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Approximate solutions of inverse Nodal problem with conformable derivative

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Abstract

Our research is about the Sturm-Liouville equation which contains conformable fractional derivatives of order $\alpha \in (0,1]$ in lieu of the ordinary derivatives. First, we present the eigenvalues, eigenfunctions, and nodal points, and the properties of nodal points are used for the reconstruction of an integral equation. Then, the Bernstein technique was utilized to solve the inverse problem, and the approximation of solving this problem was calculated. Finally, the numerical examples were introduced to explain the results. Moreover, the analogy of this technique is shown in a numerical example with the Chebyshev interpolation technique .

Keywords. Inverse problem, Conformable fractional, Nodal points, Bernstein technique, Numerical solution.

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1. Introduction

Inverse nodal problem (INP) was researched by J. R. McLaughlin [20]. In this regard, several uniqueness theorems were introduced, and it became evident that knowing about the nodal points is enough to specify the potential function related to the problem of Sturm-Liouville towards a constant on an infinite interval. The connection between the function of density and the nodal points related to the string equation was researched by Shen [27]. Hald and McLaughlin presented numerical results [11] to restructure the density function related to string vibration. Numerous researchers objectively studied inverse nodal problems [28, 30, 31].

Applied mathematics and mathematical analysis widely utilized the fractional derivative of 1695. Various kinds of fractional derivatives have been made by a lot of authors [16, 21, 23, 24]. The conformable fractional derivative was defined by Khalil et al. in 2014 [14]. In [1, 4, 12], this derivative's rudimentary features and preliminary results are introduced. A diverse range of fields, namely quantum mechanics, dynamical systems, time scale problems, etc., utilized this derivative [7, 33]. Some studies have been developed for fractional differential equations (for example see [6, 9, 29]).

In recent years, some authors solved inverse problems which include the fractional derivatives [2, 5, 13, 14, 22, 25, 26]. In the present study, the following conformable fractional Sturm-Liouville problem of order α $(0 < \alpha \leq 1)$ is regarded:

$$l_\alpha y := -T_\alpha T_\alpha y + q(x)y = \lambda^2 y, \quad 0 < x < 1, \tag{1.1}$$

$$y(0,\lambda) = y(1,\lambda) = 0. \tag{1.2}$$

Here λ is the spectral parameter, $q(x) \in W^1_{2,\alpha}[0,1]$ is a real-valued function.

The Bernstein technique has been used to compute the approximate potential function by solving inverse nodal problem in this equation; therefore, this study is considered to be new.

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2. Main results

Some notations of conformable fractional calculus are evoked, and we can see more detailed knowledge [1, 4, 14].

Definition 2.1. ([14]) Suppose that $\mathcal{G} : [0, \infty) \to \Re$ is considered. The conformable fractional derivative of $\mathcal{G}$ of order α regarding x :

$$T_\alpha \mathcal{G}(x) = \lim_{h\to 0} \frac{\mathcal{G}(x + hx^{1-\alpha}) - \mathcal{G}(x)}{h}, \quad T_\alpha \mathcal{G}(0) = \lim_{x\to 0^+} T_\alpha \mathcal{G}(x), \quad x > 0.$$

If $\mathcal{G}$ is differentiable so that

$$\mathcal{G}'(x) = \lim_{h\to 0} \frac{\mathcal{G}(x + h) - \mathcal{G}(x)}{h},$$

then

$$T_\alpha \mathcal{G}(x) = x^{1-\alpha}\mathcal{G}'(x).$$

Definition 2.2. ([1]) Conformable fractional integral with order α is determined as:

$$I_\alpha \mathcal{G}(x) = \int_0^x \mathcal{G}(t) d_\alpha t = \int_0^x t^{\alpha-1}\mathcal{G}(t)dt, \quad x > 0.$$

Theorem 2.3. *([14]) Assume that $\mathcal{F}$,$\mathcal{G}$ is α- differentiable at x, $x > 0$,*
(i) $T_\alpha(c_1\mathcal{F} + c_2\mathcal{G}) = c_1T_\alpha\mathcal{F} + c_2T_\alpha\mathcal{G}, \quad \forall c_1, c_2 \in \Re$,
(ii) $T_\alpha(x^r) = rx^{r-\alpha}, \quad \forall r \in \Re$,
(iii) $T_\alpha(c) = 0, \quad$ (c is a constant),
(iv) $T_\alpha(\mathcal{F}\mathcal{G}) = T_\alpha(\mathcal{F})\mathcal{G} + \mathcal{F}T_\alpha(\mathcal{G})$,
(v) $T_\alpha(\frac{\mathcal{F}}{\mathcal{G}}) = \frac{T_\alpha(\mathcal{F})\mathcal{G}-\mathcal{F}T_\alpha(\mathcal{G})}{\mathcal{G}^2}, \quad (\mathcal{G} \neq 0)$.

Denote by $y(x, \lambda)$ the solution of (1.1) under the conditions $y(0, \lambda) = 0$ and $T_\alpha y(0, \lambda) = 1$. Hence, one can obtain the conformable fractional Volterra integral equation as below

$$y(x, \lambda) = \frac{1}{\lambda} \sin(\frac{\lambda}{\alpha}x^\alpha) + \frac{1}{\lambda}\int_0^x \sin[\frac{\lambda}{\alpha}(x^\alpha - t^\alpha)]q(t)y(t, \lambda)d_\alpha t. \tag{2.1}$$

Using the technique of successive approximations, we deduce

$$y(x, \lambda) = \frac{1}{\lambda} \sin(\frac{\lambda}{\alpha}x^\alpha) - \frac{1}{2\lambda^2} \cos(\frac{\lambda}{\alpha}x^\alpha)\int_0^x q(t)d_\alpha t + O\left(\frac{1}{\lambda^3}\right). \tag{2.2}$$

The eigenvalues $\{\lambda_n\}_{n\geq 1}$ coincide with zeros of the characteristic function

$$\Delta(\lambda) := y(1, \lambda) = \frac{1}{\lambda} \sin(\frac{\lambda}{\alpha}) - \frac{1}{2\lambda^2} \cos(\frac{\lambda}{\alpha})\int_0^1 q(t)d_\alpha t + O\left(\frac{1}{\lambda^3}\right), \tag{2.3}$$

and we have the asymptotic formula:

$$\lambda_n = n\pi\alpha + \frac{\omega}{n\pi} + O\left(\frac{1}{n^2}\right), \qquad n \to \infty, \tag{2.4}$$

where $\omega = \frac{1}{2}\int_0^1 q(t)d_\alpha t$ (see [10, 32]).

Theorem 2.4. *According to Eq. (1.1) under the conditions*

$$y(0, \lambda) = 0, \quad T_\alpha y(0, \lambda) = 1, \tag{2.5}$$

we formulate the nodes and nodal length of the problem (1.1) *and* (2.5) *in the form of*

$$x_n^j = \left[\frac{j}{n} - \frac{\omega j}{n^3\pi^2\alpha} + \frac{1}{2n^2\pi^2\alpha}\int_0^{(\frac{j}{n})^{\frac{1}{\alpha}}} q(t)d_\alpha t + O\left(\frac{1}{n^3}\right)\right]^{\frac{1}{\alpha}}, \tag{2.6}$$

$$l_n^j = (c^j)^{1-\alpha}\left[\frac{1}{n\alpha} - \frac{\omega}{n^3\pi^2\alpha^2} + \frac{1}{2n^2\pi^2\alpha^2}\int_{(\frac{j}{n})^{\frac{1}{\alpha}}}^{(\frac{j+1}{n})^{\frac{1}{\alpha}}} q(t)d_\alpha t\right] + O\left(\frac{1}{n^3}\right),$$

for some $c^j \in (x_n^j, x_n^{j+1})$.

Proof. Using (2.2),

$$y(x,\lambda) = \frac{1}{\lambda}\sin(\frac{\lambda}{\alpha}x^\alpha) - \frac{1}{2\lambda^2}\cos(\frac{\lambda}{\alpha}x^\alpha)\int_0^x q(t)d_\alpha t + O\left(\frac{1}{\lambda^3}\right).$$

Considering the roots x_n^j, $n>1$, $j=\overline{1,n-1}$ of n-th eigenfunction, we take $\lambda=\lambda_n$ and $x = x_n^j$. Thus,

$$\frac{1}{\lambda_n}\sin(\frac{\lambda_n}{\alpha}(x_n^j)^\alpha) - \frac{1}{2\lambda_n^2}\cos(\frac{\lambda_n}{\alpha}(x_n^j)^\alpha)\int_0^{x_n^j} q(t)d_\alpha t + O\left(\frac{1}{\lambda_n^3}\right) = 0.$$

Using formula (2.4), we have

$$\sin(n\pi(x_n^j)^\alpha) + \frac{\omega}{n\pi\alpha}(x_n^j)^\alpha\cos(n\pi(x_n^j)^\alpha) - \frac{1}{2n\pi\alpha}\left[\cos(n\pi(x_n^j)^\alpha) - \frac{\omega}{n\pi\alpha}(x_n^j)^\alpha\sin(n\pi(x_n^j)^\alpha)\right]$$
$$\times\int_0^{x_n^j} q(t)d_\alpha t + O\left(\frac{1}{n^2}\right) = 0.$$

Thus,

$$\tan(n\pi(x_n^j)^\alpha) = -\frac{\omega}{n\pi\alpha}(x_n^j)^\alpha + \frac{1}{2n\pi\alpha}\int_0^{x_n^j} q(t)d_\alpha t + O\left(\frac{1}{n^2}\right).$$

Therefore,

$$(x_n^j)^\alpha = \frac{j}{n} - \frac{\omega}{n^2\pi^2\alpha}(x_n^j)^\alpha + \frac{1}{2n^2\pi^2\alpha}\int_0^{x_n^j} q(t)d_\alpha t + O\left(\frac{1}{n^3}\right), \tag{2.7}$$

then, we can write

$$(x_n^j)^\alpha = \frac{j}{n} - \frac{\omega j}{n^3\pi^2\alpha} + \frac{1}{2n^2\pi^2\alpha}\int_0^{(\frac{j}{n})^{\frac{1}{\alpha}}} q(t)d_\alpha t + O\left(\frac{1}{n^3}\right).$$

Also, we have

$$(x_n^{j+1})^\alpha - (x_n^j)^\alpha = \frac{1}{n} - \frac{\omega}{n^3\pi^2\alpha} + \frac{1}{2n^2\pi^2\alpha}\int_{(\frac{j}{n})^{\frac{1}{\alpha}}}^{(\frac{j+1}{n})^{\frac{1}{\alpha}}} q(t)d_\alpha t + O\left(\frac{1}{n^3}\right).$$

Using the mean value theorem, it follows that

$$(x_n^{j+1})^\alpha - (x_n^j)^\alpha = \alpha(c^j)^{\alpha-1}(x_n^{j+1} - x_n^j),$$

for some $c^j \in (x_n^j, x_n^{j+1})$. Hence, according to the definition of nodal length, we get

$$\begin{aligned} l_n^j =& x_n^{j+1} - x_n^j = \frac{(x_n^{j+1})^\alpha - (x_n^j)^\alpha}{\alpha}(c^j)^{1-\alpha} \\ =& (c^j)^{1-\alpha}\left(\frac{1}{n\alpha} - \frac{\omega}{n^3\pi^2\alpha^2} + \frac{1}{2n^2\pi^2\alpha^2}\int_{(\frac{j}{n})^{\frac{1}{\alpha}}}^{(\frac{j+1}{n})^{\frac{1}{\alpha}}} q(t)d_\alpha t\right) + O\left(\frac{1}{n^3}\right). \end{aligned}$$

□

We consider X to be a set of all nodes, then for each fixed $x \in (0,1)$ and $\alpha \in (0,1]$, we can choose a sequence $\{j_n\} \subset X$ so that $\lim_{n\to\infty} x_n^{j_n} = x$. Now, we will give a uniqueness theorem.

Theorem 2.5. *The potential function $q - \alpha \int_0^1 q$ is uniquely obtained by a dense subset of nodes in $(0,1)$.*

Proof. Imagine that we have two problems of the kind (1.1)-(1.2) with q, $\widetilde{q}$. Let the nodes $x_n^{j_n}$, $\widetilde{x}_n^{j_n}$ meeting $x_n^{j_n} = \widetilde{x}_n^{j_n}$ make a dense subset in $(0,1)$. Solutions (1.1)-(1.2) can be taken as y for q and $\widetilde{y}$ for $\widetilde{q}$. It follows (1.1) that

$$T_\alpha[(T_\alpha y)\widetilde{y} - y(T_\alpha \widetilde{y})] = [q - \widetilde{q} + \widetilde{\lambda}_n^2 - \lambda_n^2] y_n \widetilde{y}_n.$$

Integrating this formula from 0 to $x_n^{j_n}$, we arrive at

$$(T_\alpha y(x_n^{j_n}))\widetilde{y}(x_n^{j_n}) - y(x_n^{j_n})(T_\alpha \widetilde{y}(x_n^{j_n})) = \int_0^{x_n^{j_n}} [q - \widetilde{q} + \widetilde{\lambda}_n^2 - \lambda_n^2] y_n \widetilde{y}_n d_\alpha t.$$

Since the nodes $x_n^{j_n}$ are the roots of n-th eigenfunction, then

$$(T_\alpha y(x_n^{j_n}))\widetilde{y}(x_n^{j_n}) - y(x_n^{j_n})(T_\alpha \widetilde{y}(x_n^{j_n})) = 0.$$

By choosing a sequence $x_n^{j_n}$ accumulating at an arbitrary $x \in (0,1)$, we have

$$\int_0^x [q - \widetilde{q} - \alpha \int_0^1 (q - \widetilde{q})] y_n \widetilde{y}_n d_\alpha t = 0,$$

and this holds for all x. Since $y_n \widetilde{y}_n$ is bounded, hence we can write

$$q - \widetilde{q} - \alpha \int_0^1 (q - \widetilde{q}) d_\alpha t = 0,$$

and consequently, we come to the conclusion that $q - \alpha \int_0^1 q$ is uniquely obtained by a dense subset of nodal points (also see [17, 28]). $\square$

Furthermore, by formula (2.7), we have

$$n\pi^2\alpha \left(n(x_n^j)^\alpha - j\right) = -\omega (x_n^j)^\alpha + \frac{1}{2}\int_0^{x_n^j} q(t) d_\alpha t + O\left(\frac{1}{n}\right).$$

Similarly, one can derive

$$\lim_{n\to\infty} \left[n\pi^2\alpha \left(n(x_n^{j_n})^\alpha - j_n\right)\right] = -\omega(x)^\alpha + \frac{1}{2}\int_0^x q(t) d_\alpha t.$$

Define

$$Q(x) := \lim_{n\to\infty} \left[n\pi^2\alpha \left(n(x_n^{j_n})^\alpha - j_n\right)\right],$$

thus,

$$Q(x) = -\omega(x)^\alpha + \frac{1}{2}\int_0^x q(t) d_\alpha t = -\omega(x)^\alpha + \frac{1}{2} I_\alpha q(x).$$

Using definition of ω, we arrive at

$$q(x) - \alpha \int_0^1 q(t) d_\alpha t = 2T_\alpha Q(x).$$

Theorem 2.6. *The specification of X and ω uniquely determines the potential q where*

$$\omega = \frac{1}{2}\int_0^1 q(t) d_\alpha t.$$

Proof. Suppose that q, $\tilde{q}$ and x_n^j, $\tilde{x}_n^j$ are potential functions and nodes for two operators l_α and $\tilde{l}_\alpha$, respectively. Take $x_n^j = \tilde{x}_n^j$, $j = \overline{1, n-1}$, $n > 1$, and let $\omega = \tilde{\omega}$. Therefore, in accordance with the definition ω, we obtain $\int_0^1 q d_\alpha t = \int_0^1 \tilde{q} d_\alpha t$ and at last by using Theorem 2.5, we have $q = \tilde{q}$ a.e. on $(0, 1)$. Hence, the proof of this theorem is completed (see [17]). □

3. Numerical solution of INP

INP. Let the points x_n^j, α be given, construct potential function $q(x)$.

Using the specification of x_n^j, $j = \overline{1, n-1}$, $n > 1$,, the formula (2.2) yields

$$\int_0^{x_n^j} q(x) d_\alpha x \cong 2\lambda_n \tan\left(\frac{\lambda_n}{\alpha}(x_n^j)^\alpha\right). \tag{3.1}$$

By using the Bernstein technique, we convert the above integral equation into a linear equation system. Since the solution of (3.1) is also a solution of **INP,** we obtain the approximation of the $q(x)$ with the Bernstein technique. For the convenience of readers, we firstly present Bernstein's technique as follows:

Fractional Bernstein polynomials and convergence analysis:

Definition 3.1. ([8, 19]) The $N-$th degree Bernstein basis polynomials on $[0, 1]$ are defined as

$$\mathcal{B}_{k,N}(x) = \left(\begin{array}{c} N \\ k \end{array}\right) x^k (1-x)^{N-k}, \quad k = \overline{0, N}, \tag{3.2}$$

so that, using the binomial theorem, it is clear that

$$\mathcal{B}_{k,N}(x) = \sum_{i=0}^{N-k} (-1)^i \left(\begin{array}{c} N \\ k \end{array}\right) \left(\begin{array}{c} N-k \\ i \end{array}\right) x^{k+i}, \quad k = \overline{0, N}.$$

The fractional Bernstein polynomials on $[0, 1]$, obtained by substituting $x \to x^\alpha$ in the formula (3.2), are formulated in the form (see [3, 18])

$$\begin{aligned} \mathcal{B}_{k,N}^\alpha(x) &= \left(\begin{array}{c} N \\ k \end{array}\right) x^{k\alpha} (1-x^\alpha)^{N-k} \\ &= \sum_{i=0}^{N-k} (-1)^i \left(\begin{array}{c} N \\ k \end{array}\right) \left(\begin{array}{c} N-k \\ i \end{array}\right) x^{(k+i)\alpha}, \quad k = \overline{0, N}, \end{aligned} \tag{3.3}$$

and the arbitrary function $\mathcal{F}(x)$ defined on $[0, 1]$ can be approximated by the fractional Bernstein polynomials as

$$\mathcal{F}(x) \cong \mathcal{B}_N^{\mathcal{F},\alpha}(t) := \sum_{k=0}^{N} \mathcal{F}(\frac{k}{N}) \mathcal{B}_{k,N}^\alpha(x), \tag{3.4}$$

where $0 < \alpha \leq 1$ and N is any positive integer.
Substituting x^α to x in the process used in proof of theorem 1.1.1 in [19], leads to an intersting result.
Let the function $\mathcal{F}(x)$ be α-integrable on $[0, 1]$ and take $F(x) = \int_0^x \mathcal{F}(s) d_\alpha s$. then, by using the same process with [19], we can write

$$\begin{aligned}
P_N^{\mathcal{F},\alpha}(x) &= T_\alpha \mathcal{B}_{N+1}^{F,\alpha}(x) \\
&= \sum_{k=0}^{N+1} F(\frac{k}{N+1}) \left(\begin{array}{c} N+1 \\ k \end{array} \right) T_\alpha \left(x^{k\alpha}(1-x^\alpha)^{N-k} \right) \\
&= \sum_{k=1}^{N+1} F(\frac{k}{N+1}) \left(\begin{array}{c} N+1 \\ k \end{array} \right) k\alpha x^{\alpha(k-1)}(1-x^\alpha)^{N+1-k} \\
&- \sum_{k=0}^{N+1} F(\frac{k}{N+1}) \left(\begin{array}{c} N+1 \\ k \end{array} \right) \alpha(N+1-k) x^{\alpha k}(1-x^\alpha)^{N-k} \\
&= \sum_{k=0}^{N} F(\frac{k+1}{N+1}) \left(\begin{array}{c} N+1 \\ k+1 \end{array} \right) (k+1)\alpha x^{\alpha k}(1-x^\alpha)^{N-k} \\
&- \sum_{k=0}^{N} F(\frac{k}{N+1}) \left(\begin{array}{c} N+1 \\ k \end{array} \right) \alpha(N+1-k) x^{\alpha k}(1-x^\alpha)^{N-k} \\
&= \sum_{k=0}^{N} \left[F\left(\frac{k+1}{N+1}\right) - F\left(\frac{k}{N+1}\right) \right] \alpha(N+1) \left(\begin{array}{c} N \\ k \end{array} \right) x^{\alpha k}(1-x^\alpha)^{N-k} \\
&= \sum_{k=0}^{N} \alpha(N+1) \left(\begin{array}{c} N \\ k \end{array} \right) x^{\alpha k}(1-x^\alpha)^{N-k} \int_{\frac{k}{N+1}}^{\frac{k+1}{N+1}} \mathcal{F}(s) d_\alpha s \qquad (3.5) \\
&= \sum_{k=0}^{N} C_k \mathcal{B}_{k,N}^{\alpha}(x),
\end{aligned}$$

where $C_k = \alpha(N+1) \int_{\frac{k}{N+1}}^{\frac{k+1}{N+1}} \mathcal{F}(s) d_\alpha s$. Therefore, if function $\mathcal{F}(x)$ is α-integrable on $[0,1]$, then we have

$$P_N^{\mathcal{F},\alpha}(x) = \sum_{k=0}^{N} C_k \mathcal{B}_{k,N}^{\alpha}(x). \qquad (3.6)$$

Theorem 3.2. *At any point x of $[0,1]$ where $\mathcal{F}(x)$ is fractional derivative of its indefinite fractional integral of order α, which is almost everywhere,*

$$\lim_{N\to\infty} P_N^{\mathcal{F},\alpha}(x) = \mathcal{F}(x).$$

Proof. Like the process applied in [19], this theorem can be proved. □

The formula (3.5) can be considered as

$$P_N^{\mathcal{F},\alpha}(x) = \sum_{k=0}^{N} \int_{\frac{k}{N+1}}^{\frac{k+1}{N+1}} K_N(x,s)\mathcal{F}(s) d_\alpha s = \int_0^1 K_N(x,s)\mathcal{F}(s) d_\alpha s, \qquad (3.7)$$

where for $0 \le x \le 1$

$$K_N(x,s) = \alpha(N+1) \left(\begin{array}{c} N \\ k \end{array} \right) x^{\alpha k}(1-x^\alpha)^{N-k}, \quad \frac{k}{N+1} < s \le \frac{k+1}{N+1}, \quad k = \overline{0,N}.$$

Substituting x^α instead of x in the process used in proof of theorem 2.1.2 in [19], it can be shown that for any $\mathcal{F} \in L_{p,\alpha}$, the polynomial $P_N^{\mathcal{F},\alpha}(x)$ is strongly convergent towards $\mathcal{F}$.

From (3.7), we consider $P_N^{\mathcal{F},\alpha}(x) = \int_0^1 K_N(x,s)\mathcal{F}(s)d_\alpha s$, where for $0 \le x \le 1$, $\frac{k}{N+1} < s \le \frac{k+1}{N+1}$, $k = \overline{0,N}$,

$$K_N(x,s) = \alpha(N+1)\left(\begin{array}{c} N \\ k \end{array}\right) x^{\alpha k}(1-x^\alpha)^{N-k}.$$

So, for $0 \le x \le 1$, $\frac{k}{N+1} < s \le \frac{k+1}{N+1}$, $k = \overline{0,N}$, we have

$$\begin{aligned}
\int_0^1 |K_N(x,s)| d_\alpha s &= \sum_{k=0}^{N} \int_{\frac{k}{N+1}}^{\frac{k+1}{N+1}} \alpha(N+1)\left(\begin{array}{c} N \\ k \end{array}\right) x^{\alpha k}(1-x^\alpha)^{N-k} d_\alpha s \\
&= \sum_{k=0}^{N} \alpha(N+1)\left(\begin{array}{c} N \\ k \end{array}\right) x^{\alpha k}(1-x^\alpha)^{N-k} \int_{\frac{k}{N+1}}^{\frac{k+1}{N+1}} d_\alpha s \\
&= \sum_{k=0}^{N} \alpha(N+1)\left(\begin{array}{c} N \\ k \end{array}\right) x^{\alpha k}(1-x^\alpha)^{N-k} \left[\frac{s^\alpha}{\alpha}\right]_{\frac{k}{N+1}}^{\frac{k+1}{N+1}} \\
&= \sum_{k=0}^{N} (N+1)\left(\begin{array}{c} N \\ k \end{array}\right) x^{\alpha k}(1-x^\alpha)^{N-k} \left[\left(\frac{k+1}{N+1}\right)^\alpha - \left(\frac{k}{N+1}\right)^\alpha\right]. \qquad (3.8)
\end{aligned}$$

Using the mean value theorem, we have

$$\left(\frac{k+1}{N+1}\right)^\alpha - \left(\frac{k}{N+1}\right)^\alpha = \frac{\alpha}{N+1} c_k^{\alpha-1},$$

for some $c_k \in \left(\frac{k}{N+1}, \frac{k+1}{N+1}\right)$, $k = \overline{0,N}$. Since $c_0 = \min\{c_0, c_1, ..., c_N\}$ and $0 < \alpha \le 1$, it follows from (3.8) that

$$\begin{aligned}
\int_0^1 |K_N(x,s)| d_\alpha s &= \alpha \sum_{k=0}^{N} c_k^{\alpha-1} \left(\begin{array}{c} N \\ k \end{array}\right) x^{\alpha k}(1-x^\alpha)^{N-k} \\
&\le \alpha c_0^{\alpha-1} \sum_{k=0}^{N} \left(\begin{array}{c} N \\ k \end{array}\right) x^{\alpha k}(1-x^\alpha)^{N-k} = \alpha c_0^{\alpha-1},
\end{aligned}$$

and also for $\frac{k}{N+1} < s \le \frac{k+1}{N+1}$,

$$\begin{aligned}
\int_0^1 |K_N(x,s)| d_\alpha x &= \int_0^1 \alpha(N+1)\left(\begin{array}{c} N \\ k \end{array}\right) x^{\alpha k}(1-x^\alpha)^{N-k} d_\alpha x \\
&= \alpha(N+1)\left(\begin{array}{c} N \\ k \end{array}\right) \int_0^1 x^{\alpha k}(1-x^\alpha)^{N-k} d_\alpha x \\
&= \alpha(N+1)\left(\begin{array}{c} N \\ k \end{array}\right) \int_0^1 x^{\alpha k}(1-x^\alpha)^{N-k} x^{\alpha-1} dx \\
&= (N+1)\left(\begin{array}{c} N \\ k \end{array}\right) \int_0^1 t^k(1-t)^{N-k} dt \\
&= (N+1)\left(\begin{array}{c} N \\ k \end{array}\right) \frac{(N-k)!}{(N+1)N...(k+2)(k+1)} = 1.
\end{aligned}$$

Take $M = \max\{1, \alpha c_0^{\alpha-1}\}$, assume that $\mathcal{F} \in H$ and H is a set of continuous functions in $L_{p,\alpha}$. According to Theorem 3.2, we can write $P_N^{\mathcal{F},\alpha} \to \mathcal{F}$ and since H is an everywhere dense set in $L_{p,\alpha}$ (see [19]), thus for $\mathcal{F} \in L_{p,\alpha}$, $P_N^{\mathcal{F},\alpha} \to \mathcal{F}$ in $L_{p,\alpha}$.

Therefore, the approximation solution of function $q(x) \in L_{2,\alpha}(0,1)$, can be computed by the fractional Bernstein technique. In fact, according to (3.4), we have

$$q(x) \cong \sum_{k=0}^{N} c_k \mathcal{B}_{k,N}^{\alpha}(x) = X^T \phi(x), \tag{3.9}$$

where

$$X = [c_0, c_1, \dots, c_N]^T, \quad \phi(x) = [\mathcal{B}_{0N}^{\alpha}(x), \mathcal{B}_{1N}^{\alpha}(x), \dots, \mathcal{B}_{NN}^{\alpha}(x)]^T.$$

Substituting (3.9) into (3.1), for $n > 1$, $j = 1, ..., n-1$, we arrive at

$$\sum_{k=0}^{N} c_k \int_0^{x_n^j} \mathcal{B}_{k,N}^{\alpha}(x) d_\alpha x \cong 2\lambda_n \tan\left(\frac{\lambda_n}{\alpha}(x_n^j)^\alpha\right).$$

Consequently, potential function $q(x)$ can be created using the numerical algorithm as below:

Numerical algorithm:

1. Choose the values n, α. Set $N = n-2$ and let the nodal points $\{x_n^j\}_{j=1}^{n-1}$, be given.

2. Define the matrixes A and B in the form

$$A = \begin{pmatrix} \int_0^{x_n^1} \mathcal{B}_{0,N}^{\alpha}(x) d_\alpha x & \int_0^{x_n^1} \mathcal{B}_{1,N}^{\alpha}(x) d_\alpha x & \dots & \int_0^{x_n^1} \mathcal{B}_{N,N}^{\alpha}(x) d_\alpha x \\ \int_0^{x_n^2} \mathcal{B}_{0,N}^{\alpha}(x) d_\alpha x & \int_0^{x_n^2} \mathcal{B}_{1,N}^{\alpha}(x) d_\alpha x & \dots & \int_0^{x_n^2} \mathcal{B}_{N,N}^{\alpha}(x) d_\alpha x \\ \vdots & & \dots & \vdots \\ \int_0^{x_n^{n-1}} \mathcal{B}_{0,N}^{\alpha}(x) d_\alpha x & \int_0^{x_n^{n-1}} \mathcal{B}_{1,N}^{\alpha}(x) d_\alpha x & \dots & \int_0^{x_n^{n-1}} \mathcal{B}_{N,N}^{\alpha}(x) d_\alpha x \end{pmatrix},$$

$$B = \begin{pmatrix} 2\lambda_n \tan\left(\frac{\lambda_n}{\alpha}(x_n^1)^\alpha\right) \\ 2\lambda_n \tan\left(\frac{\lambda_n}{\alpha}(x_n^2)^\alpha\right) \\ \vdots \\ 2\lambda_n \tan\left(\frac{\lambda_n}{\alpha}(x_n^{n-1})^\alpha\right) \end{pmatrix}.$$

3. Compute the vector $X = [c_0, c_1, \dots, c_N]^T$ by the following linear system:

   $AX = B.$

4. Calculate the values $q(x_i)$, $i = \overline{0, N}$, by the formula

   $[q(x_i)] = X^T \Phi,$

where

$$x_i = \frac{i}{N}, \quad i = 0, 1, ..., N, \quad \Phi = [\phi(x_0), \phi(x_1), ..., \phi(x_N)].$$

4. Numerical examples

In this section, the Bernstein technique is applied to compute the approximate solution of **INP**, and the accuracy of the presented technique is determined by furnishing the numerical examples. Further, a comparison of this technique with Chebyshev interpolation technique can be seen in Example 4.1. To obtain the numerical results, we apply Matlab software.

Example 4.1. Let the function $q(x) = \cos 3\pi x$, and the value of $\alpha = \frac{2}{3}$ be given. Since

$$\omega = \frac{1}{2}\int_0^1 q(x) d_\alpha x = \frac{1}{2}\int_0^1 x^{\alpha-1}\cos 3\pi x dx = \frac{3}{4} hypergeom\left(\frac{1}{3}, \left[\frac{1}{2}, \frac{4}{3}\right], \frac{-9\pi^2}{4}\right),$$

and

$$\begin{aligned} q_1(x_n^j) &= \frac{1}{2}\int_0^{x_n^j} q(x) d_\alpha x = \frac{1}{2}\int_0^{x_n^j} x^{\alpha-1}\cos 3\pi x dx \\ &= \frac{3}{4}(x_n^j)^{\frac{2}{3}} hypergeom\left(\frac{1}{3}, \left[\frac{1}{2}, \frac{4}{3}\right], \frac{-9\pi^2 (x_n^j)^2}{4}\right), \end{aligned}$$

so, it follows from (2.6) that

$$(x_n^j)^{\frac{2}{3}} = \frac{j}{n} - \frac{9j}{8\pi^2 n^3}\left[hypergeom\left(\frac{1}{3}, \left[\frac{1}{2}, \frac{4}{3}\right], \frac{-9\pi^2}{4}\right) - hypergeom\left(\frac{1}{3}, \left[\frac{1}{2}, \frac{4}{3}\right], \frac{-9\pi^2 j^3}{4n^3}\right)\right].$$

Take $n = 15$, then according to the above formula, the numerical values of nodes x_n^j, $j = \overline{1, n-1} = \overline{1, 14}$ are shown in Table 1.

Table 1. The nodes x_n^j in Example 4.1.

j	1	2	3	4	5	6	7
x_n^j	0.01722494	0.04871865	0.08949781	0.13777955	0.19252978	0.25305038	0.31883341
j	8	9	10	11	12	13	14
x_n^j	0.38949422	0.46473226	0.54430123	0.62798395	0.71557563	0.80688198	0.90173291

Now, suppose that the above nodes are given, next, we calculate the approximation of function q as a solution of inverse nodal problem by the Bernstein technique. We draw an exact solution and numerical approximation for $n = 15$ denoted in Figure 1.

Also, the exact solution and the numerical approximations for $n = 10$ and $\alpha = 0.40, 0.67, 0.80$ are shown in Figure 2.

Moreover, the absolute errors obtained with $n = 5, 10, 15$, and $\alpha = \frac{2}{3}$ by Bernstein technique and comparison of Bernstein technique with Chebyshev interpolation technique for $n = 10$ and $\alpha = \frac{2}{3}$ are seen in Figure 3 and Figure 4, respectively.

In Figure 4, for $n = 10$, it can be seen the Chebyshev interpolation technique is better than the Bernstein technique and has less error, but due to the fact that the interpolation technique has a low speed, it is not recommended for large n values. In other words, for a large n value, we propose the Bernstein technique due to its high speed.

Example 4.2. Assume that the function $q(x) = x^2 \sin 7\pi x$, and the value of $\alpha = 0.75$ are given. Take $n = 15$, then according to the formula (2.6), the numerical values of nodes x_n^j, $j = \overline{1, n-1} = \overline{1, 14}$ are shown in Table 2.

Suppose that the nodes with $n = 40$ are given, we calculate approximation of q, by Bernstein technique that is shown in Figure 5. Exact solution and also numerical approximations for $n = 30$ and $\alpha = 0.35, 0.55, 0.75, 0.95$ and the absolute errors obtained with $n = 20, 30, 40$, and $\alpha = 0.75$ by Bernstein technique are seen in Figure 6 and Figure 7, respectively.

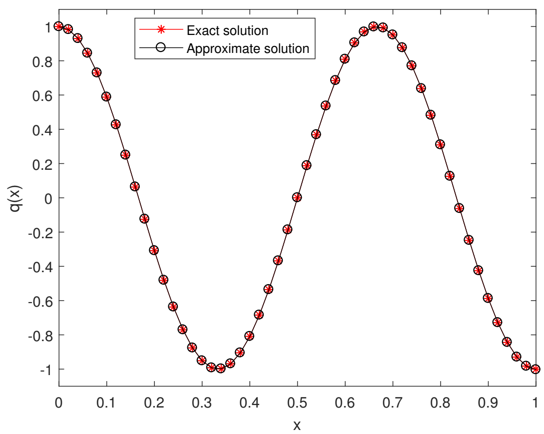

FIGURE 1. Exact and approximate solutions of function $q(x)$ with $n = 15$ by Bernstein technique in Example 4.1.

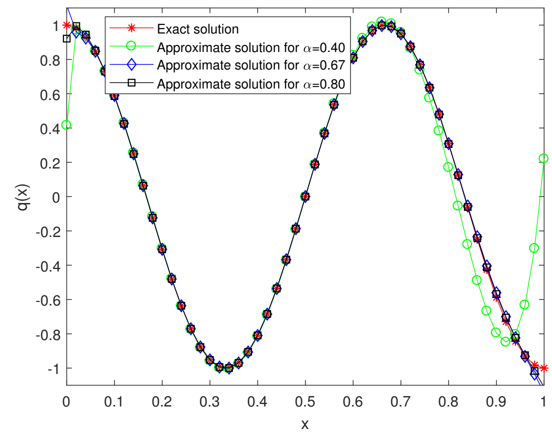

FIGURE 2. Exact solution and numerical approximations of function $q(x)$ for $n = 10$ and different values of α by Bernstein technique in Example 4.1.

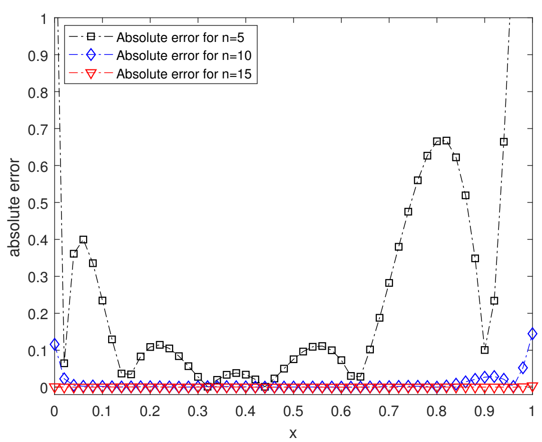

FIGURE 3. Absolute errors between approximate and exact solutions for $\alpha = \frac{2}{3}$ by Bernstein technique in Example 4.1.

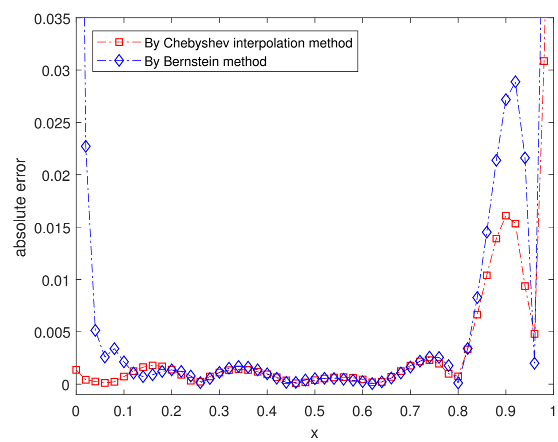

FIGURE 4. Comparison of Bernstein technique with Chebyshev interpolation technique for $n = 10$ and $\alpha = \frac{2}{3}$ in Example 4.1.

Example 4.3. Assume that the function $q(x) = x^3 - \frac{9}{8}x^2 + \frac{5}{16}x - \frac{3}{128}$, and the value of $\alpha = 0.5$ are given. Take $n = 15$, then according to the formula (2.6), the numerical values of nodes x_n^j, $j = \overline{1, n-1} = \overline{1, 14}$ are shown in Table 3.

If the above nodes are given, next, an approximation of function q, by Bernstein technique with $n = 15$, is shown in Figure 8.

Also, the numerical approximations calculated for $n = 5$ and $\alpha = 0.5, 0.6, 0.7$ and the absolute errors obtained with $n = 5, 10, 15$, are shown in Figure 9 and Figure 10, respectively.

TABLE 2. The nodes x_n^j in Example 4.2.

j	1	2	3	4	5	6	7
x_n^j	0.02703152	0.06811520	0.11695879	0.17163963	0.23111553	0.29471570	0.36196676
j	8	9	10	11	12	13	14
x_n^j	0.43250753	0.50604903	0.58237062	0.66129711	0.74264819	0.82627150	0.91209315

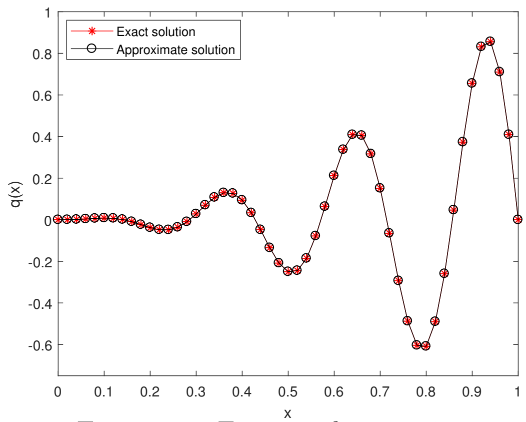

FIGURE 5. Exact and approximate solutions of function $q(x)$ with $n = 40$ by Bernstein technique in Example 4.2.

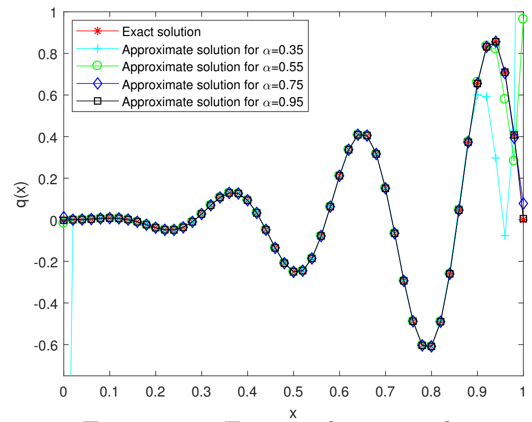

FIGURE 6. Exact solution and numerical approximations of function $q(x)$ for $n = 30$ and different values of α by Bernstein technique in Example 4.2.

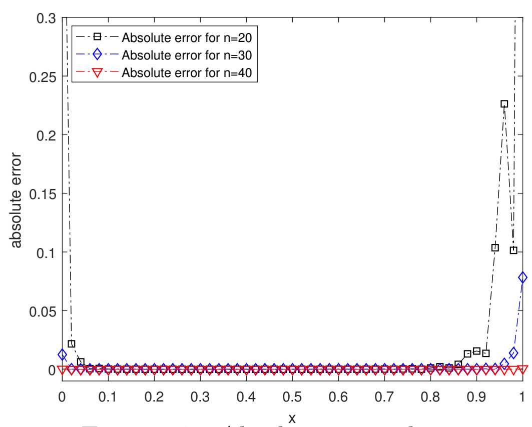

FIGURE 7. Absolute errors between approximate and exact solutions for $\alpha = 0.75$ by Bernstein technique in Example 4.2.

5. CONCLUSION

In this perusal, we considered the Sturm-Liouville equation with conformable fractional derivatives of order $0 < \alpha \leq 1$ and reconstructed an integral equation using the property that the nodes are zeroes of eigenfunctions. Then,

TABLE 3. The nodes x_n^j in Example 4.3.

j	1	2	3	4	5	6	7
x_n^j	0.00444427	0.01777713	0.03999869	0.07110910	0.11110850	0.15999696	0.21777443
j	8	9	10	11	12	13	14
x_n^j	0.28444070	0.35999545	0.44443834	0.53776920	0.63998839	0.75109733	0.87109920

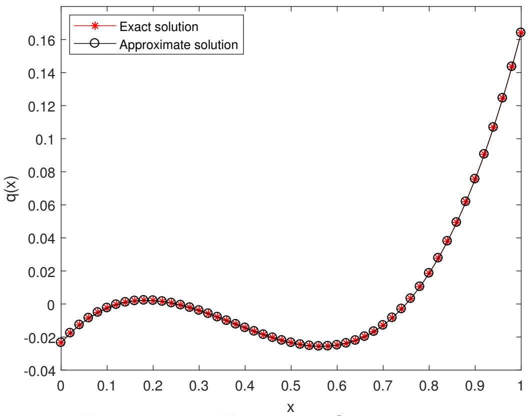

FIGURE 8. Exact and approximate solutions of function $q(x)$ with $n = 15$ by Bernstein technique in Example 4.3.

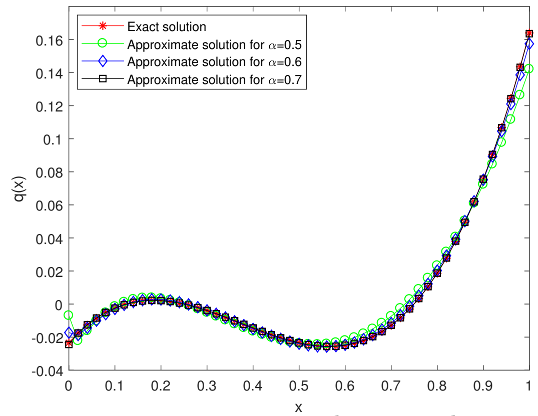

FIGURE 9. Exact solution and numerical approximations of function $q(x)$ for $n = 5$ and different values of α by Bernstein technique in Example 4.3.

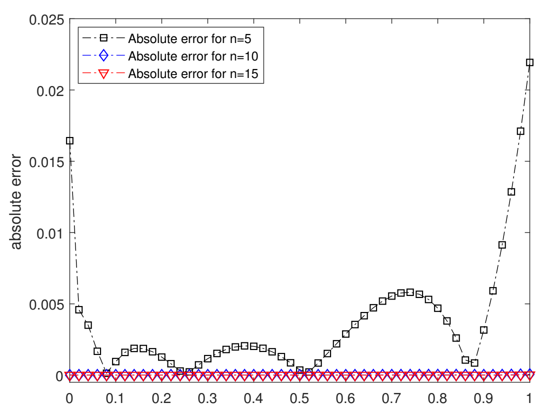

FIGURE 10. Absolute errors between approximate and exact solutions for $\alpha = 0.5$ by Bernstein technique in Example 4.3.

the Bernstein technique was used to solve inverse nodal problem and the approximation of the solution to this problem was calculated. Moreover, a comparison of this technique with the Chebyshev interpolation technique was shown in a numerical example and it was concluded that both techniques are good numerical techniques however Chebyshev interpolation technique has the less errors. Nevertheless, due to the fact that the interpolation technique has a low speed for $n >> 1$, it is not recommended. In other words, in this study, we propose the Bernstein technique for a large value of n due to this high speed.

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