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Modified simple equation method (MSEM) for solving nonlinear (3+1)-dimensional space-time fractional equations

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Abstract

In the present paper, modified simple equation method (MSEM) is implemented for obtaining exact solutions of three nonlinear $(3+1)$-dimensional space-time fractional equation, namely three types of modified Korteweg-de-Vries (mKdV) equations. Here, the derivatives are of the type of conformable fractional derivatives. The solving process produces a system of algebraic equations which is possible to be easily with no need of using software for determining unknown coefficients. Results show that this method can supply a powerful mathematical tool to construct exact solutions of mKdV equations and it can be employed for other nonlinear $(3+1)$ - dimensional space-time fractional equations.

Keywords. Modified simple equation method (MSEM), Exact solutions, ($3+1$)- Dimensional fractional equations, Conformable fractional derivative.

2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. Introduction

In almost every field of science and engineering, including nuclear, plasma, chemical and solid state physics, optical fibers, fluid mechanics, biology, etc. Nonlinear phenomena are revealed. Mathematical modeling of most physics systems results in nonlinear evolution equations (NLEE). Indeed, a lot of NLEEs are broadly utilized to explain these physical phenomena. Accordingly, trying to find solutions to NLEEs is highly essential, and developing efficient methods to get analytic and numerical solutions of such equations have been popular by researchers. For example see [1, 5, 13, 17, 30].

Exact solutions to nonlinear evolution equations play a significant role in nonlinear physical science, because various natural phenomena, such as solitons, vibrations, and propagation with a finite speed can be described by these solutions well [29]. There are abundant powerful methods to obtain the exact solutions of such equations including, Hirota's method [19], extended tanh-function method [2], (G'/G)-expansion method [6, 25], exp-function method [15, 32], homogeneous balance method [16, 28], modified auxiliary equation method [4], Jacobi elliptic function expansion method [21], optimal perturbation iteration method [27], Weierstrass elliptic function method [14], modified simple equation method (MSEM) [10, 20], generalized projective Riccati equations method [3], and so on.

The KdV equation as a celebrated NLEE plays an important role in modeling various phenomena. It is the governing equation for shallow waves of water interacting weakly and nonlinearly, long internal waves in a density-stratified ocean, ion-acoustic waves in a plasma, and acoustic waves on a crystal lattice. Additionally, it describes a string in the Fermi-PastaUlam-Tsingou problem in the continuum limit. This equation is a famous model for solitons and it is a significant foundation to investigate other equations [9]. Thus, investigating the exact solutions of this equation is significant because they are useful to understand the physics behind them. Also, the higher the dimensions of the model, the more realistic; in literature multiple modifications to the KdV equation have been suggested including, the $(3+1)$-dimensional modified KdV equations by [18] and [31] as follows.

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$$u_t + 6u^2 u_x + u_{xyz} = 0, \tag{1.1}$$

$$u_t + 6u^2 u_y + u_{xyz} = 0, \tag{1.2}$$

$$u_t + 6u^2 u_z + u_{xyz} = 0. \tag{1.3}$$

These equations play an important role in three-dimensional non-linear dispersion problems [26].

Fractional differential equations (FDEs) are generalizations of ordinary differential equations to an arbitrary order. The theory of derivatives and integrals of fractional order allows us to describe physical phenomena more accurately[23]. Many phenomena in engineering, physics, chemistry, biology, and some other sciences may be modeled using FDEs. For example, [7] have transformed the classical model of a cubic isothermal auto-catalytic chemical system (CIACS) into a new fractional form by using three different and special fractional operators. [8] have formulated a fractional optimal control model of spread of severe acute respiratory syndrome-coronavirus 2 (SARS-CoV-2) in Atangana-Baleanu-Caputo derivative sense. [11] have obtained a new model for Chua's circuit by transforming the classical model of Chua's circuit into novel forms of various fractional derivatives. [13] have proposed an optimal perturbation iteration procedure with the Laplace transform to solve the fractional type of damped Burgers' equation which is obtained by remodeling the classical damped Burgers' equation to fractional differential form via the Atangana-Baleanu fractional derivatives described with the help of the Mittag-Leffler function. [26] introduced the conformable fractional order derivative of Equations ((1.1)-(1.3)).

This work aims to employ MSEM to examine exact solutions to three types of (3+1) dimensional fractional modified Korteweg-de-Vries (mKdV) equation introduced by [26].

The outline of the current paper is as follows: The description of the conformable fractional derivative and its properties are presented in Section 2. The MSEM is introduced in Section 3. The method is implemented for the $(3 + 1)$-dimensional fractional (mKdV) equation in Section 4. In the end, the conclusions are supplied in Section 5.

2. The conformable fractional derivative and its properties

In this section, the conformable fractional derivative is defined. Then, some of its properties are presented according to what was first reported by [22].

Definition 2.1. [26].
Suppose that $y : [0, \infty) \to \mathbb{R}$. Then the conformable fractional derivative of y of order α is described as follows.

$$D_t^{\alpha}(y(x)) = \lim_{\epsilon \to 0} \frac{y\left(x + \epsilon x^{1-\alpha}\right) - y(x)}{\epsilon},$$

for all $x > 0$ and $\alpha \in (0, 1]$.

Theorem 2.2. *[22].*
If a function $y : [0, \infty) \to \mathbb{R}$ is α-differentiable at $x_0 > 0, \alpha \in (0, 1]$, then y is continuous at x_0.

Proof. [22].
Suppose that y and z are α-differentiable at $x > 0$ and $\alpha \in (0, 1]$. Some properties of the conformable fractional derivative is presented in the following [26].

(*i*) $D_x^{\alpha}(x^c) = cx^{c-\alpha}$ for all $c \in \mathbb{R}$.
(*ii*) $D_x^{\alpha}(a) = 0$, for all constant functions $y(x) = a$.
(*iii*) $D_x^{\alpha}(ay(x) + bz(x)) = aD_x^{\alpha}y(x) + bD_x^{\alpha}z(x)$ for all $a, b \in \mathbb{R}$.

(vi) $D_x^\alpha(y(x)z(x)) = D_x^\alpha y(x)(z(x)) + z(x)D_t^\alpha(y(x))$.
(v) $D_x^\alpha\left(\frac{y(x)}{z(x)}\right) = \frac{z(x)D_x^\alpha y(x) - y(x)D_x^\alpha z(x)}{z^2(x)}$, $z(x) \neq 0$.
(iv) If y is differentiable then, $D_x^\alpha(y(x)) = x^{1-\alpha}\frac{dy}{dx}$.

□

3. Description of MSEM

It is well-known that MSEM is employed to solve a nonlinear ordinary differential equation (ODE) as follows [20].

$$Q\left(Y(\xi), \frac{d}{d\xi}Y(\xi), \frac{d^2}{d\xi^2}Y(\xi), \ldots\right) = 0, \tag{3.1}$$

where Q is a polynomial of the unknown function Y and its derivatives. To solve Eq. (3.1), its solution $U(\xi)$ is expanded in the following finite series.

$$Y(\xi) = \sum_{j=0}^{N} c_j \left(\frac{\omega'(\xi)}{\omega(\xi)}\right)^j, \tag{3.2}$$

in which $c_0, \ldots, c_N$ are constants which must be specified, provided that $c_N \neq 0$, and ω is an unknown function which is specified subsequently, so that $\omega' \neq 0$. The simple equation method is basic designed on the presumption that some special functions satisfy some ODEs. These ODEs are attributed as the simplest equations. The simplest equation has two main properties: firstly, the order equation is lesser than the order of Eq. (3.1); secondly, the general solution of this equation is known. This implies that we can obtain the exact solutions $Y(\xi)$ of (3.2) by a finite series (3.1) in the general solution of the simplest equation [20]

3.1. **Algorithm of MSEM.** To express the algorithm of MSEM, the following general nonlinear fractional differential equation is considered.

$$G\left(u, D_t^\alpha u, D_x^{\alpha_1} u, D_y^{\alpha_2} u, \ldots\right) = 0, \tag{3.3}$$

where $\alpha, \alpha_1, \alpha_2$ are fractional orders and G is a polynomial in u and its fractional partial derivatives. The principal stages of the MSEM for solving Eq. (3.3) are as follows.

1: Substitute the wave transformation (Nuruddeen 2018)

$$u(x,t) = Y(\xi), \quad \xi = a\frac{x^{\alpha_1}}{\alpha_1} + b\frac{y^{\alpha_2}}{\alpha_2} - r\frac{t^\alpha}{\alpha}, \tag{3.4}$$

where a, b and r are nonzero constants, into Eq. (3.3). This leads to a reduced ODE of the polynomial form as (3.1).

2: Assume that the solution of Eq. (3.1) can be presented by a polynomial in $\frac{\omega'}{\omega}$ as Eq. (3.2).

3: Balance the highest order derivatives with the highest order nonlinear term in Eq. (3.1) to determine the positive integer N in (3.2). To find out this procedure we focus on the leading terms of (3.1). These are the terms that lead to the least positive p when a monomial $Y = a/\xi^p$ is substituted in all the components of the equation. The value of N is found by the homogeneous balancing between the leading terms.

4: Substitute Eq.(3.2) in Eq.(3.1) and collect all terms with the same power of $\omega^{-i}, i = 0, \ldots, N$ together. In this case, a polynomial of $\frac{\omega'}{\omega}$ and its derivatives is achieved. Then, equate each coefficient to zero. This process results in producing a system of equations that is possible to be solve with no need of software to determine $c_0, \ldots, c_N$ and ω. Consequently, an exact solution of Eq. (3.1) and subsequently an exact solution of Eq. (3.3) can be achieved.

4. Application

In this section, by applying the algorithm described in the previous section, some exact solutions for the $(3+1)$-dimensional fractional modified Korteweg-de-Vries (mKdV) equations. Graphical representations are carried out by Maple software.

4.1. $(3+1)$-dimensional modified Korteweg-de-Vries equation (mKdV).

4.1.1. *The first equation.* Consider the first $(3+1)$-dimensional mKdV equation as follows [26]

$$D_t^{\alpha} u + 6 D_x^{\alpha} u^3 + D_{xyz}^{3\alpha} u = 0. \tag{4.1}$$

Using the wave transformation,

$$u(x,y,z,t) = Y(\xi), \quad \xi = a\frac{x^{\alpha}}{\alpha} + b\frac{y^{\alpha}}{\alpha} + c\frac{z^{\alpha}}{\alpha} - r\frac{t^{\alpha}}{\alpha}, \tag{4.2}$$

gives

$$-rY' + 6a\left(Y^3\right)' + abcY''' = 0. \tag{4.3}$$

Integrating Eq. (4.3) with zero constant of integration leads to

$$-rY + 6aY^3 + abcY'' = 0. \tag{4.4}$$

In addition, we have $N = 1$ by using the homogeneous balance method. So, Eq. (3.2) has the following form.

$$Y(\xi) = c_0 + c_1\frac{\omega'(\xi)}{\omega(\xi)}, \tag{4.5}$$

by computing Y^3 and Y'' according to Eq. (4.5), we have

$$Y^3 = c_0^3 + 3c_0^2 c_1 \frac{\omega'}{\omega} + 3c_0 c_1^2\left(\frac{\omega'}{\omega}\right)^2 + c_1^3\left(\frac{\omega'}{\omega}\right)^3, \tag{4.6}$$

and

$$Y'' = c_1\left(\frac{\omega'''}{\omega} - 3\frac{\omega'\omega''}{\omega^2} + 2\left(\frac{\omega'}{\omega}\right)^3\right). \tag{4.7}$$

Substituting Eqs. (4.5), (4.7) into Eq. (4.4), and then putting all the coefficients of ω^0, ω^{-1}, ω^{-2}, and ω^{-3} equal to zero, leads to:

$$\omega^0 : -rc_0 + 6ac_0^3 = 0, \tag{4.8}$$

$$\omega^{-1} : abcc_1\omega''' - rc_1\omega' + 18ac_0^2 c_1\omega' = 0, \tag{4.9}$$

$$\omega^{-2} : 18ac_0c_1^2\omega'^2 - 3abcc_1\omega'\omega'' = 0. \tag{4.10}$$

and

$$\omega^{-3} : \quad 6ac_1^3\omega'^3 + 2abcc_1\omega'^3 = 0. \tag{4.11}$$

By simplifying the equations ((4.8), (4.11)), we have

$$c_0 = \pm \frac{\sqrt{r}}{\sqrt{6a}}, \quad a, r > 0, \tag{4.12}$$

$$abc\omega''' + 2r\omega' = 0, \tag{4.13}$$

$$\omega' = \frac{bc}{6c_0c_1}\omega'', \tag{4.14}$$

and

$$c_1 = \pm \frac{\sqrt{-bc}}{\sqrt{3}}, \quad bc < 0. \tag{4.15}$$

Substituting (4.12) and (4.15) into Eq. (4.14) gives

$$\omega' = -\frac{\sqrt{-abc}}{\sqrt{2r}}\omega''. \tag{4.16}$$

By substituting (4.16) into Eq. (4.13), we have

$$\omega''' + \sqrt{2r}d\omega'' = 0, \tag{4.17}$$

where $d = \frac{1}{\sqrt{-abc}}$. A solution of Eq. (4.17) is given by

$$\omega(\xi) = A + Be^{-\gamma\xi}, \tag{4.18}$$

where $\gamma = \sqrt{2r}d$, and A and B are arbitrary free parameters.

Consequently, substituting Eq. (4.18) for $\omega(\xi)$ into Eq. (4.4) for $u(x,y,z,t)$ gives the following exact solution.

$$u(x,y,z,t) = \pm\left(\frac{\sqrt{r}}{\sqrt{6a}} + \frac{\sqrt{-bc}}{\sqrt{3}}\frac{B\gamma e^{-\gamma\left(a\frac{x^\alpha}{\alpha}+b\frac{y^\alpha}{\alpha}+c\frac{z^\alpha}{\alpha}-r\frac{t^\alpha}{\alpha}\right)}}{A + Be^{-\gamma\left(a\frac{x^\alpha}{\alpha}+b\frac{y^\alpha}{\alpha}+c\frac{z^\alpha}{\alpha}-r\frac{t^\alpha}{\alpha}\right)}}\right). \tag{4.19}$$

For $a = b = r = A = B = 1$ and $c = -1$, we have

$$u(x,y,z,t) = \pm\left(\frac{\sqrt{6}}{6} + \frac{\sqrt{3}}{3}\frac{\sqrt{2}e^{\frac{-\sqrt{2}}{\alpha}(x^\alpha+y^\alpha-z^\alpha-t^\alpha)}}{1+e^{\frac{-\sqrt{2}}{\alpha}}(x^\alpha+y^\alpha-z^\alpha-t^\alpha)}\right). \tag{4.20}$$

The solution $u(x,y,z,t)$ in Eq. (4.21) with positive sign, at $y = z = t = 1$ and for $\alpha = 0.25, 0.5, 0.75, 1$ is represented in Figure 1. Also, the solution $u(x,y,z,t)$ in Eq. (4.21) at $y = z = 1$ and for $\alpha = 1$ is shown in Figure 2.

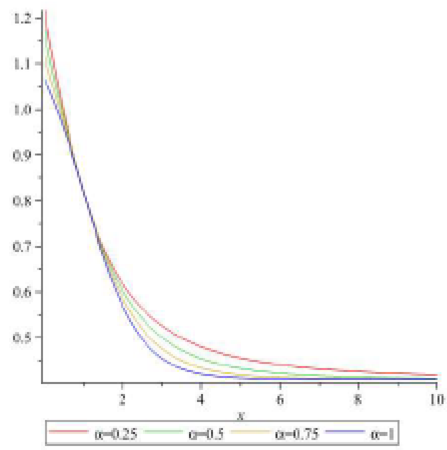

FIGURE 1. The soluation $u(x, 1, 1, 1)$ in Eq. (4.21) for different values of α.

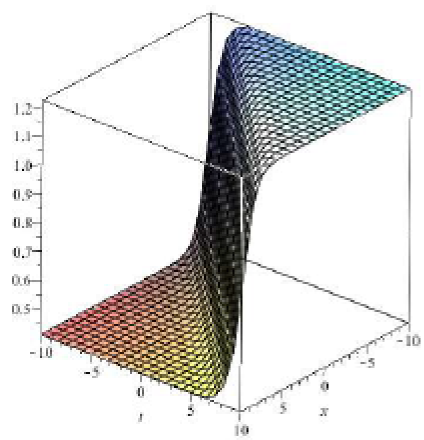

FIGURE 2. The solution $u(x, 1, 1, t)$ in Eq. (4.21) for $\alpha = 1$.

4.1.2. **The second equation.** Consider the second $(3+1)$-dimensional fractional mKdV equation as follows [26].

$$D_t^{\alpha} u + 6D_y^{\alpha} u^3 + D_{xyz}^{3\alpha} u = 0. \tag{4.21}$$

Applying the wave transformation (4.2) and integrating the resulting equation with zero constant of integration, gives

$$-rY + 6bY^3 + abcY'' = 0. \tag{4.22}$$

Besides, we obtain $N = 1$ by homogeneous balancing. So, Eq. (4.22) has a solution as Eq. (4.5). Substituting Eqs. (4.5), (4.6) into Eq. (4.22), and setting all the coefficients of $\omega^0, \omega^{-1}, \omega^{-2}$, and ω^{-3} equal to zero, leads to:

$$\omega^0 : -rc_0 + 6bc_0^3 = 0, \tag{4.23}$$
$$\omega^{-1} : abcc_1\omega''' - rc_1\omega' + 18bc_0^2c_1\omega' = 0, \tag{4.24}$$
$$\omega^{-2} : 18bc_0c_1^2\omega'^2 - 3abcc_1\omega'\omega'' = 0, \tag{4.25}$$

and

$$\omega^{-3} : \quad 6bc_1^3\omega'^3 + 2abcc_1\omega'^3 = 0. \tag{4.26}$$

From equations ((4.23)-(4.26)), we can get the following results:

$$c_0 = \pm\frac{\sqrt{r}}{\sqrt{6b}}, \quad b, r > 0, \tag{4.27}$$
$$abc\omega''' + 2r\omega' = 0, \tag{4.28}$$
$$\omega' = \frac{ac}{6c_0c_1}\omega'', \tag{4.29}$$

and

$$c_1 = \pm\frac{\sqrt{-ac}}{\sqrt{3}}, \quad ac < 0. \tag{4.30}$$

Substituting (4.27) and (4.30) into Eq. (4.29) gives

$$\omega' = -\frac{\sqrt{-abc}}{\sqrt{2r}}\omega'', \tag{4.31}$$

which is the same as Eq. (4.16). Substituting (4.31) into Eq. (4.28), gives a similar equation as Eq. (4.17). Hence, Eq. (4.21) has a solution as follows.

$$u(x,y,z,t) = \pm\left(\frac{\sqrt{r}}{\sqrt{6b}} + \frac{\sqrt{-ac}}{\sqrt{3}}\frac{B\gamma e^{-\gamma\left(a\frac{x^\alpha}{\alpha}+b\frac{y^\alpha}{\alpha}+c\frac{z^\alpha}{\alpha}-r\frac{t^\alpha}{\alpha}\right)}}{A+Be^{-\gamma\left(a\frac{x^\alpha}{\alpha}+b\frac{y^\alpha}{\alpha}+c\frac{z^\alpha}{\alpha}-r\frac{t^\alpha}{\alpha}\right)}}\right). \tag{4.32}$$

For $b = a = 1, r = 6, A = B = 1$ and $c = -3$, we have

$$u(x,y,z,t) = \pm\left(1 + 2\frac{e^{\frac{-2}{\alpha}(x^\alpha+y^\alpha-3z^\alpha-6t^\alpha)}}{1+e^{\frac{-2}{\alpha}(x^\alpha+y^\alpha-3z^\alpha-6t^\alpha)}}\right). \tag{4.33}$$

The solution $u(x,y,z,t)$ in Eq. (4.33) with positive sign, at $x = z = t = 1$ and for $\alpha = 0.4, 0.6, 0.8, 1$ is shown in Figure 3. Also, in Figure 4, the solution $u(x,y,z,t)$ in Eq. (4.33) at $x = z = 1$ and for $\alpha = 1$ is plotted.

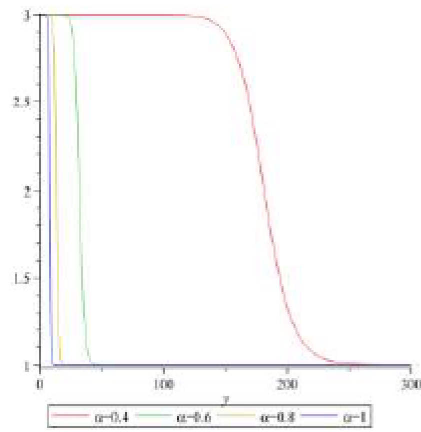

FIGURE 3. The solution $u(1, y, 1, 1)$ in Eq. (4.33) for different values of α.

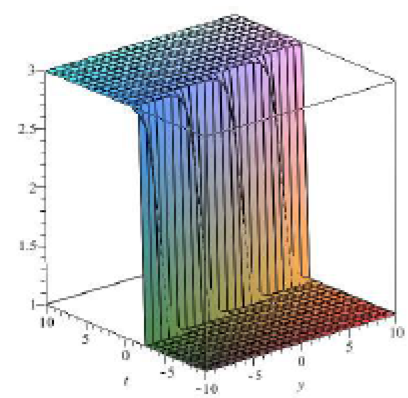

FIGURE 4. The solution $u(1, y, 1, t)$ in Eq. (4.33) for $\alpha = 1$.

4.1.3. **The third equation.** Consider the third (3+1)-dimensional conformable space-time fractional mKdV equation as follows [26].

$$D_t^{\alpha} u + 6D_z^{\alpha} u^3 + D_{xyz}^{3\alpha} u = 0. \tag{4.34}$$

Applying the wave transformation (4.2) and integrating the resulting equation with zero constant of integration, gives

$$-rY + 6cY^3 + abcY'' = 0. \tag{4.35}$$

Also, homogeneous balancing results in $N = 1$. So, Eq. (4.35) has a solution as Eq. (4.5).

By substituting Eqs. (4.5)-(4.6) into Eq. (4.35), and setting all the coefficients of ω^0, ω^{-1}, ω^{-2},and ω^{-3} equal to zero, we get:

$$\omega^0 : -rc_0 + 6cc_0^3 = 0, \tag{4.36}$$

$$\omega^{-1} : abcc_1\omega''' - rc_1\omega' + 18cc_0^2c_1\omega' = 0, \tag{4.37}$$

$$\omega^{-2} : 18cc_0c_1^2\omega'^2 - 3abcc_1\omega'\omega'' = 0, \tag{4.38}$$

and

$$\omega^{-3} : \quad 6cc_1^3\omega'^3 + 2abcc_1\omega'^3 = 0. \tag{4.39}$$

Manipulating equations ((4.36), (4.39)), gives the following results.

$$c_0 = \pm\frac{\sqrt{r}}{\sqrt{6c}}, \quad c, r > 0, \tag{4.40}$$

$$abc\omega''' + 2r\omega' = 0, \tag{4.41}$$

$$\omega' = \frac{ab}{6c_0c_1}\omega'', \tag{4.42}$$

and

$$c_1 = \pm\frac{\sqrt{-ab}}{\sqrt{3}}, \quad ab < 0. \tag{4.43}$$

Substituting (4.40) and (4.43) into Eq. (4.42) and then substituting the resulted equation into Eq. (4.41), gives a similar equation as Eq. (4.17). Finally, Eq. (4.34) has a solution as follows.

$$u(x,y,z,t) = \pm\left(\frac{\sqrt{r}}{\sqrt{6c}} + \frac{\sqrt{-ab}}{\sqrt{3}}\frac{B\gamma e^{-\gamma\left(a\frac{x^\alpha}{\alpha}+b\frac{y^\alpha}{\alpha}+c\frac{z^\alpha}{\alpha}-r\frac{t^\alpha}{\alpha}\right)}}{A + Be^{-\gamma\left(a\frac{x^\alpha}{\alpha}+b\frac{y^\alpha}{\alpha}+c\frac{z^\alpha}{\alpha}-r\frac{t^\alpha}{\alpha}\right)}}\right). \tag{4.44}$$

For $a = 4, c = 0.5, r = 3, A = B = 1$ and $b = -3$, we have

$$u(x,y,z,t) = \pm\left(1 + 2\frac{e^{\frac{-1}{\alpha}(4x^\alpha - 3y^\alpha + 0.5z^\alpha - 3t^\alpha)}}{1 + e^{\frac{-1}{\alpha}(4x^\alpha - 3y^\alpha + 0.5z^\alpha - 3t^\alpha)}}\right). \tag{4.45}$$

The solution $u(x,y,z,t)$ in Eq. (4.45) with positive sign, at $x = y = t = 1$ and for $\alpha = 0.4, 0.6, 0.8, 1$ is shown in Figure 1. Also, in Figure 2, the solution $u(x,y,z,t)$ in Eq. (4.45) at $x = y = 1$ and for $\alpha = 1$ is plotted.

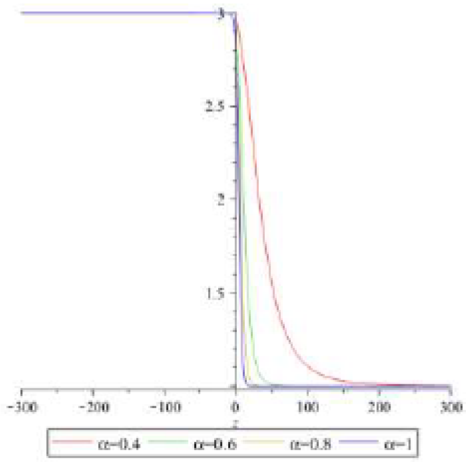

FIGURE 5. The solution $u(1, 1, z, 1)$ in Eq. (4.45) for different values of α.

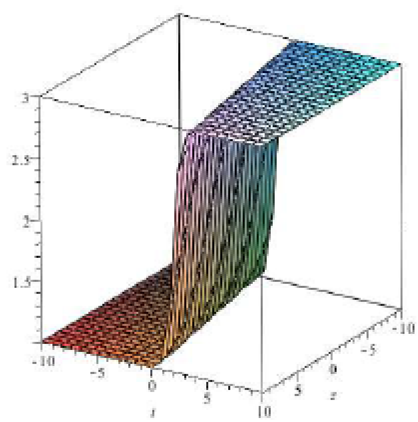

FIGURE 6. The solution $u(1, 1, z, t)$ in Eq. (4.45) for $\alpha = 1$.

5. CONCLUSION

In this work, MSEM has been utilized to obtain exact solutions for three types of $(3 + 1)$ dimensional space-time conformable fractional derivatives mKdV equations. By this method, a system of algebraic equations is derived which is possible to be solved easily with no need of using software. Also, 3-dimensional (3D) and 2-dimensional representations (2D) of all studied three types of $(3 + 1)$-dimensional space-time conformable fractional derivatives mKdV equations have been given in Figures 1-6. These figures show the kink soliton solutions for different values of α, parameters, and variables. Results show that this method can provide a reliable technique for constructing exact solutions for mKdV equations and it can be employed for other nonlinear $(3 + 1)$ - dimensional space-time fractional equations.

References

[1] P. Agarwal, S. Deniz,and S. Jain, AA. Alderremy ,and S. Aly., *A new analysis of a partial differential equation arising in biology and population genetics via semi analytical techniques*, Physica A: Statistical Mechanics and its Applications, *15*(542) (2020) , 122769.

[2] M. S. Ahmed, A. A. Zaghrout, and H. M. Ahmed, *Travelling wave solutions for the doubly dispersive equation using improved modified extended tanh-function method*, Alexandria Engineering Journal, *61*(10) (2022), 7987–7994.

[3] G. Akram, M. Sadaf, S. Arshed, and F. Sameen, *Bright, dark, kink singular and periodic soliton solutions of Lakshmanan-Porsezian-Daniel model by generalized projective Riccati equations method*, Optik, *241* (2021), 167051.

[4] G. Akram, M. Sadaf, and I. Zainab, *The dynamical study of Biswas-Arshed equation via modified auxiliary equation method*, Optik, *255* (2022), 168614.

[5] N. H. Aljahdaly and F. O. Al Zobidi, *On the Schrödinger equation for deep water waves using the Padé-Adomian decomposition method*, Journal of Ocean Engineering and Science, (2022).

[6] J. Biazar and Z. Ayati, *Improved G'/G-expansion method and comparing with tanh-coth method*, Applications and Applied Mathematics: An International Journal (AAM), *6*(1) (2011), 20.

[7] N. Bildik and S. Deniz, *A comparative study on solving fractional cubic isothermal autocatalytic chemical system via new efficient technique*, Chaos, Solitons & Fractals, *132* (2020), 109555.

[8] E. Bonyah, A. K. Sagoe, D. Kumar, and S. Deniz, *Fractional optimal control dynamics of coronavirus model with Mittag-Leffler law*, Ecological Complexity, 45 (2021), 100880.

[9] J. L. Brent, E. N. Onder, and A. P. Andrew, *Chapter 12 - Nonlinear differential equations, Advanced Mathematics for Engineering Students*, Butterworth-Heinemann, (2022), 329–347.

[10] C. Chen, *Singular solitons of Biswas-Arshed equation by the modified simple equation method*, Optik, *184* (2019), 412–420.

[11] M. De la Sen, S. Deniz, and H. Sözen, *A new efficient technique for solving modified Chua's circuit model with a new fractional operator.*, Advances in Difference Equations, *1* (2021), 1–16.

[12] S. Deniz, A. Konuralp, and M. De la Sen, *Optimal perturbation iteration method for solving fractional model of damped Burgers' equation*, Symmetry, *12*(6) (2020), 958.

[13] S. Deniz and M. Sezer, *Rational Chebyshev collocation method for solving nonlinear heat transfer equations*, International Communications in Heat and Mass Transfer, *114* (2020), 104595.

[14] A. El Achab, *Constructing of exact solutions to the nonlinear Schrödinger equation (NLSE) with power-law nonlinearity by the Weierstrass elliptic function method*, Optik, *127*(3) (2016), 1229–1232.

[15] E. M. Eskandari and N. Taghizadeh, *Exact Solutions of Two Nonlinear Space-time Fractional Differential Equations by Application of Exp-function Method.*, Applications and Applied Mathematics: An International Journal (AAM), *15*(2) (2020), 15.

[16] M. Eslami and M. Mirzazadeh, *Exact solutions of modified Zakharov-Kuznetsov equation by the homogeneous balance method*, Ain Shams Engineering Journal, *5*(1) (2014), 221–225.

[17] O. Gonzalez-Gaxiola, A. Biswas, M. Ekici, and S. Khan, *Highly dispersive optical solitons with quadratic-cubic law of refractive index by the variational iteration method*, Journal of Optics, *51*(1) (2022), 29–36.

[18] W. Hereman, R. Martino, J. Miller, L. Hong, S. Formenac, and A. Menz, *Exact solutions of nonlinear partial differential equations The tanh/sech method*, Math. Visiting Scholar Grant Program Wolfram Research Inc., Champaign, Illinois, *25* (2000), 1–26.

[19] R. Hirota, *Exact envelope-soliton solutions of a nonlinear wave equation*, Journal of Mathematical Physics, *14*(7) (1973), 805–809.

[20] A. J. A. M. Jawad, M. D. Petković, and A. Biswas, *Modified simple equation method for nonlinear evolution equations*, Applied Mathematics and Computation, *217*(2) (2010), 869–877.

[21] T. A. Khalil, N. Badra, H. M. Ahmed, and W. B. Rabie, *Optical solitons and other solutions for coupled system of nonlinear Biswas-Milovic equation with Kudryashov's law of refractive index by Jacobi elliptic function expansion method*, Optik, (2022), 168540.

[22] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, *A new definition of fractional derivative*, Journal of computational and applied mathematics, *264* (2014), 65–70.

[23] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, *204* (2006). elsevier.

[24] W. X. Ma, *Soliton solutions by means of Hirota bilinear forms*, Partial Differential Equations in Applied Mathematics, *5* (2022), 100220.

[25] S. K. Mohanty, S. Kumar, A. N. Dev, M. K. Deka, D. V. Churikov, and O. V. Kravchenko, *An efficient technique of* G′G-*expansion method for modified* KdV *and Burgers equations with variable coefficients*, Results in Physics, (2022), 105504.

[26] R. I. Nuruddeen, *Multiple soliton solutions for the* $(3 + 1)$ *conformable space-time fractional modified korteweg-de-vries equations*, Journal of Ocean Engineering and Science, *3*(1) (2018), 11–18.

[27] H. M. Srivastava, S. Deniz, and K. M. Saad, *An efficient semi-analytical method for solving the generalized regularized long wave equations with a new fractional derivative operator*, Journal of King Saud University-Science, *33*(2) (2021), 101345.

[28] N. Taghizadeh, M. Mirzazadeh, and F. Farahrooz, *Exact soliton solutions for second-order Benjamin-Ono equation*, Applications and Applied Mathematics: An International Journal (AAM), *6*(1) (2011), 31.

[29] N. Taghizadeh, M. Mirzazadeh, AS. Paghaleh, and J. Vahidi, *Exact solutions of nonlinear evolution equations by using the modified simple equation method*, Ain Shams Engineering Journal, *3*(3) (2012), 321–5.

[30] A. C. Varsoliwala and T. R. Singh, *Mathematical modeling of atmospheric internal waves phenomenon and its solution by Elzaki Adomian decomposition method*, Journal of Ocean Engineering and Science, *7*(3) (2022), 203–212.

[31] A. M. Wazwaz, *Exact soliton and kink solutions for new* $(3+1)$*-dimensional nonlinear modified equations of wave propagation*, Open Engineering, *7*(1) (2017), 169–174.

[32] A. Zulfiqar and J. Ahmad, *Soliton solutions of fractional modified unstable Schrödinger equation using Exp-function method*, Results in Physics, *19* (2020), 103476.