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Rumor spread dynamics and its sensitivity analysis under the influence of the Caputo fractional derivatives

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Abstract

Rumor spreading is the circulation of doubtful messages on the social network. Fact retrieving process that aims at preventing the spread of the rumor, appears to have a significant global impact. In this research, we have investigated a mathematical model projecting rumor spread by considering six groups of individuals namely ignorant, exposed, intentional rumor spreader, unintentional rumor spreader, stifler, and fact retriever. To represent the current abnormal and fast pattern of the message spread around various platforms, in the projected model, we have implemented the fractional derivative in the Caputo context. Using the existing theory of the fractional derivative, we have examined the theoretical aspects such as the existence and uniqueness of the solutions, the existence and stability of the rumor-free and rumor equilibrium points, and the global stability of the rumor-free equilibrium point. Computing basic reproduction numbers, we have analyzed the existence and stability of points of equilibrium. The sensitivity of basic reproduction numbers is also examined. Importance of the fact retrieving drive is highlighted by relating it to the basic reproduction number. Finally, by applying the Adams-Bashforth-Moulton method, we have presented the numerical results by capturing the profile of each of the groups under the influence of fractional derivative and investigated the impact of rumor verification rate and contact rate in controlling and preventing the rumor. With the Caputo fractional operator in the projected model, the current research highlights the significance of the fact retriever and the curb in individual contact and captures the relevant consequences.

Keywords. Rumor spread, Sensitivity analysis, Basic reproduction number.
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1. Introduction

Rumors are messages that circulate without being verified. The people who spread these rumors neither bother about their impact nor bother to determine if it is true or not. Conventionally, rumors or gossip are spread from one person to another person verbally or digitally and as they do so, they tend to shift slightly. Information spreading and retrieval has gradually moved from offline to online sources for a few years due to the rapid growth of Web and mobile internet technology. Along with the newspapers, Weibo, WeChat, Twitter, Facebook, and other popular social media platforms have evolved in the dissemination of crucial information [44]. When dealing with complex subjects, some online users may lack rational and optimism [40], which can cause untimely debunking of information or further propagation of the rumors. All kinds of social contradictions on social media become more obvious if an emergency occurs. False information and reports about these emergencies have spread far and wide in seemingly endless streams. Some individuals with ulterior motives use these situations to spread false information, which harms people's lives as well as the peace and stability of society. Examples of this include the Tianjin explosion accident in 2015, the issue with the vaccine in 2018, the COVID-19 epidemic starting at the end of 2019, the effects of the Syria-Turkey earthquake accompanied by false news, and many more. Here are some rumors associated with the earthquake in Syria, Turkey, and Lebanon that have later been verified by BBC [24]. According to one message from a verified Twitter user, the

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earthquake in Turkey caused a nuclear reactor to explode. More than 1.2 million people viewed the video. However, a fact-check discovered that they had been from the aftermath of the August 2020 Beirut blast, which claimed at least 200 lives. A video that claimed to show a building collapsing in Turkey was circulated online early on Monday, garnering more than a million views on Twitter. However, it was in fact from Florida in 2021. Omid Djalili, a British-Iranian comedian, posted a video of what he described as a "tsunami after the earthquake hit the shore of Turkey" on his Twitter account. There have been almost 300,000 views. However, the video was shot in Indonesia and was released in September 2018. Some people invent and disseminate misleading information or unfavorable commentary in the event of a crisis, jeopardizing societal order while also having a significant detrimental impact on people's daily lives [19]. For instance, a claim that salt might become contaminated as a result of the Fukushima nuclear accident in Japan was extensively circulated online in 2011 and led to a panicked buying of salt. Disease Transmission, being one of the most explored areas mathematically, has opened the opportunities for many researchers to study other areas in that context [4, 20, 30, 31, 42, 46]. Researchers have formed models that elaborate phenomena such as viral marketing, dissemination of information, and panics caused by epidemics or emergencies [2, 5, 7, 10, 27, 34, 35, 43, 52, 57]. These articles inspired us to take up rumors spread in novel directions. Sometimes rumors are spread unintentionally (as time passes) and sometimes intentionally for individual profit, promotion, attention, revenge, etc. In the book 'Rumors and Rumor Control', Kimmel gives a brief description of understanding and controlling rumors [33]. Many times rumor spreading can have bad impacts such as inducing panic psychology, and economic loss in unexpected events [58]. Because most rumors are based on misunderstandings and incomplete information, rumors about the subject of the gossip may cause melancholy, anxiety, and even suicidal thoughts. [14, 25, 26, 60]. The study on rumor spreading is as important as the study on diseases since rumors also harm society and can affect people's lifestyles. Therefore, it is preferable to make speculations about how rumors might be managed and avoided. DK model [13] is the first model in rumor spreading proposed by Daley and Kendall. Based on mathematical theory, Maki and Thomson [15] and Murray [41] discovered the rumor-spreading model. In [21, 50], authors have analyzed the rumor-spreading process by applying the SIR model. Motivated by the already explored literature, we have formulated the model regarding the spread of the rumor via individuals in a population compartmentalizing the rumor-spreading process as ignorant, exposed, intentional rumor spreader, unintentional rumor spreader, stifler, and fact retriever.

Fractional derivatives (FDs) are useful tools to capture the hereditary properties and prior information and experience of various activities that link to each individual. Evolution in the area of fractional calculus has appeared as an added advantage in this process [1, 6, 8, 23, 29, 51]. The Riemann-Liouville, Caputo, Atangana-Baleanu derivative, Caputo-Fabrizio, Grünwald Letnikov, and Jumarie fractional operators [3, 12, 22, 32, 38, 39, 45] are some of the FDs that have gained their importance among researchers and their theories are explored worldwide. Among these derivatives, the Caputo derivative is the one, which has attracted the researchers the most in evaluating various dynamical models because it has the advantage of allowing standard initial and boundary conditions to be incorporated in the model formulation [11]. Results obtained in modeling the real phenomena having memory effect are better. A Hysteresis is a form of memory where the current state of the system is influenced by both current and prior conditions, meaning that past events have an impact on current dynamics. The hysteresis effect is introduced into the biological model [47] because certain live organisms' defense mechanisms are triggered by this. The FDs are significant since the past is seen to be the source of the present. The effect of memory is analyzed in fractional differential equations [9, 54]. It is known that the derivatives of positive integer orders are determined by the properties of the differentiable function only in an infinitesimal neighborhood of the considered point. As a result, differential equations with integer order derivatives cannot describe processes with memory. A powerful tool for describing the effects of power-law memory is fractional calculus. Again, the Caputo fractional derivative has the added advantage of considering initial and boundary conditions with integer order derivatives. Previous experiences suggest that during a rumor spreading, verification of a message is a defensive action to limit the circulation of an uncertain rumor in the social network. When a fact has a long-term memory impact, it causes an individual to defend himself through memory. Therefore, rumor can be a subject of the hysteresis effect and can be studied well under the influence of fractional derivatives. However, factors such as speed of propagation, stifling as well as contact rates are difficult to estimate due to intentional rumor spreading and the lack of fewer rumor verifiers. As a result, employing classical differential equations to model the dynamics of the populations involved may be ineffective. The ability to capture the heterogeneous and memory-based properties of FDs motivated us to incorporate fractional operators in the projected rumor-spreading model. Looking at the effect of memory impact on the spread/control of rumor, in this paper, we have incorporated the Caputo operator to study the dynamics of ignorant, exposed, intentional rumor spreader, unintentional rumor spreader, stifler, and fact retriever population. Whenever any rumor contagion erupts among the population, it hardly disappears completely. It keeps resurfacing via different media at some interval of time. This phenomenon of coexistence can be very well depicted by fractional calculus. In this paper, we have taken fact retriever which brings novelty to the projected model. The fact retrievers can retrieve the facts through standard media, apps, or websites like www.thecut.com, www.cisa.gov, www.factcheck.org, www.snopes.com, etc. The fact retriever retrieves the latest, reliable, and verified content through which rumor spreading can be controlled in our society since the purpose of the fact retriever is to verify and disseminate the information which is helpful to make the best possible decisions. We have presented the numerical simulation of rumor spread and the impact of various factors on it. The influence of facts in controlling and preventing the rumor is projected by determining the basic reproduction (BR) number. Since one of the major reasons for the circulation of rumors is direct or digital contact among people, we have investigated the role of contact rate in controlling the spread. The numerical results are computed by using the generalized fractional Adams-Bashforth Moulton technique compatible with the Caputo fractional derivative [16–18, 36].

2. Some Essential Theorems

The theoretical results presented in this paper are supported by the theorems mentioned here. Here, $D^{\alpha}_{t_0}$ denotes the Caputo fractional derivative.

Definition 2.1. (Caputo Fractional Derivative) [48] The Caputo derivative with fractional order $0 < \alpha < 1$ of function $g(\mathfrak{t})$ is defined by:

$$D^{\alpha}_{\mathfrak{t}_\circ} g(\mathfrak{t}) = \frac{1}{\Gamma(1-\alpha)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\tau)^{-\alpha} g'(\tau) d\tau,$$

where $\Gamma(\cdot)$ refers to the Gamma function.

Definition 2.2. The fractional order integral operator of Riemann-Liouville is defined as: [48]

$$J^{\alpha}_{v} f(v) = \frac{1}{\Gamma(\alpha)} \int_0^{v} \frac{f(\mathfrak{t})}{(v-\mathfrak{t})^{1-\alpha} d\mathfrak{t}}, \alpha > 0,$$

$$J^0 f(v) = f(v).$$

Lemma 2.3. *Consider the system[53]:*

$$D^{\alpha}_{\mathfrak{t}_\circ} x(\mathfrak{t}) = g(\mathfrak{t}, x), \mathfrak{t} > \mathfrak{t}_0,$$

choosing the initial condition as $v(\mathfrak{t}_0)$, where $0 < \alpha \leq 1$ and $g : [\mathfrak{t}_0, \infty) \times \Omega \to \mathbb{R}^n, \Omega \in \mathbb{R}^n$. When $g(\mathfrak{t}, v)$ holds the locally Lipchitz conditions concerning to v, it has a unique solution on $[\mathfrak{t}_0, \infty) \times \Omega$.

Lemma 2.4. *[37] We assume that $g(\mathfrak{t})$ is a continuous function on $[\mathfrak{t}_0, +\infty)$ which satisfies:*

$$D^{\alpha}_{\mathfrak{t}_\circ} g(\mathfrak{t}) \leq -\lambda g(\mathfrak{t}) + \xi, g(\mathfrak{t}_0) = g_{\mathfrak{t}_0},$$

here $0 < \alpha \leq 1$, $(\lambda, \xi) \in \mathbb{R}^2$ and $\lambda \neq 0$ and consider $\mathfrak{t}_0 \geq 0$ as the initial time. Now,

$$g(\mathfrak{t}) \leq (g(\mathfrak{t}_0) - \frac{\xi}{\lambda}) E_{\alpha}[-\lambda(\mathfrak{t}-\mathfrak{t}_0)^{\alpha}] + \frac{\xi}{\lambda}.$$

Lemma 2.5. *[37] Let $v(\mathfrak{t}) \in \mathbb{R}_+$ be a derivable and continuous function. Then, at any time $\mathfrak{t} > \mathfrak{t}_0$,*

$$D^{\alpha}_{\mathfrak{t}_\circ}(v(\mathfrak{t}) - v^* - v^* ln\frac{v(\mathfrak{t})}{v^*}) \leq (1 - \frac{v^*}{v}) D^{\alpha}_{\mathfrak{t}_\circ} v(t), \quad v^* \in \mathbb{R}_+, \quad \forall \alpha \in (0,1).$$

3. Model Formulation

We have formulated a rumor-spreading model on social networks subdividing the population into various groups of people. Ignorant (S) are those who are likely to get influenced by rumors; exposed (Q) to rumors are those who are going to hear the rumors but in such a stage, in which they have not heard of the rumors yet; intentional rumor (X) spreaders intentionally spread the rumors for profit, relationship, politics, position, promotion, etc.; unintentional rumor (U) spreaders are those who circulates the rumors unwillingly and unconsciously; stiflers (F) are people, who have heard the rumor but do not spread it; fact retriever (V) retrieves and circulate fact information through some of the social media apps, websites, some standard media or verbally, to control rumors.

$$
\begin{aligned}
D_t^\alpha S &= N-(\phi_1+\phi_2)\nu SX-(\phi_1+\phi_2)\nu SU-\mu S,\\
D_t^\alpha Q &= (\phi_1+\phi_2)\nu SX+(\phi_1+\phi_2)\nu SU-\psi Q-\delta Q,\\
D_t^\alpha X &= \eta\psi Q-rX,\\
D_t^\alpha U &= \psi(1-\eta)Q-rU-\rho U,\\
D_t^\alpha F &= rX+rU-\beta F,\\
D_t^\alpha V &= \mu S+\delta Q+\rho U+\beta F-\xi V.
\end{aligned}
\tag{3.1}
$$

with initial condition $S(t)>0$, $Q(t)>0$, $X(t)>0$, $U(t)>0$, $F(t)>0$ and $V(t)>0$ where t_0 is the initial time. All the parameters $N,\nu,\psi,\mu,\phi_1,\phi_2,\delta,\eta,r,\rho,\beta,\xi$ are positive whose meanings of symbols are mentioned in Table 1. The conversion flow of the population is represented by the flow diagram 1.

Table 1. Description of the Projected Rumor Model.

N	The total population in Social Network
ν	Rumor transmission ratio
ϕ_1	Verbal contact rate
ϕ_2	Digital contact rate
μ	Fact retrieving rate
ψ	Rate of exposed population being heard of rumor
η	Rate at which exposed population become spreader
r	The probability that individual spreader become stifler
ρ	Verifying rate of rumor by unintentional rumor spreader
β	Verifying rate of rumor by stifler
δ	Rate of which E progresses to F
ξ	Forgetting rate

The assumptions of the projected rumor-spreading model can be stated as follows:

(a) Ignorants when contacted by either of the spreaders, are exposed to rumors and with probability, at the rate of $\nu S(\phi_1+\phi_2)(U+X)$ move to the exposed compartment.

(b) When exposed individuals heard the rumor, they move to the spreader compartment at the rate of ψQ

(c) When intentional spreaders spread the rumor, he/she will be either done with work or sometimes not. Hence at a certain point in time, the intentional spreader will gradually convert to stifler at the rate of rX.

(d) Unintentional rumor spreader loses the temptation of circulating the rumor after some days on the same topic and hence loses interest in spreading. Then unintentional rumor spreaders become stifler at the rate of rU.

(e) Except for the intentional rumor spreader, every other population retrieves the facts and disseminates the information at the rate of $\mu S, \delta Q, \rho U$, and βF.

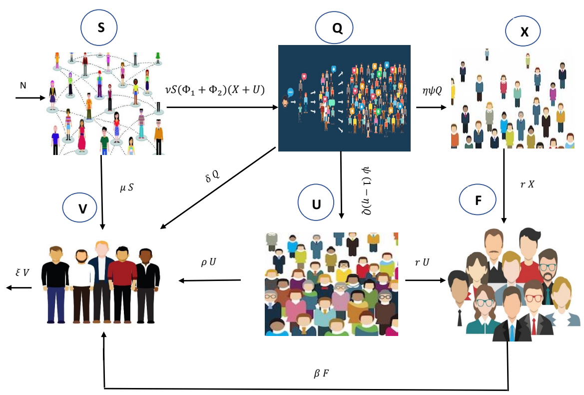


FIGURE 1. Schematic diagram of interaction among various groups of population.

4. Existence of solutions

Here we establish the existence of the solutions of the proposed model (3.1) by using the fixed-point theorem. There are no precise algorithms or approaches for evaluating the exact solutions because the model is complex and non-local. However, the existence of the solution is guaranteed if certain conditions are met. The system (3.1) can be re-written as follows:

$$\begin{aligned} D_t^\alpha[S(t)] &= \mathfrak{G}_1(t,S), \\ D_t^\alpha[Q(t)] &= \mathfrak{G}_2(t,Q), \\ D_t^\alpha[X(t)] &= \mathfrak{G}_3(t,X), \\ D_t^\alpha[U(t)] &= \mathfrak{G}_4(t,U), \\ D_t^\alpha[F(t)] &= \mathfrak{G}_5(t,F), \\ D_t^\alpha[V(t)] &= \mathfrak{G}_6(t,V). \end{aligned} \tag{4.1}$$

Since the change of the fact retriever population is directly proportional to only the ignorant population while verifying the existence and uniqueness of the solution, we have not considered the last equation of the system (3.1). Hence the above system can be transformed into a Volterra type integral equation:

$$\begin{aligned} S(t) - S(t_0) &= \frac{1}{\Gamma(\alpha)} \int_0^t \mathfrak{G}_1(\vartheta, G(\vartheta))(t-\vartheta)^{\alpha-1} d\vartheta, \\ Q(t) - Q(t_0) &= \frac{1}{\Gamma(\alpha)} \int_0^t \mathfrak{G}_1(\vartheta, Q(\vartheta))(t-\vartheta)^{\alpha-1} d\vartheta, \\ X(t) - X(t_0) &= \frac{1}{\Gamma(\alpha)} \int_0^t \mathfrak{G}_1(\vartheta, X(\vartheta))(t-\vartheta)^{\alpha-1} d\vartheta, \\ U(t) - U(t_0) &= \frac{1}{\Gamma(\alpha)} \int_0^t \mathfrak{G}_1(\vartheta, U(\vartheta))(t-\vartheta)^{\alpha-1} d\vartheta, \\ F(t) - F(t_0) &= \frac{1}{\Gamma(\alpha)} \int_0^t \mathfrak{G}_1(\vartheta, F(\vartheta))(t-\vartheta)^{\alpha-1} d\vartheta, \\ V(t) - V(t_0) &= \frac{1}{\Gamma(\alpha)} \int_0^t \mathfrak{G}_1(\vartheta, V(\vartheta))(t-\vartheta)^{\alpha-1} d\vartheta. \end{aligned} \tag{4.2}$$

Theorem 4.1. *The kernel* $\mathfrak{G}_1$ *holds the Lipschitz condition and contraction if* $0 \le (\nu(\phi_1+\phi_2)(\epsilon_3+\epsilon_4)+\mu) < 1$ *holds true.*

Proof. By considering the two functions G and G_1 such as:

$$\begin{aligned} ||\mathfrak{G}_1(t,\bar{S})-\mathfrak{G}_1(t,S)|| &= ||N-(\phi_1+\phi_2)\nu S(t)X(t)-(\phi_1+\phi_2)\nu S(t)U(t)-\mu S(t)-(N- \\ &\quad (\phi_1+\phi_2)\nu \bar{S(t)}X(t)-(\phi_1+\phi_2)\nu \bar{S(t)}U(t)-\mu \bar{S(t)})|| \\ &= ||-(\phi_1+\phi_2)\nu X(S-\bar{S})-(\phi_1+\phi_2)\nu U(S-\bar{S})+\mu(S-\bar{S})|| \\ &\le (\nu(\phi_1+\phi_2)(\epsilon_3+\epsilon_4)+\mu)||G(t)-G_1(t)|| \\ &= \zeta_1||G(t)-G_1(t)||. \end{aligned} \tag{4.3}$$

Taking $\zeta_1 = \nu(\phi_1+\phi_2)(\epsilon_3+\epsilon_4)+\mu$, where $||X(t)|| \le \epsilon_3$ and $||U(t)|| \le \epsilon_4$ are bounded functions, implies that

$$||\mathfrak{G}_1(t,S)-\mathfrak{G}_1(t,\bar{S})|| \le \zeta_1||S(t)-\bar{S}(t)||. \tag{4.4}$$

We have $||S(t)|| \le \epsilon_1, ||Q(t)|| \le \epsilon_2, ||X(t)|| \le \epsilon_3, ||U(t)|| \le \epsilon_4, ||F(t)|| \le \epsilon_5$ and $||V(t)|| \le \epsilon_6$ are bounded functions and therefore, the Lipschitz condition is satisfied for $\mathfrak{G}_1$ and if $0 \le \nu(\phi_1+\phi_2)(\epsilon_3+\epsilon_4) < 1$, then it follows a contraction. Similarly, the remaining cases can be proved and represented as follows:

$$\begin{aligned} ||\mathfrak{G}_2(t,Q)-\mathfrak{G}_2(t,\bar{Q})|| &\le \zeta_2||Q(t)-\bar{Q}(t)||, \\ ||\mathfrak{G}_3(t,X)-\mathfrak{G}_3(t,\bar{X})|| &\le \zeta_3||X(t)-\bar{X}(t)||, \\ ||\mathfrak{G}_4(t,U)-\mathfrak{G}_4(t,\bar{U})|| &\le \zeta_4||U(t)-\bar{U}(t)||, \\ ||\mathfrak{G}_5(t,F)-\mathfrak{G}_5(t,\bar{F})|| &\le \zeta_5||F(t)-\bar{F}(t)||, \\ ||\mathfrak{G}_6(t,V)-\mathfrak{G}_6(t,\bar{V})|| &\le \zeta_6||V(t)-\bar{V}(t)||, \end{aligned} \tag{4.5}$$

where $\zeta_2 = \psi+\delta, \quad \zeta_3 = r, \quad \zeta_4 = r+\rho, \quad \zeta_5 = \beta$and $\zeta_6 = \xi$. ☐

Theorem 4.2. *The solution of the system* (3.1) *will exist and is unique, if we obtain some* t_0 *such that*

$$\frac{1}{\Gamma(\alpha)}\zeta_i t_0 < 1, \text{ for } i = 1,2,3,...,6.$$

Proof. The proof of the theorem is derived in 3 steps.
Step 1. Now, by system (4.2), the recursive form can be written as

$$\begin{aligned} \aleph_{1,n}(t) &= S_n(t)-S_{n-1}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (\mathfrak{G}_1(\vartheta,S_{n-1})-\mathfrak{G}_1(\vartheta,S_{n-2}))(t-\vartheta)^{\alpha-1}d\vartheta, \\ \aleph_{2,n}(t) &= Q_n(t)-Q_{n-1}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (\mathfrak{G}_1(\vartheta,Q_{n-1})-\mathfrak{G}_1(\vartheta,Q_{n-2}))(t-\vartheta)^{\alpha-1}d\vartheta, \\ \aleph_{3,n}(t) &= X_n(t)-X_{n-1}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (\mathfrak{G}_1(\vartheta,X_{n-1})-\mathfrak{G}_1(\vartheta,X_{n-2}))(t-\vartheta)^{\alpha-1}d\vartheta, \\ \aleph_{4,n}(t) &= U_n(t)-U_{n-1}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (\mathfrak{G}_1(\vartheta,U_{n-1})-\mathfrak{G}_1(\vartheta,U_{n-2}))(t-\vartheta)^{\alpha-1}d\vartheta, \\ \aleph_{5,n}(t) &= F_n(t)-F_{n-1}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (\mathfrak{G}_1(\vartheta,F_{n-1})-\mathfrak{G}_1(\vartheta,F_{n-2}))(t-\vartheta)^{\alpha-1}d\vartheta, \\ \aleph_{6,n}(t) &= V_n(t)-V_{n-1}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (\mathfrak{G}_1(\vartheta,V_{n-1})-\mathfrak{G}_1(\vartheta,V_{n-2}))(t-\vartheta)^{\alpha-1}d\vartheta. \end{aligned} \tag{4.6}$$

The initial conditions are

$$\begin{aligned} &S_0(t) = S(t_0), \quad Q_0(t) = Q(t_0), \quad X_0(t) = X(t_0), \\ &U_0(t) = U(t_0), \quad F_0(t) = F(t_0), \quad V_0(t) = V(t_0). \end{aligned}$$

By applying the norm on the system (4.6), we have

$$\begin{aligned}
||\aleph_{1,n}(t)|| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t ||(\mathfrak{G}_1(\vartheta, S_{n-1}) - \mathfrak{G}_1(\vartheta, S_{n-2}))(t-\vartheta)^{\alpha-1}d\vartheta||,\\
||\aleph_{2,n}(t)|| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t ||(\mathfrak{G}_1(\vartheta, Q_{n-1}) - \mathfrak{G}_1(\vartheta, Q_{n-2}))(t-\vartheta)^{\alpha-1}d\vartheta||,\\
||\aleph_{3,n}(t)|| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t ||(\mathfrak{G}_1(\vartheta, X_{n-1}) - \mathfrak{G}_1(\vartheta, X_{n-2}))(t-\vartheta)^{\alpha-1}d\vartheta||,\\
||\aleph_{4,n}(t)|| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t ||(\mathfrak{G}_1(\vartheta, U_{n-1}) - \mathfrak{G}_1(\vartheta, U_{n-2}))(t-\vartheta)^{\alpha-1}d\vartheta||,\\
||\aleph_{5,n}(t)|| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t ||(\mathfrak{G}_1(\vartheta, F_{n-1}) - \mathfrak{G}_1(\vartheta, F_{n-2}))(t-\vartheta)^{\alpha-1}d\vartheta||,\\
||\aleph_{6,n}(t)|| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t ||(\mathfrak{G}_1(\vartheta, V_{n-1}) - \mathfrak{G}_1(\vartheta, V_{n-2}))(t-\vartheta)^{\alpha-1}d\vartheta||. \qquad (4.7)
\end{aligned}$$

Using the Lipchitz condition, we get

$$\begin{aligned}
||\aleph_{1,n}(t)|| &\leq \frac{1}{\Gamma(\alpha)}\zeta_1\int_0^t ||\aleph_{1,n-1}(\vartheta)d\vartheta||,\\
||\aleph_{2,n}(t)|| &\leq \frac{1}{\Gamma(\alpha)}\zeta_2\int_0^t ||\aleph_{2,n-1}(\vartheta)d\vartheta||,\\
||\aleph_{3,n}(t)|| &\leq \frac{1}{\Gamma(\alpha)}\zeta_3\int_0^t ||\aleph_{3,n-1}(\vartheta)d\vartheta||,\\
||\aleph_{4,n}(t)|| &\leq \frac{1}{\Gamma(\alpha)}\zeta_4\int_0^t ||\aleph_{4,n-1}(\vartheta)d\vartheta||,\\
||\aleph_{5,n}(t)|| &\leq \frac{1}{\Gamma(\alpha)}\zeta_5\int_0^t ||\aleph_{5,n-1}(\vartheta)d\vartheta||,\\
||\aleph_{6,n}(t)|| &\leq \frac{1}{\Gamma(\alpha)}\zeta_6\int_0^t ||\aleph_{6,n-1}(\vartheta)d\vartheta||. \qquad (4.8)
\end{aligned}$$

Then by the successive terms difference, we have noticed that,

$$S_n(t) = \sum_{i=1}^{n}\aleph_{1,i}(t), Q_n(t) = \sum_{i=1}^{n}\aleph_{2,i}(t), X_n(t) = \sum_{i=1}^{n}\aleph_{3,i}(t),$$

$$U_n(t) = \sum_{i=1}^{n}\aleph_{4,i}(t), F_n(t) = \sum_{i=1}^{n}\aleph_{5,i}(t), V_n(t) = \sum_{i=1}^{n}\aleph_{6,i}(t).$$

From Eq. (4.8) and recursively we get,

$$\begin{aligned}
||\aleph_{1,i}(t)|| &\leq ||S_n(t_0)||\left[\frac{\zeta_1 t}{(\Gamma\alpha)}\right]^n,\\
||\aleph_{2,i}(t)|| &\leq ||Q_n(t_0)||\left[\frac{\zeta_2 t}{(\Gamma\alpha)}\right]^n,\\
||\aleph_{3,i}(t)|| &\leq ||X_n(t_0)||\left[\frac{\zeta_3 t}{(\Gamma\alpha)}\right]^n,\\
||\aleph_{4,i}(t)|| &\leq ||U_n(t_0)||\left[\frac{\zeta_4 t}{(\Gamma\alpha)}\right]^n,\\
||\aleph_{5,i}(t)|| &\leq ||F_n(t_0)||\left[\frac{\zeta_5 t}{(\Gamma\alpha)}\right]^n,\\
||\aleph_{6,i}(t)|| &\leq ||V_n(t_0)||\left[\frac{\zeta_6 t}{(\Gamma\alpha)}\right]^n. \qquad (4.9)
\end{aligned}$$

Hence, both the existence and continuity are shown for the obtained solutions.

Step 2. By using the above theorem, we prove the following results by assuming the following. To prove that the relation (4.9) is the solution for (3.1), we consider:

$$\begin{aligned}
S(t)-S(t_0) &= S_n(t)-\mathfrak{W}_{1n}(t),\\
Q(t)-Q(t_0) &= Q_n(t)-\mathfrak{W}_{2n}(t),\\
X(t)-X(t_0) &= X_n(t)-\mathfrak{W}_{3n}(t),\\
U(t)-U(t_0) &= U_n(t)-\mathfrak{W}_{4n}(t),\\
F(t)-F(t_0) &= F_n(t)-\mathfrak{W}_{5n}(t),\\
V(t)-V(t_0) &= V_n(t)-\mathfrak{W}_{6n}(t).
\end{aligned}$$

In order to get the required results, we set

$$\begin{aligned}
||\mathfrak{W}_{1n}(t)|| &= ||\frac{1}{\Gamma(\alpha)}\int_0^t (\mathfrak{G}_1(\vartheta,S)-\mathfrak{G}_1(\vartheta,S_{n-1}))d\vartheta||\\
&\leq \frac{1}{\Gamma(\alpha)}\zeta_1||S-S_{n-1}||t. \qquad (4.10)
\end{aligned}$$

Continuing the same procedure, at t_0, we get

$$||\mathfrak{W}_{1n}(t)|| \leq \left(\frac{t_0}{\Gamma(\alpha)}\right)^{n+1}\zeta_1^{n+1}M. \qquad (4.11)$$

From Eq. (6.2), we can see that as n tends to ∞, $||\mathfrak{W}_{1n}(t)||$ approaches 0 provided $\frac{\zeta_1 t_0}{\Gamma(\alpha)} < 1$. In the same way, it can be proved that all $||\mathfrak{W}_{2n}(t)||, ||\mathfrak{W}_{3n}(t)||, ||\mathfrak{W}_{4n}(t)||, ||\mathfrak{W}_{5n}(t)||, ||\mathfrak{W}_{6n}(t)||$ tends to 0.

Step 3. Now, for the proposed model, we shall prove the uniqueness of the solution. On the contrary, if there exists other set of solutions $S^*(t), Q^*(t), X^*(t), U^*(t), F^*(t), V^*(t)$, then it follows from the first equation that,

$$S(t)-S^*(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (\mathfrak{G}_1(\vartheta,S)-\mathfrak{G}_1(\vartheta,S^*))d\vartheta.$$

The above equation after employing the norm becomes,

$$\begin{aligned}
||S(t)-S^*(t)|| &= ||\frac{1}{\Gamma(\alpha)}\int_0^t (\mathfrak{G}_1(\vartheta,S)-\mathfrak{G}_1(\vartheta,S^*))d\vartheta||\\
&\leq \frac{1}{\Gamma(\alpha)}\zeta_1||S(t)-S^*(t)||. \qquad (4.12)
\end{aligned}$$

On simplification

$$||S(t) - S^*(t)||\left(1 - \frac{\zeta_1 t_0}{\Gamma(\alpha)}\right) \leq 0.$$

Since

$$\left(1 - \frac{1}{\Gamma(\alpha)}\zeta_1 t_0\right) \geq 0, \tag{4.13}$$

from the above inequality, it is clear that $S(t) - S^*(t) = 0$. Hence, the Eq. (4.13) proves the required result. □

5. Boundedness

In this Section, we establish that the solutions of the system (3.1) are bounded.

Theorem 5.1. *The solutions of the system (3.1) are uniformly bounded.*

Proof. Define a function, $\mathfrak{L}(t) = S(t) + Q(t) + X(t) + U(t) + F(t) + V(t)$.
On applying fractional derivatives, we get

$$\begin{aligned} D^\alpha_{t_0}\mathfrak{L}(t) + \beta\mathfrak{L}(t) &= D^\alpha_{t_0}[S(t) + Q(t) + X(t) + U(t) + F(t) + V(t)] + \beta[S(t) + Q(t) + X(t) + U(t) \\ &\quad +F(t) + V(t)] \qquad (5.1)\\ &= N - (\phi_1 + \phi_2)\nu SX - (\phi_1 + \phi_2)\nu SU - \mu S + (\phi_1 + \phi_2)\nu SX + (\phi_1 + \phi_2)\nu SU - \\ &\quad \psi Q - \delta Q + \psi(1 - \eta)Q - rU - \rho U + rX + rU - \beta F + \mu S + \delta Q + \rho U + \beta F - \xi V + \\ &\quad \beta S + \beta Q + \beta X + \beta U + \beta F + \beta V \\ &\leq N + \beta(S + Q + X + U + F + V). \qquad (5.2) \end{aligned}$$

The solution exists and is unique in

$$\mho = \{(S, Q, X, U, F, V) : max\{|S|, |Q|, |X|, |U|, |F|, |V|\} \leq m\}.$$

The above inequality yields,

$$D^\alpha_{t_0}\mathfrak{L}(t) + \beta\mathfrak{L}(t) \leq N + 6\beta m.$$

By the Lemma 2.4, we get

$$D^\alpha_{t_0}\mathfrak{L}(t) \leq (\mathfrak{L}(t_0) - \frac{1}{\beta}N + 6\beta m)E_\alpha[-\beta(t - t_0)^\alpha] + \frac{1}{\beta}(N + 6\beta m) \to N + 6\beta m,$$

as $t \to \infty$. Therefore, all the solutions of the system (3.1) that initiates in $\mho$ remained bound in

$$\Theta = \{(S, Q, X, U, F, V) \in \mho_+|\mathfrak{L}(t) \leq N + 6\beta m + \epsilon, \quad \epsilon > 0\}.$$

□

6. Equilibrium Analysis

We analyze the rumor-free equilibrium and rumor equilibrium state of the system (3.1) here. Since the sixth equation of system (3.1) is a part of the Ignorant population, therefore while computing the equilibrium point we have not considered this equation of the system (3.1). The basic reproduction number (BR) and its sensitivity are examined. To find the points of equilibrium, from Eq. (3.1) we consider

$$\begin{aligned} N-(\phi_1+\phi_2)\nu SX-(\phi_1+\phi_2)\nu SU-\mu S &= 0,\\ (\phi_1+\phi_2)\nu SX+(\phi_1+\phi_2)\nu SU-\psi Q-\delta Q &= 0,\\ \eta\psi Q-rX &= 0,\\ \psi(1-\eta)Q-rU-\rho U &= 0,\\ rX+rU-\beta F &= 0,\\ \mu S+\delta Q+\rho U+\beta F-\xi V &= 0. \end{aligned} \tag{6.1}$$

If all the eigenvalues Θ_i, $i=1,2,...,\beta$, of the Jacobian matrix $J(\bar{E})$, $\bar{E}$ being the point of equilibrium, satisfying the condition

$$|arg(eig(J(\bar{E})))| = |arg(\Theta_i)| > \frac{\alpha\pi}{2},\ i=1,2,...,\beta. \tag{6.2}$$

then the $\bar{E}$ is a stable point of equilibrium. We determine these eigenvalues by solving the characteristic equation $|J(\bar{E})-\Theta_i I|=0$.

Lemma 6.1. *[42] Define the following characteristic equation*

$$P(\Theta)=\Theta^{\beta}+A_1\Theta^{\beta-1}+A_2\Theta^{\beta-2}+...A_{\beta}=0. \tag{6.3}$$

The conditions stated below make all the roots of the characteristic Eq. (6.3) satisfy the Eq. (6.2):

(1) For $\beta=1$, the condition for Eq. (6.2) is $A_1>0$.

(2) For $\beta=2$, the conditions for Eq. (6.2) are either Routh-Hurwitz conditions or $A_1>0$, $4A_2>A_1^2$, $|tan^{-1}\frac{\sqrt{4A_2-A_1^2}}{A_1}|>\frac{\alpha\pi}{2}$.

(3) For $\beta=3$, if the discriminant of the polynomial $P(\Theta)$ is positive then necessary and sufficient conditions to satisfy the Eq. (6.2) are

$$A_1>0,\ A_2>0,\ A_1A_2>A_3.$$

If the discriminant of the polynomial $P(\Theta)$ is negative then necessary and sufficient conditions to satisfy the Eq. (6.2) are

$$A_1>0,\ A_2>0,\ A_1A_2=A_3.$$

(4) For general β, $A_\beta>0$ is the necessary condition for Eq. (6.2) to be satisfied.

6.1. **Basic reproduction (BR) number.** On solving the system (6.1), we obtain the Rumor-free equilibrium (RFE) point $E_0=(\bar{S},0,0,0,0,0)=(\frac{N}{\mu},0,0,0,0,0)$. We have computed the BR number by evaluating the next-generation matrix. To compute BR number R_0, we consider

$$D_t^{\alpha}(\Delta(t)) = F(t)-V(t), \tag{6.4}$$

where

$$F(t)=\begin{pmatrix} S(X+U)\nu(\phi_1+\phi_2)\\ 0\\ 0\end{pmatrix} \text{ and } V(t)=\begin{pmatrix} Q\delta+Q\psi\\ rX-Q\eta\psi\\ \rho U+rU-Q(1-\eta)\psi\end{pmatrix}.$$

At RFE P_0, the Jacobian matrix of $F(t)$ and $G(t)$ are given as

$$J_F=\begin{pmatrix} 0 & \frac{\nu N(\phi_1+\phi_2)}{\mu} & \frac{\nu N(\phi_1+\phi_2)}{\mu}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \text{ and } J_V=\begin{pmatrix} \delta+\psi & 0 & 0\\ -\eta\psi & r & 0\\ (\eta-1)\psi & 0 & \rho+r\end{pmatrix}.$$

$$J_F J_V^{-1} = \begin{pmatrix} \frac{\nu N(r\psi - r\eta\psi)(\phi_1+\phi_2)}{\mu(\rho+r)(r\delta+r\psi)} + \frac{\nu N\eta\psi(\rho+r)(\phi_1+\phi_2)}{\mu(\rho+r)(r\delta+r\psi)} & \frac{N\nu(\delta+\psi)(\phi_1+\phi_2)}{\mu r(\delta+\psi)} & \frac{N\nu(\phi_1+\phi_2)}{\mu(\rho+r)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

is the next-generation matrix and the BR number is

$$R_0 = \frac{N\nu\psi(r+\rho\eta)(\phi_1+\phi_2)}{\mu r(\rho+r)(\delta+\psi)}. \tag{6.5}$$

6.2. **Stability of RFE.**

Theorem 6.2. *The RFE point E_0 of the projected fractional-order rumor model (3.1) is locally asymptotically stable if $R_0 < 1$ and is unstable if $R_0 > 1$.*

Proof. The Jacobian matrix at E_0 is

$$J(E_0) = \begin{pmatrix} -\mu & 0 & -\frac{N\nu(\phi_1+\phi_2)}{\mu} & -\frac{N\nu(\phi_1+\phi_2)}{\mu} & 0 & 0 \\ 0 & -(\delta+\psi) & \frac{N\nu(\phi_1+\phi_2)}{\mu} & \frac{N\nu(\phi_1+\phi_2)}{\mu} & 0 & 0 \\ 0 & \eta\psi & -r & 0 & 0 & 0 \\ 0 & (1-\eta)\psi & 0 & -(\rho+r) & 0 & 0 \\ 0 & 0 & r & r & -\beta & 0 \\ \mu & \delta & 0 & \rho & \beta & -\xi \end{pmatrix}.$$

Thus, the RFE E_0 is locally asymptotically stable if all the eigenvalues Θ_i, $i = 1, 2, \cdots, 5$, of the Jacobian matrix $J(E_0)$ satisfy the condition

$$|arg(eig(J(E_0)))| = |arg(\Theta_i)| > \frac{\alpha\pi}{2}, i = 1, 2, \cdots, 5. \tag{6.6}$$

The characteristic equation is

$$(\beta+\Theta)(\mu+\Theta)(\xi+\Theta)(\Theta^3 + A_1\Theta^2 + A_2\Theta + A_3) = 0.$$

This yields $\Theta_1 = -\delta < 0$, $\Theta_2 = -\lambda < 0$, and

$$\Theta^3 + A_1\Theta^2 + A_2\Theta + A_3 = 0, \tag{6.7}$$

where

$$\begin{aligned} A_1 &= \rho + 2r + \delta + \psi > 0, \\ A_2 &= r(\rho+r)\left(1 + \frac{\delta+\psi}{\rho+r}(\rho+2r) - R_0\frac{\delta+\psi}{r+\rho\eta}\right) > 0, \\ A_3 &= r(1-R_0)(r+\rho)(\delta+\psi). \end{aligned}$$

Clearly $A_1 > 0$, $A_2 > 0$. We have $A_3 > 0$ if $R_0 < 1$. This concludes that RFE point E_0 is stable if $R_0 < 1$. □

6.3. **Existence and stability of rumor equilibrium point.** The rumor equilibrium point is $E_1 = (S^*, Q^*, X^*, U^*, F^*, V^*)$, where

$$\begin{aligned}
S^* &= \frac{rN}{r\mu + 2Q\eta\nu\psi\phi_1 + 2Q\eta\nu\psi\phi_2} = \frac{N}{\mu R_0},\\
Q^* &= \frac{(R_0-1)\mu r}{2\eta\nu\psi(\phi_1+\phi_2)},\\
X^* &= \frac{Q\eta\psi}{r} = \frac{(R_0-1)\mu}{2\nu(\phi_1+\phi_2)},\\
U^* &= \frac{Q(\eta-1)\psi}{(\rho+r)} = \frac{(R_0-1)\mu r(\eta-1)}{2\eta r(\phi_1+\phi_2)(\rho+r)},\\
F^* &= \frac{(Q(r+\rho\eta)\psi}{(\rho+r)\beta} = \frac{(R_0-1)(r+\rho\eta)\mu r}{2\eta\nu(\rho+r)(\phi_1+\phi_2)},\\
V^* &= \frac{Q(\delta+\psi) + \frac{Nr\mu}{r\mu+2Q\eta\nu\psi(\phi_1+\phi+2)}}{\xi} = \frac{(R_0-1)\mu r(\delta+\psi)}{2\eta\nu\psi(\phi_1+\phi_2)} + \frac{N}{R_0}.
\end{aligned}$$

The rumor equilibrium point exists if $R_0 > 1$.

Theorem 6.3. *The rumor equilibrium E_1 of the projected fractional-order model (3.1) is locally asymptotically stable if $R_0 > 1$ and unstable otherwise.*

Proof. The Jacobian matrix at E_1 is

$$J(E_1) = \begin{pmatrix}
-\mu - \frac{(R_0-1)\mu}{2} + \frac{(R_0-1)r(\eta-1)}{2(\rho+r)\eta} & 0 & -\frac{N\nu(\phi_1+\phi_2)}{\mu+(R_0-1)\mu} & -\frac{N\nu(\phi_1+\phi_2)}{\mu+(R_0-1)\mu} & 0 & 0\\
\frac{(R_0-1)\mu}{2} - \frac{(R_0-1)r(\eta-1)\mu}{2(\rho+r)\eta} & -(\delta+\psi) & \frac{N\nu(\phi_1+\phi_2)}{\mu+(R_0-1)\mu} & \frac{N\nu(\phi_1+\phi_2)}{\mu+(R_0-1)\mu} & 0 & 0\\
0 & \eta\psi & -r & 0 & 0 & 0\\
0 & (1-\eta)\psi & 0 & -(\rho+r) & 0 & 0\\
0 & 0 & r & r & -\beta & 0\\
\mu & \delta & 0 & \rho & \beta & -\xi
\end{pmatrix}.$$

The characteristic polynomial is

$$\frac{1}{R_0}\frac{1}{(\rho+r)\eta\mu}(\Theta+\beta)(\Theta+\xi)(\Theta^4 + B_1\Theta^3 + B_2\Theta^2 + B_3\Theta^+B_4) = 0, \tag{6.8}$$

where

$$\begin{aligned}
B_1 &= R_0(\rho+r)\eta\mu,\\
B_2 &= \mu R_0\eta(p^2 + p(3r+\delta+\mu+)(R_0-1)\mu + 2\psi) + r(2r+\delta+\mu+\psi)),\\
B_3 &= R_0\mu\eta(p^2(r+\delta+\mu+(R_0-1)\mu+\psi+r^2(r+2\delta+\frac{(R_0-1)}{\eta}\mu+2\rho)+\\
&\quad \rho r(3\delta+(R_0-1)(\frac{\mu}{2\mu}+1)) + \mu(\delta+\psi)(r\frac{(R_0-1)}{2\eta}+1) + (R_0+1)\rho) + (\mu+\psi)(2r^2+3\rho r)\\
&\quad -\frac{N}{R_0\mu}(\rho+r)\nu\psi(\phi_1+\phi_2)),\\
B_4 &= R_0\mu\eta(\mu(\delta+\psi)(\rho^2\frac{(R_0+1)}{2} + r^2(\frac{(R_0-1)}{\eta}+2) + \rho r(\frac{(R_0-1)}{2\eta}+R_0+2)) +\\
&\quad +(\mu+\psi)(r^3+\rho r^2) + \rho^2 r(\frac{\delta}{\eta} + \frac{(R_0+1)\mu}{2\eta}) + 2r^2\rho\delta + \frac{(R_0-1)}{2}(\frac{1}{\eta}+1)\mu\rho r^2).
\end{aligned}$$

□

By Lemma 6.1, the rumor equilibrium point is stable if $R_0 > 1$.

6.4. **Global Stability.** We have presented the global asymptotic stability analysis of rumor-free equilibrium point in this section.

Theorem 6.4. *The RFE point E_0 is globally asymptotically stable if $Q > 0$, where $Q = m_7\xi - \mu m_2 - (\phi_1 + \phi_2)\nu\frac{N}{\mu}$.*

Proof. From positive definite function, we have

$$\begin{aligned}
H(S,Q,X,U,F,V) &= \left(S - \bar{S} - \bar{S}ln\left(\frac{S}{\bar{S}}\right)\right) + Q + X + U + F + V. \\
D_t^\alpha H(S,Q,X,U,F,V) &\leq \left(1 - \frac{\bar{S}}{S}\right) D_t^\alpha G + D_t^\alpha E + D_t^\alpha I + D_t^\alpha U + D_t^\alpha S \\
&= \left(\frac{S-\bar{S}}{S}\right)(N - (\phi_1+\phi_2)\nu SX - (\phi_1+\phi_2)\nu SU - \mu S) + (\phi_1+\phi_2)\nu SX + \\
&\quad (\phi_1+\phi_2)\nu SU - \psi Q - \delta Q + \eta\psi Q - rX + \psi(1-\eta)Q - rU - \rho U + rX \\
&\quad + rU - \beta F + \mu S + \delta Q + \rho U + \beta F - \xi V \\
&= \left(\frac{S-\bar{S}}{S}\right)(-\mu(S-\bar{S}) - (\phi_1+\phi_2)\nu S(X+U)) + (\phi_1+\phi_2)\nu S(X+U) + \\
&\quad \mu S - \xi V.
\end{aligned}$$

Let $m_1 < S < m_2$, $m_3 < X < m_4$, $m_5 < U < m_6$ and $m_7 < V < m_8$.
Then,

$$\begin{aligned}
D_t^\alpha H &\leq -\frac{\mu}{m_2}(S-\bar{S})^2 - \Big(m_7\xi - \mu m_2 - (\phi_1+\phi_2)\nu\frac{N}{\mu}) \\
&= -\frac{\mu}{m_2}(S-\bar{S})^2 - Q,
\end{aligned}$$

where

$$Q = m_7\xi - \mu m_2 - (\phi_1+\phi_2)\nu\frac{N}{\mu}.$$

Therefore, $D_t^\alpha H \leq 0 \quad \text{if} \quad Q > 0.$ □

6.5. **Sensitivity analysis of R_0.** Sensitivity indices are useful to compute the relative changes of a variable when a parameter changes. The sensitivity index of a variable is defined as the ratio of the relative change in the variable to the relative change in the parameter. The normalized forward sensitivity index of R_0, which depends on differentiably on a parameter p, is defined by

$$\mathfrak{B}_p^{R_0} = \frac{\partial R_0}{\partial p} \times \frac{p}{R_0}.$$

In any infection model sensitivity analysis is always performed by pointing out which parameters that affect the BR number and how they affect it [28, 55, 56]. In this study, we have analyzed the sensitivity of the BR number by evaluating the first derivative of the BR number concerning various parameters.

$$\begin{aligned}
\frac{\partial R_0}{\partial N} &= \frac{1}{N},\\
\frac{\partial R_0}{\partial \nu} &= \frac{1}{\nu},\\
\frac{\partial R_0}{\partial \phi_1} &= \frac{1}{\phi_1+\phi_2},\\
\frac{\partial R_0}{\partial \phi_2} &= \frac{1}{\phi_1+\phi_2},\\
\frac{\partial R_0}{\partial \psi} &= \frac{\delta(\delta+\psi)}{\psi^3(\frac{\delta}{\psi}+1)^2},\\
\frac{\partial R_0}{\partial \rho} &= \frac{r(\rho-1)(1-\eta)}{(\rho+r)(r+\rho\eta)},\\
\frac{\partial R_0}{\partial \mu} &= -\frac{1}{\mu},\\
\frac{\partial R_0}{\partial r} &= \frac{(r-1)(\rho^2\eta+\eta\rho(r+1)+r)}{r(\rho+r)(r+\rho\eta)},\\
\frac{\partial R_0}{\partial \delta} &= -\frac{(1+\psi)}{(\delta+\psi)},\\
\frac{\partial R_0}{\partial \eta} &= \frac{\rho}{(r+\rho\eta)}.
\end{aligned}$$

From the above computation, we observe that the BR number is directly proportional to N, ν, ϕ_1, ϕ_2, ψ and inversely proportional to μ, δ. BR number proportionality to ρ and r can be controlled by making the rate η at which the exposed population progress towards rumor spreading population, verifying the rate of the rumor by unintentional rumor spreader ρ and the probability that individual spreader becomes stifler r, less than 1. In other words, we have identified the specific conditions under which the verifying rate of rumor by unintentional rumor spreaders and the probability of an individual spreader becoming a stifler exhibit negative sensitivity indices. These findings presented in Table 2 indicate the scenarios in which the propagation of rumors is more likely to be increased.

Table 2. Sensitivity indices of parameter values.

Parameters	**Description**	**Sensitivity Indices**
N	The total population in Social Network	+ ve
ϕ_1	Verbal contact rate	+ ve
ϕ_2	Digital contact rate	+ ve
ν	Rumor transmission ratio	+ ve
μ	Fact retrieving rate	- ve
ψ	The rate at exposed populations being heard of rumor	+ ve
δ	The rate of which E progresses to F	-ve
η	Rate at which exposed population become spreader	+ ve
ρ	Verifying the rate of rumor by unintentional rumor spreader	-ve ($0 \leq \rho < 1$ & $0 \leq \eta < 1$)
r	The probability that individual spreader becomes stifler	-ve ($0 \leq r < 1$)
ξ	Forgetting rate	+ve

7. Numerical Method

In this part, we have presented the numerical method employed to solve the rumor-spreading model (3.1). Consider the nonlinear equation below

$$D_t^\alpha x(t) = f(t, x(t)), \quad 0 \le t \le T,$$

$$x^{(m)}(0) = x_0^{(m)}, \quad m = 0, 1, 2, 3, ..., \nu..., \nu = [\alpha].$$

The corresponding Volterra integral equation may be written as

$$x(t) = \sum_{m=0}^{\nu-1} x_0^{(m)} \frac{t^m}{m!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds. \tag{7.1}$$

To integrate Eq. (7.1), the Adams-Bashforth Moulton method has been used. Setting $h = \frac{T}{\mathbb{N}}$, $t_n = nh$, $n = 0, 1, 2, ..., \mathbb{N} \in Z^+$ and using the Adams-Bashforth Moulton technique, the solution to the system (3.1) can be presented as:

$$\begin{aligned}
S_{n+1} &= S_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \left(N - (\phi_1+\phi_2)\nu S_{n+1}^p X_{n+1}^p - (\phi_1+\phi_2)\nu S_{n+1}^p U_{n+1}^p - \mu S_{n+1}^p\right) \\
&\quad + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} a_{i,n+1} \left(N - (\phi_1+\phi_2)\nu S_i X_i - (\phi_1+\phi_2)\nu S_i U_i - \mu S_i\right), \\
Q_{n+1} &= Q_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \left((\phi_1+\phi_2)\nu S_{n+1}^p X_{n+1}^p + (\phi_1+\phi_2)\nu S_{n+1}^p U_{n+1}^p - \psi Q_{n+1}^p - \delta Q_{n+1}^p\right) \\
&\quad + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} a_{i,n+1} \left((\phi_1+\phi_2)\nu S_i X_i + (\phi_1+\phi_2)\nu S_i U_i - \psi Q_i - \delta Q_i\right), \\
X_{n+1} &= X_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \left(\eta\psi Q_{n+1}^p - r X_{n+1}^p] + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} a_{i,n+1} [\eta\psi Q_i - r X_i\right), \\
U_{n+1} &= U_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \left(\psi(1-\eta) Q_{n+1}^p - r U_{n+1}^p - \rho U_{n+1}^p\right) \\
&\quad + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} a_{i,n+1} \left(\psi(1-\eta) Q_i - r U_i - \rho U_i\right), \\
F_{n+1} &= F_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \left(r X_{n+1}^p + r U_{n+1}^p - \beta F_{n+1}^p\right) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} a_{i,n+1} \left(r X_i + r U_i - \beta F_i\right), \\
V_{n+1} &= V_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \left(\mu S_{n+1}^p + \delta Q_{n+1}^p + \rho U_{n+1}^p + \beta F_{n+1}^p - \xi V_{n+1}^p\right) \\
&\quad + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} a_{i,n+1} \left(\mu S_i + \delta Q_i + \rho U_i + \beta F_i - \xi V_i\right),
\end{aligned}$$

where

$$
\begin{aligned}
S_{n+1}^{p} &= S_0 + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{i=0}^{n} b_{i,n+1} \left(N - (\phi_1+\phi_2)\nu S_i X_i - (\phi_1+\phi_2)\nu S_i U_i - \mu S_i\right), \\
Q_{n+1}^{p} &= Q_0 + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{i=0}^{n} b_{i,n+1} \left((\phi_1+\phi_2)\nu S_i X_i + (\phi_1+\phi_2)\nu S_i U_i - \psi Q_i - \delta Q_i\right), \\
X_{n+1}^{p} &= X_0 + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{i=0}^{n} b_{i,n+1} \left(\eta_I \xi_E E_i - r_I I_i - \delta I_i\right), \\
U_{n+1}^{p} &= U_0 + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{i=0}^{n} b_{i,n+1} \left(\psi(1-\eta) Q_i - r U_i - \rho U_i\right), \\
F_{n+1}^{p} &= F_0 + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{i=0}^{n} b_{i,n+1} \left(r X_i + r U_i - \beta F_i\right), \\
V_{n+1}^{p} &= V_0 + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{i=0}^{n} b_{i,n+1} \left(\mu S_i + \delta Q_i + \rho U_i + \beta F_i - \xi V_i\right),
\end{aligned}
\tag{7.2}
$$

in which

$$
a_{i,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & i = 0, \\ (n-i+2)^{\alpha+1} + (n-i)^{\alpha+1} - 2(n-i+1)^{\alpha+1}, & 1 \le i \le n, \\ 1, & i = n+1, \end{cases}
$$

and

$$
b_{i,n+1} = ((n-i+1)^{\alpha} - (n-i)^{\alpha}), \quad 0 \le i \le n.
$$

TABLE 3. Estimated and adopted parameter values and variables used in the simulation of the model.

Parameters	**Value**	**Source**
N	10	Assumed
ϕ_1	0.0398	Assumed
ϕ_2	0.00155	Assumed
ν	0.74	[49]
μ	0.028	Assumed
ψ	0.5	[59]
δ	0.5	Assumed
η	0.955	Assumed
β	0.5	Assumed
ρ	0.5	Assumed
r	0.74	[49]
ξ	0.167	[49]

8. Numerical Simulation

The numerical simulation of the fractional rumor-spreading model (3.1) is discussed in this component. We have taken the population in the social network. For the simulation, we used the fractional predictor-corrector method and Mathematica software. We have assumed the initial conditions for the six groups of populations as below:
$S_0(t) = 7.04$, $Q_0(t) = 0.2873$, $X_0(t) = 0.1432$, $U_0(t) = 2.875$, $F_0(t) = 0.082$, $V_0(t) = 0.208$. Using these initial data along with the data tabulated in Table 3, we have plotted the graphs of ignorant, exposed, intentional rumor spreader, unintentional rumor spreaders, stifler, and fact retriever for $\alpha = 1$, 0.9, 0.8, 0.75 and 0.6 to see the effect of FDs on

different population groups. In Figure 2, we have presented the influence of fractional order derivative on the rumor spread dynamics of the proposed model. In each category of population, we observe that the incorporation of the fractional order derivative accelerates the rate of convergence of the trajectories. From this figure, one can notice that, when $\alpha = 1$, rumors decline quickly in society. This case is not possible in most events since once the rumor spreads widely, it takes some time to come out of people's thinking. When rumors reach their peak and things get intense, using fractional order derivatives, we can show that they gradually settle down to a stable situation. Figures 3, 4 and 5 represent the effect of different digital contact rate $\phi_1 = 0.0298, 0.0398, 0.0498, 0.598, 0.698$ on ignorants, exposed, intentional rumor spreader, and unintentional rumor spreader, stifler and fact retriever population. In Figure 6, 7, and 8, we have intended to figure out the impact of verbal contact rate in the spread of rumor contagion for $\phi_2 = 0.00100,\ 0.00155,\ 0.00255, 0.00355$, and 0.00455. From Figure 3 to 8, concerning different fractional order $\alpha = 1, \alpha = 0.9, \alpha = 0.8$, we can notice that, as the contact rate increases, exposed, intentional rumor spreader, unintentional rumor spreader, and fact retrievers also increase and ignorants population declines faster than integer order derivative. These figures show that whenever the contact rate increases, ignorants are becoming exposed. Hence exposed, intentional, unintentional rumor spreader population increases and also it can be seen that when the contact rate is higher, rumor spreading will be more. Consequently, the rumor will be heard by more people in less time. As a result, there is more chance that people get to hear the same rumor repeatedly which leads to a loss of interest and curiosity to spread, causing increase in the stifler population. One of the significant observations here we can make out is, as the contact rate increases, fact retriever also increases in order to control and prevent the rumors. The fractional operator in the system and its numerical simulation using the predictor-corrector approach, have important implications for examining and forecasting the future of the suggested model. Figure 9, 10, and 11 indicates the different verifying rate of the rumor by unintentional rumor spreader for $\rho = 0.3,\ 0.5,\ 0.6,\ 0.8$ and 1 at $\alpha = 1, 0.85$ and 0.6. As the verifying rate of rumors increases, we can observe a decline in rumor spread. Also, we notice slow convergence towards stability. Incorporation of the fractional derivative fasten the convergence of the system towards the point of stability. Therefore, to represent a rumor spread situation, where an increase in the verifying rate speeds up the stability of the system, the current system (3.1) shall act as an efficient mathematical model.

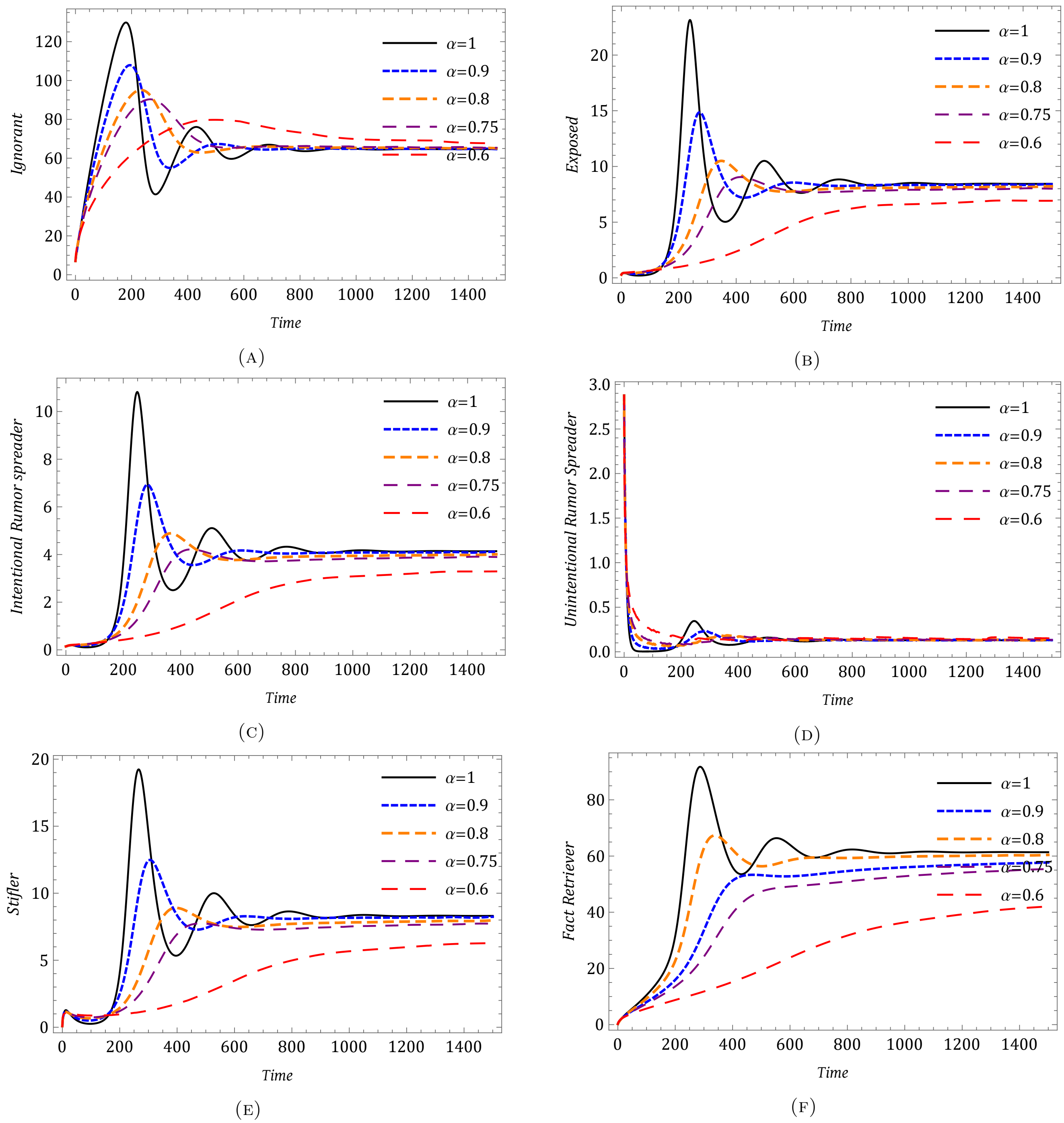


FIGURE 2. Numerical simulation for different values of α.

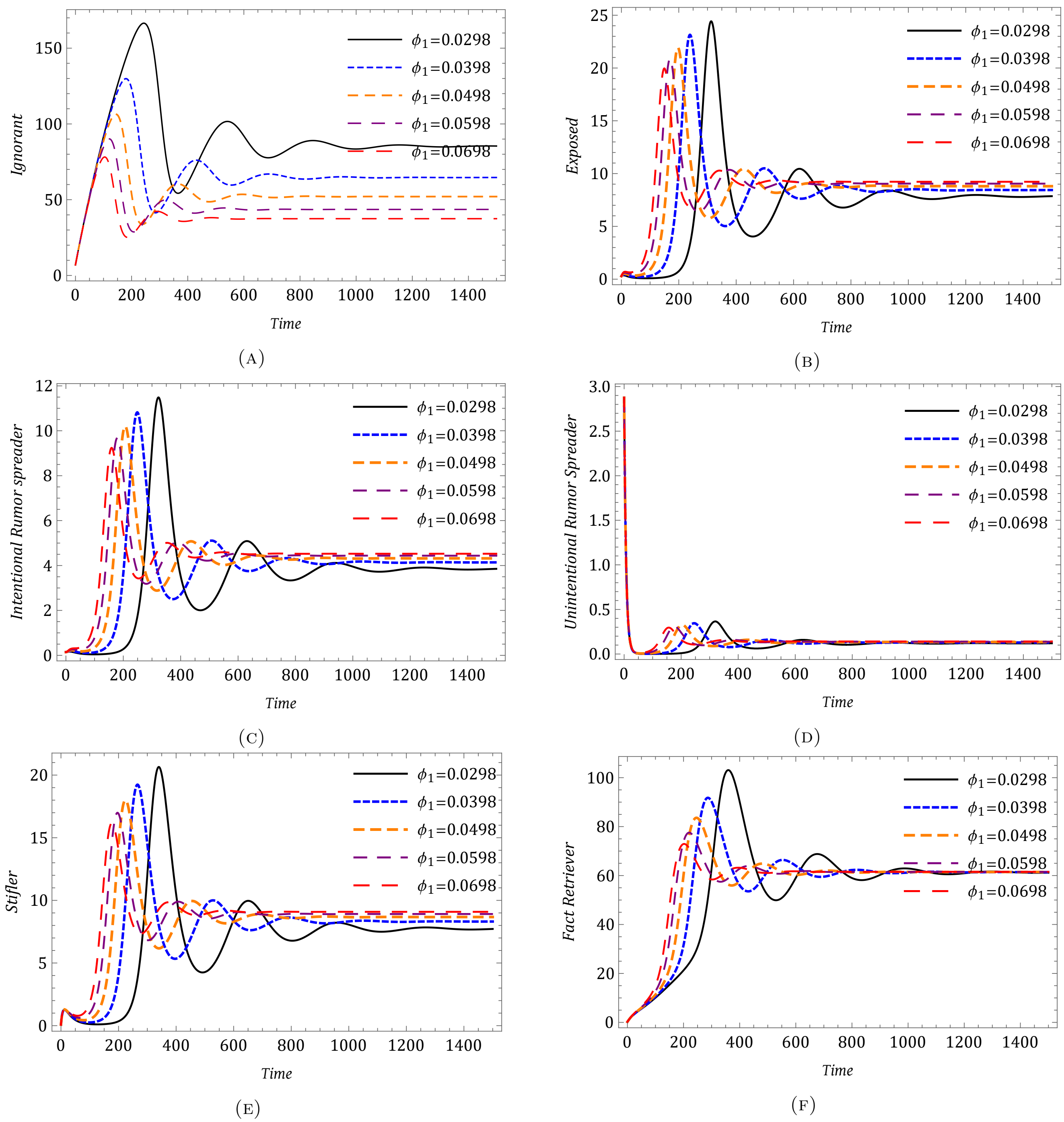


FIGURE 3. Numerical simulation for $\alpha = 1$ concerning different digital contact rate.

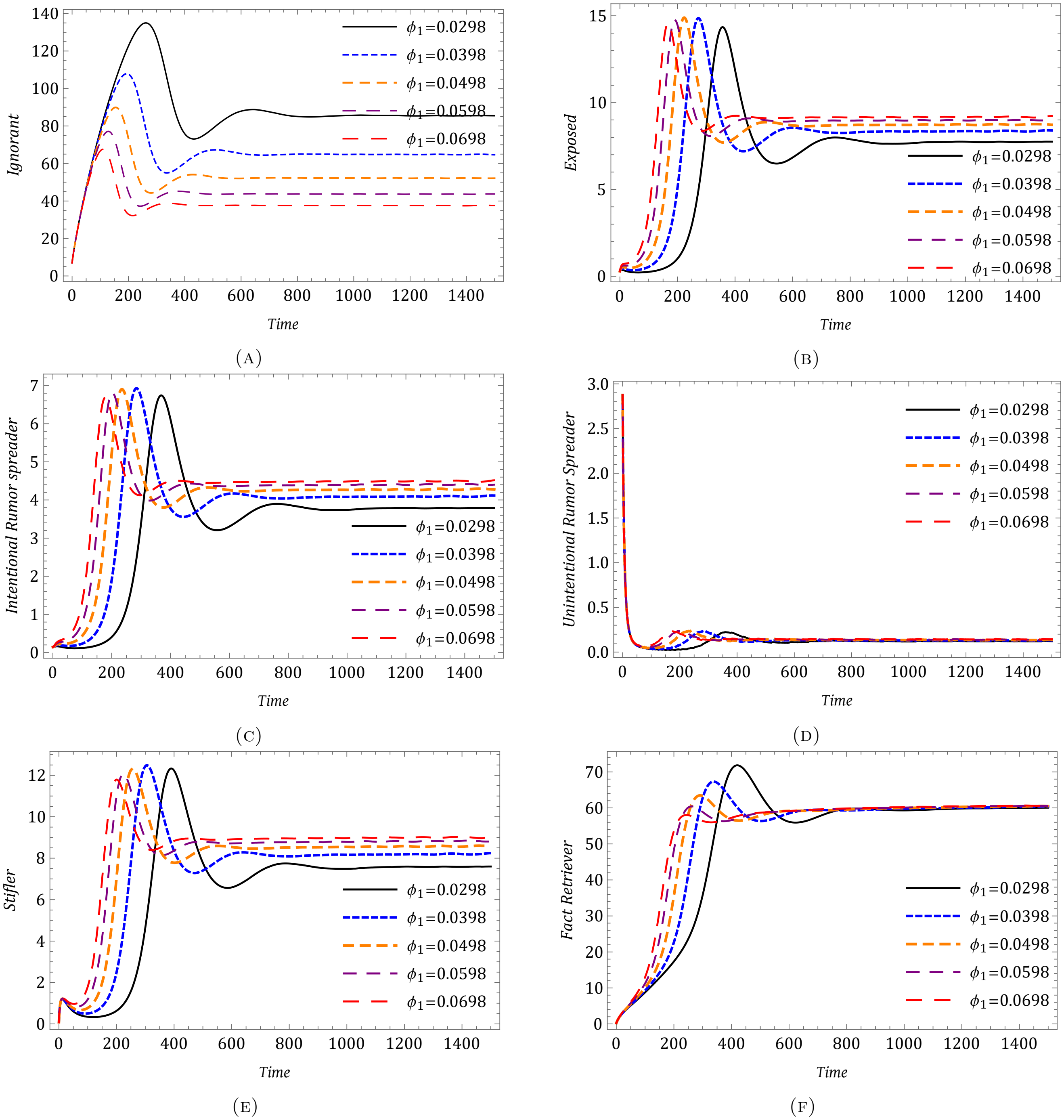


FIGURE 4. Numerical simulation for $\alpha = 0.9$ with respect to different digital contact rate.

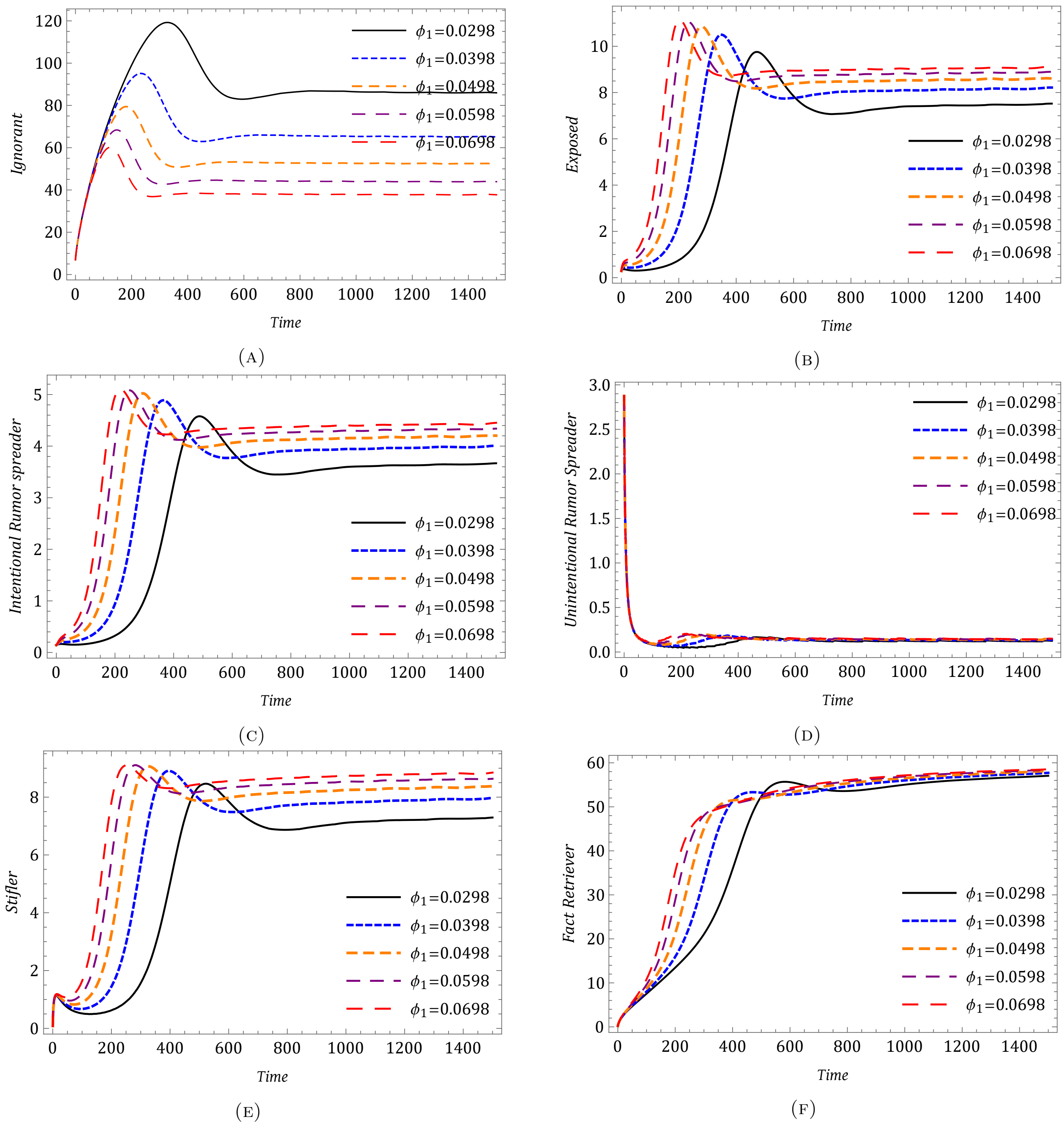
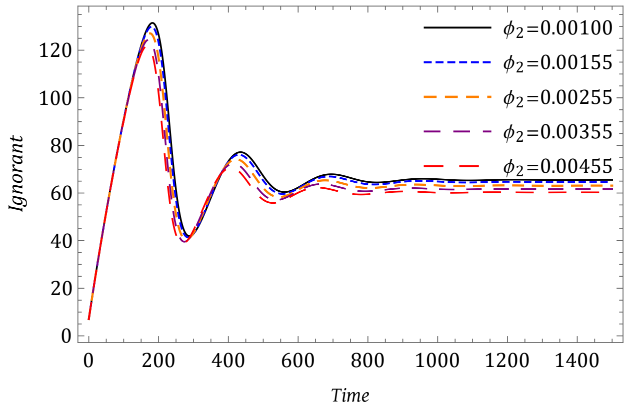
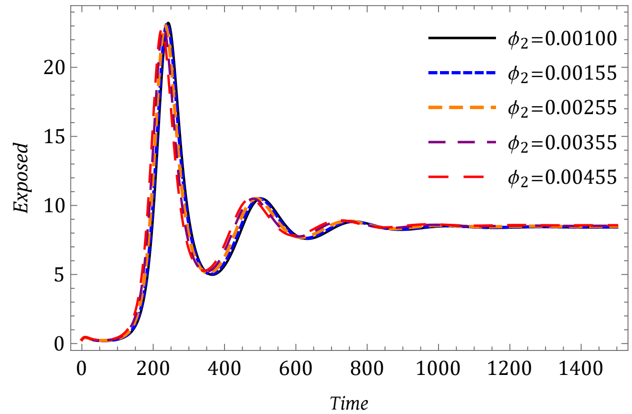


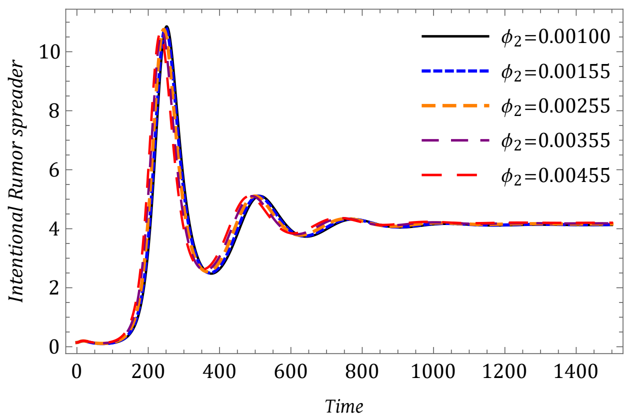
FIGURE 5. Numerical simulation for $\alpha = 0.8$ with respect to different digital contact rate.



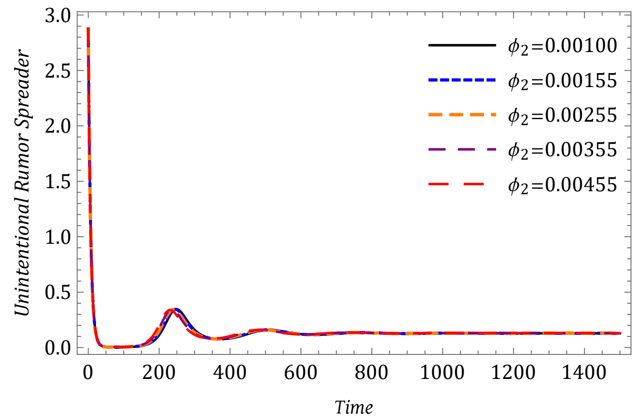
(A)



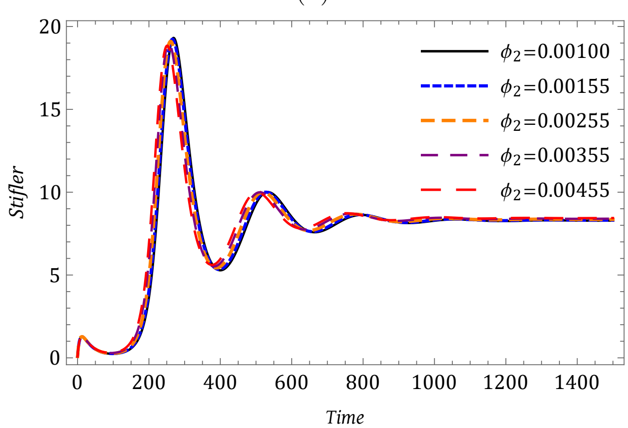
(B)



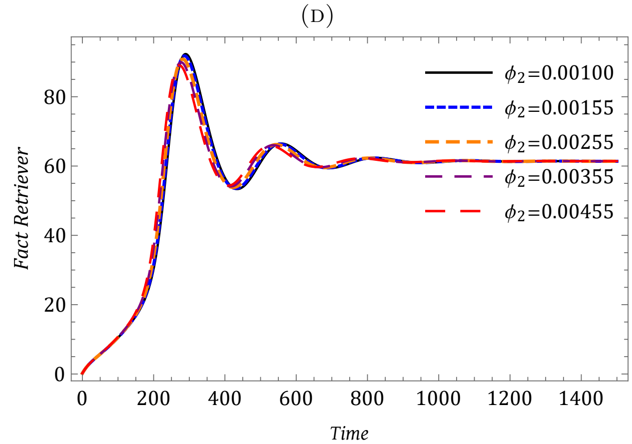
(C)



(D)



(E)



(F)

FIGURE 6. Numerical simulation for $\alpha = 1$ with respect to different verbal contact rate.

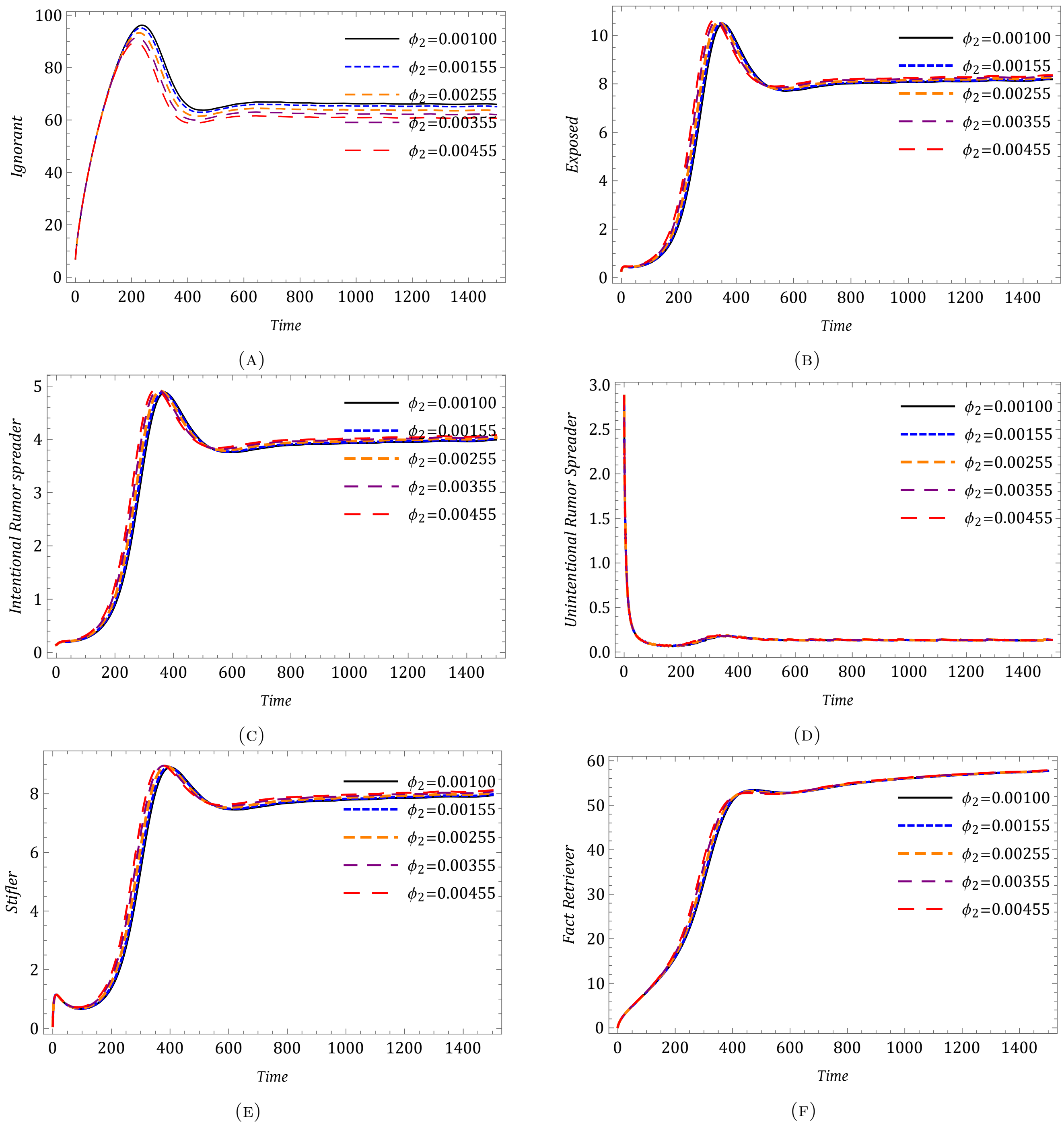


Figure 7. Numerical simulation for $\alpha = 0.9$ with respect to different verbal contact rate.

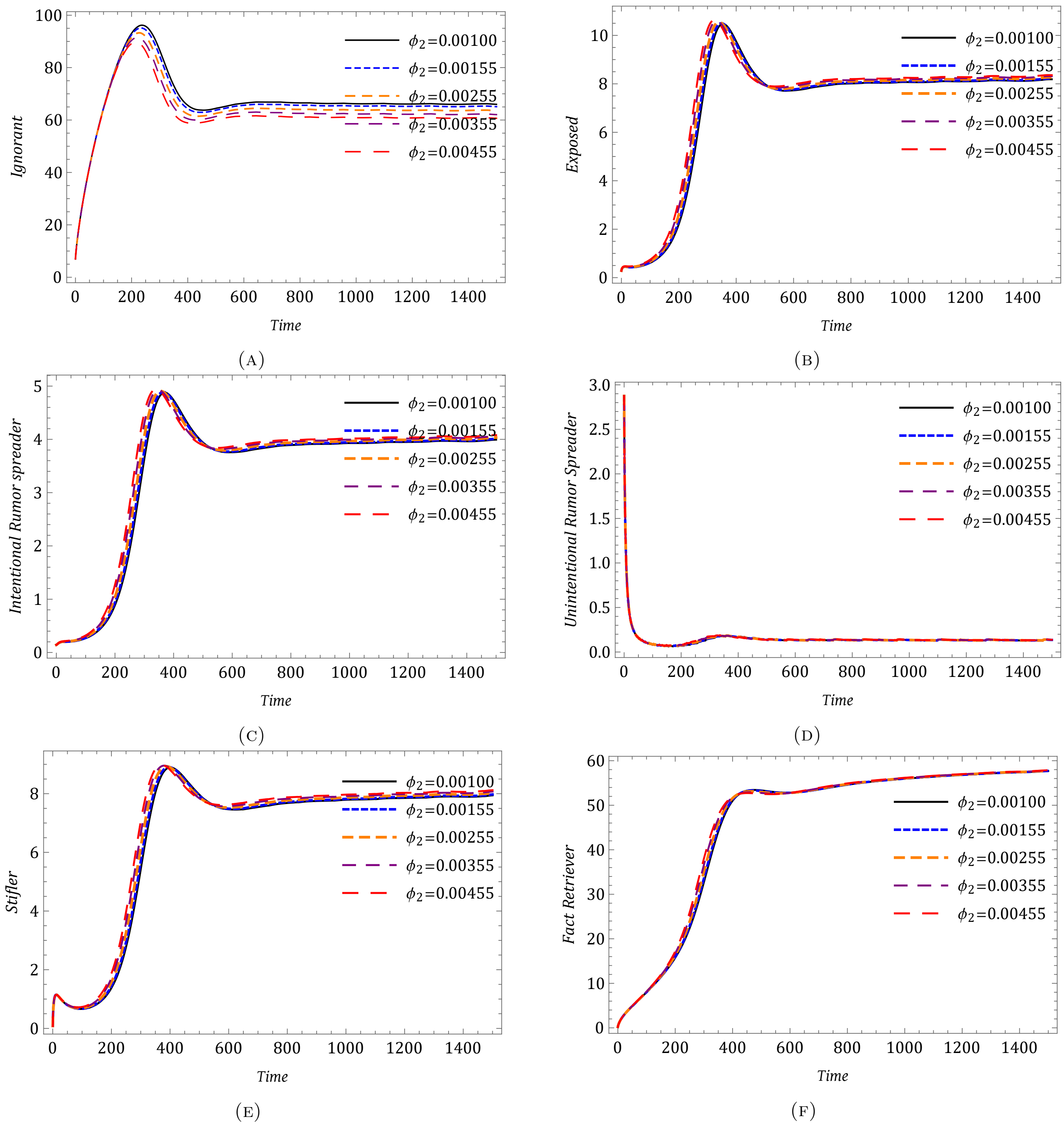


FIGURE 8. Numerical simulation for $\alpha = 0.8$ with respect to different verbal contact rate.

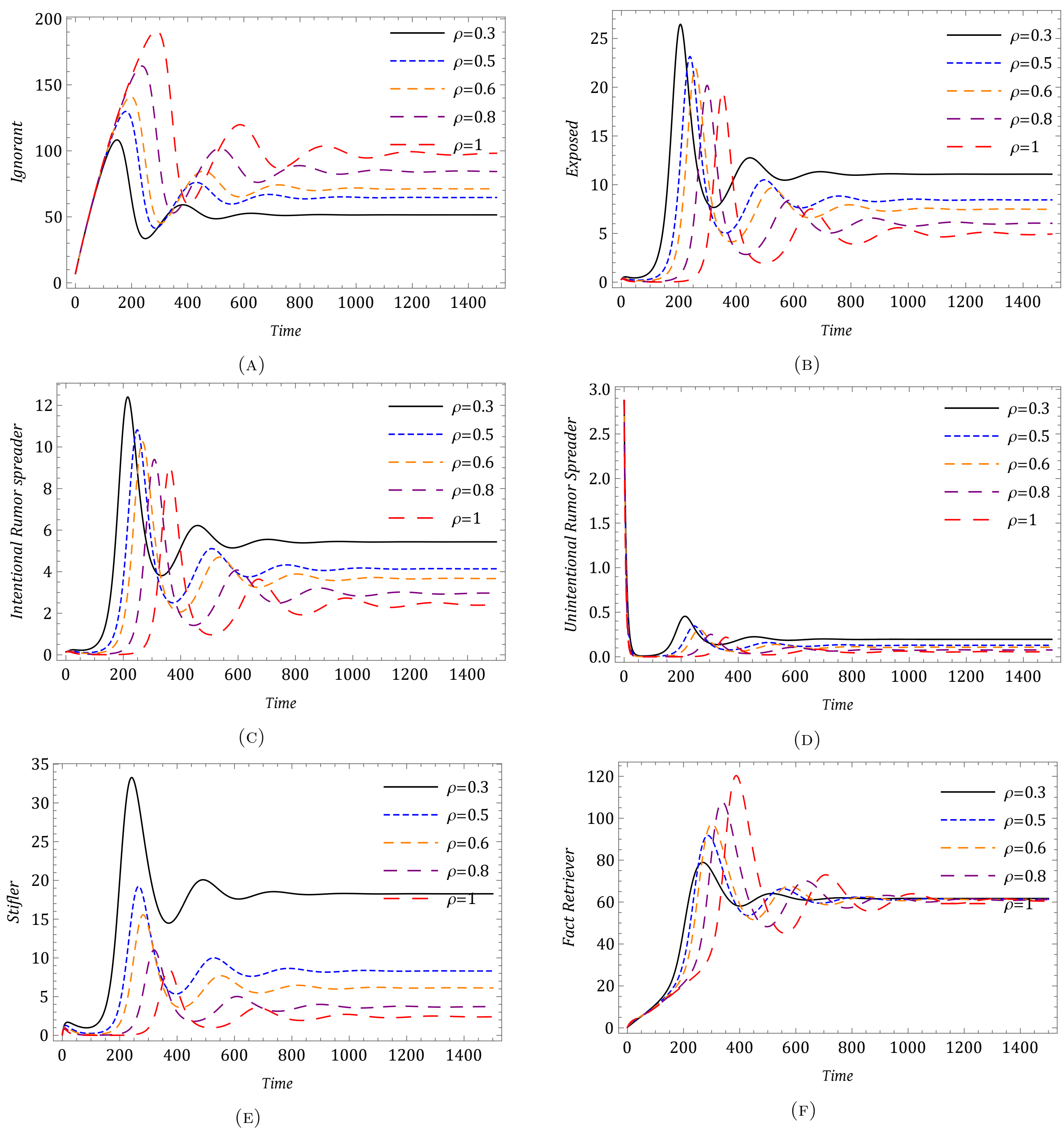
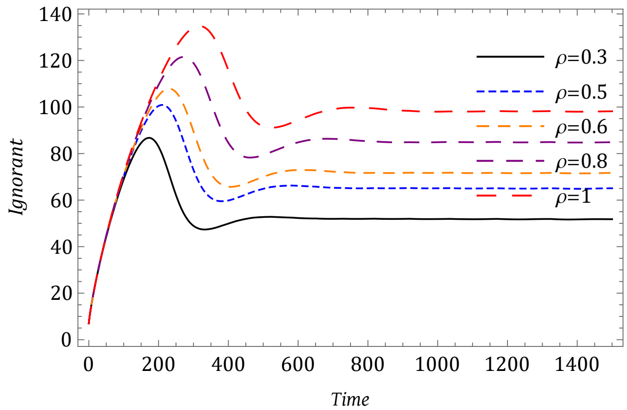
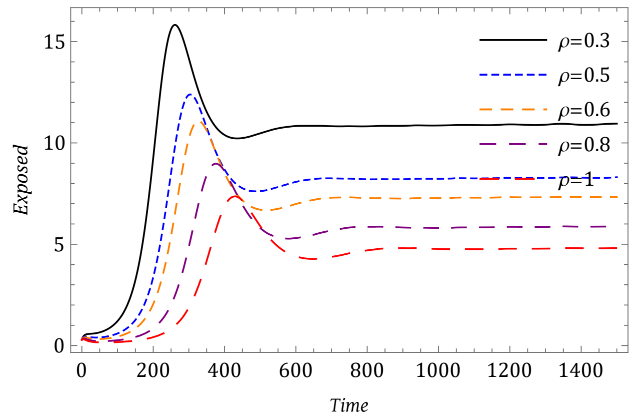


FIGURE 9. Numerical simulation for $\alpha = 1$ with respect to different verifying rate of rumor by unintentional rumor spreader.



(A)



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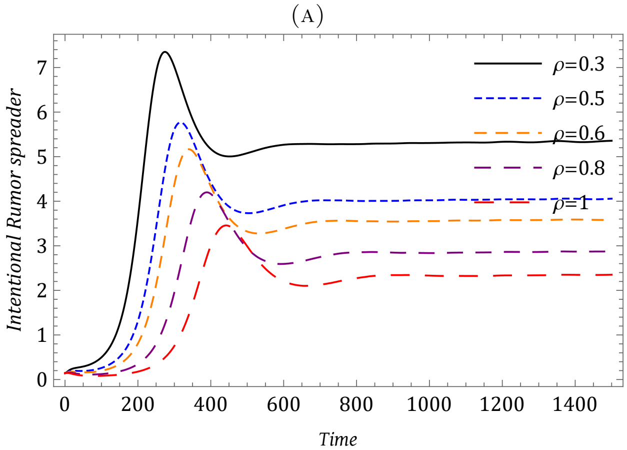

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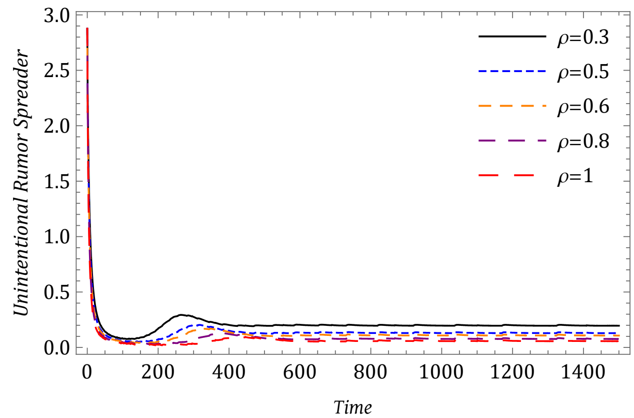

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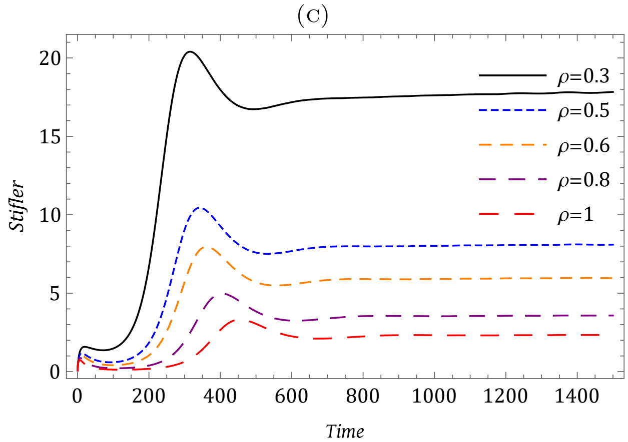

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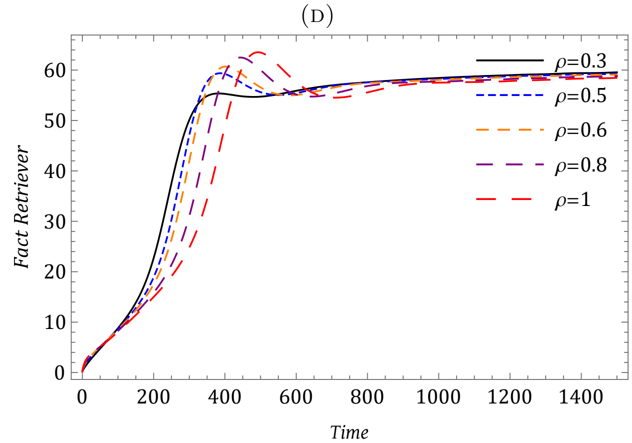

(F)

Figure 10. Numerical simulation for $\alpha = 0.85$ with respect to different verifying rate of rumor by unintentional rumor spreader.

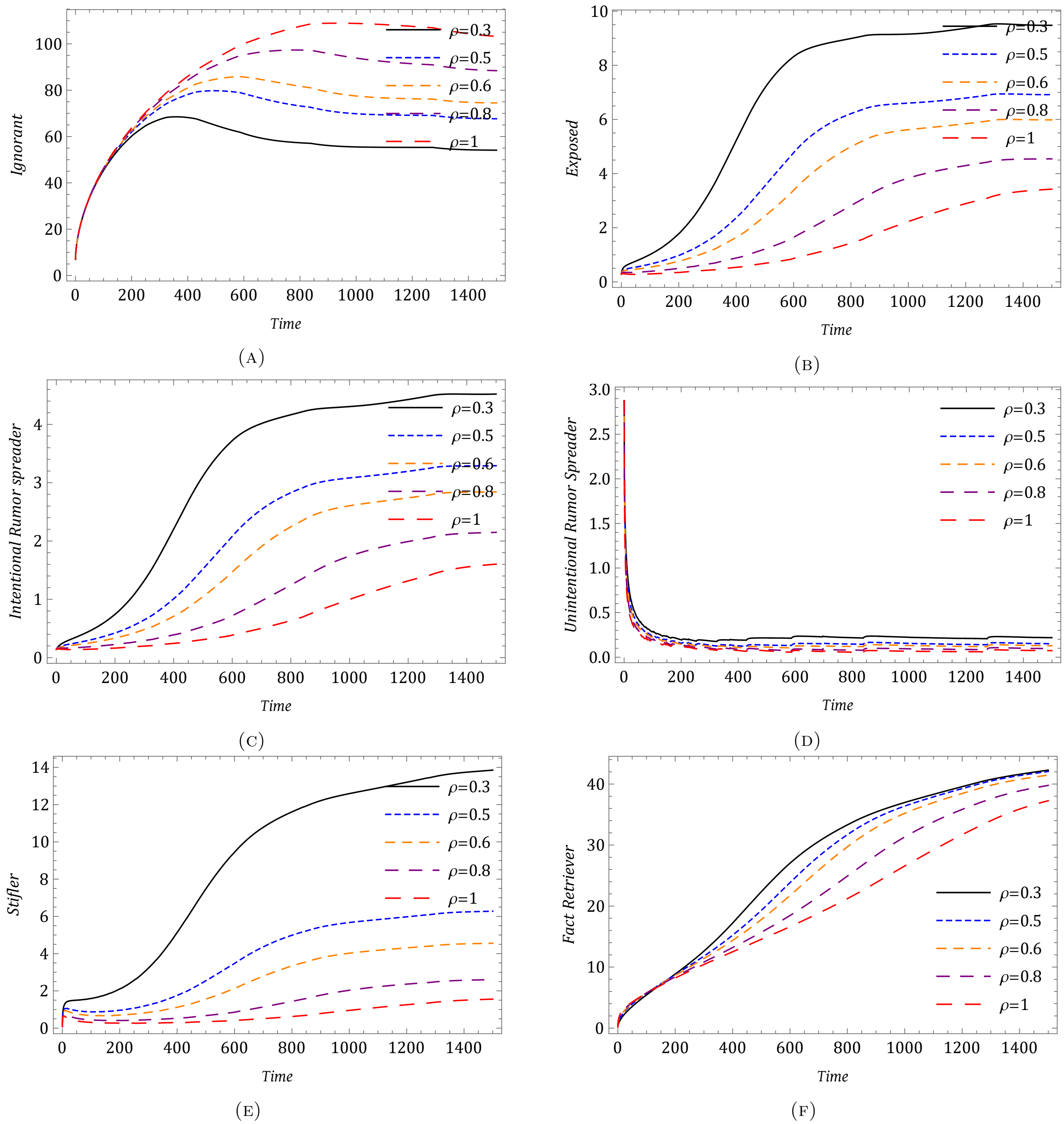

FIGURE 11. Numerical simulation for $\alpha = 0.6$ with respect to different verifying rate of the rumor by unintentional rumor spreader.

9. Conclusion

The main contribution of this study is the analysis of a mathematical model for propagating the rumor spreading in the presence of a fact retriever in the frame of the Caputo fractional-order derivative. We studied the existence and uniqueness of the solutions using the fixed point theory. The BR number R_0 was calculated using the next-generation matrix approach, and it acts as a threshold parameter in rumor circulation, determining whether the rumor continues or disappears from the population. The existence of a rumor equilibrium point has been investigated. For the projected fractional-order system, the local asymptotic stability conditions of the RFE and rumor equilibrium points have been discussed in terms of BR number, and it has been established that the rumor-free equilibrium point is stable if $R_0 < 1$, and the rumor equilibrium point is stable if $R_0 > 1$. We have developed the essential requirements for achieving global stability of equilibrium points by creating the Lyapunov function. In the numerical investigation, we examine the evolution of rumor, the effect of fact retriever, and the effect of contact rate in its spread by evaluating the connection of Caputo fractional operator with Ignorants, Exposed, Intentional rumor spreaders, Unintentional rumor spreaders, Stifler and Fact retriever in the contagion of the rumor. Fact verification and contact effect are exemplified to understand the evolution and forecast its development globally. Incidences of serious loss caused by rumor have decreased as fact retrieve rates have increased; rumor spreading among the people who came to know what is fact infrequent and cases of major loss like due to war, suicide, depression, and panic psychology are now mostly seen in those population who never heard of facts. This influence of fact retrieving in curbing the spread of the rumor is depicted by computing the BR number. Numerical simulation obtained by using the predictor-corrector technique and the computational mathematical tool Mathematica records convincing results. It is observed that plots presented using the fractional-order value of the derivative reveal a more realistic pattern of the rumor spread than the integer-order derivative. The current study emphasizes the importance of using a mathematical model when discussing real-world problems and the performance of the fractional operator under consideration. Furthermore, the projected solution procedure for solving the system of fractional differential equations is highly methodical and effective.

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