsilon-1} \frac{\xi_m}{\Gamma_q(m+1)}(\varsigma-\varsigma_0)^m_q\, {}_{\Delta_q}E^m_{\alpha,((1+\beta)/\alpha),((\beta+m)/\alpha)}(\lambda, \varsigma-\varsigma_0). \tag{3.18}
$$

Proof. Using Lemma 2.2, Eq. (2.19), and the successive approximation with

$$
\psi_0(\varsigma) = \sum_{m=0}^{\upsilon-1} \frac{(\varsigma-\varsigma_0)^m_q}{\Gamma_q(m+1)}\Delta_q g(\varsigma_0), \tag{3.19}
$$

Observe that Corollary 3.1 is retrieved when $0 < \alpha < 1$, that is, $\upsilon = 1$.

After that, we solve Eq. (1.1) in a nonhomogeneous form.

Lemma 3.6. *Let $g : \mathbb{T}_q \to \mathbb{R}$, and $m \in \mathbb{N}$, $\alpha > 0$. Then*

$$
I^{\alpha}_{\Delta_q,\varsigma_0} g(q^{-m}\varsigma) = q^{m\alpha}(I^{\alpha}_{\Delta_q,q^{-m}\varsigma_0} g)(q^{-m}\varsigma_0), \forall \varsigma \in \mathbb{T}_q. \tag{3.20}
$$

Proof. Let

$$
\begin{aligned}
I^{\alpha}_{\Delta_q,\varsigma_0} g(q^{-m}\varsigma) &= \frac{1}{\Gamma_q(\alpha)}\int_{\varsigma_0}^{\varsigma}(\varsigma-q\vartheta)^{\alpha-1}_q g(q^{-m}\vartheta)\Delta_q\vartheta \\
&= \frac{q^m}{\Gamma_q(\alpha)}\int_{q^{-m}\varsigma_0}^{q^{-m}\varsigma}(\varsigma-qq^m\vartheta)^{\alpha-1}_q g(\vartheta)\Delta_q\vartheta \\
&= \frac{q^{m\alpha}}{\Gamma_q(\alpha)}\int_{q^{-m}\varsigma_0}^{q^{-m}\varsigma}(q^{-m}\varsigma-q\vartheta)^{\alpha-1}_q g(\vartheta)\Delta_q\vartheta \\
&= q^{m\alpha}(I^{\alpha}_{\Delta_q,q^{-m}\varsigma} g)(q^{-m}\varsigma).
\end{aligned}
$$

Now, Consider the following Caputo delta $q-$fractional dynamic equation

$$
\begin{aligned}
&{}^{C}D^{\alpha}_{\Delta_q,0}\psi(\varsigma) = \lambda\,\varsigma^{\beta}\,\psi(q^{-\beta}\varsigma) + g(\varsigma), \\
&\qquad \psi(0) = \xi,
\end{aligned} \tag{3.21}
$$

where $0 < \alpha < 1$, $\beta > -\alpha$, $\beta \in \mathbb{N}_0$, and $\lambda, \xi \in \mathbb{R}$.

By applying Lemma 3.6 and the successive approximation as in Example 3.1, we can state the following

Theorem 3.7. *The solution of Eq.* (3.21) *is*

$$
\begin{aligned}
\psi(\varsigma) &= \xi_{\Delta_q}E_{\alpha,(1+(\beta/\alpha)),\beta/\alpha}(\lambda,\varsigma) \\
&\quad + \sum_{\eta=0}^{\infty}\frac{\lambda^\eta}{\Gamma_q(\alpha\eta+\alpha)}q^{-\alpha\beta(\eta(\eta+1)/2)}\int_0^{\varsigma}(\varsigma-q\vartheta)^{\alpha\eta+\alpha}_q g(q^{-\eta\beta}\vartheta)\Delta_q\vartheta.
\end{aligned} \tag{3.22}
$$

Observe that, if in Eq. (3.21) we set $\beta = 0$, then Example 3.1 in [31] is recovered for $\varsigma_0 = 0$.

Conclusion

Mittag-Leffler functions are important in studying solutions of fractional differential equations, and they are associated with a wide range of problems in many areas of mathematics and physics. The importance and great considerations of Mittag-Leffler functions have led many researchers in the theory of special functions to explore possible generalizations and applications. However, there is no paper that has dealt with a generalized delta $q-$fractional Mittag-Leffler function. This paper extends our previous work in the paper [31] and establishes a generalized version of delta $q-$fractional Mittag-Leffler function.

Conflicts of interests

The authors certify that they have no financial or non-financial interests (such as honoraria, educational grants, participation in speakers bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements) in this manuscript.

Data availability

No data was used for the research described in the article.

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