Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 12, No. 2, 2024, pp. 329-337
DOI:10.22034/cmde.2023.57413.2401

A new Bernstein-reproducing kernel method for solving forced Duffing equations with integral boundary conditions

Azam Ghasemi and Abbas Saadatmandi*
Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Kashan, Kashan 87317-53153, Iran.

Abstract

In the current work, a new reproducing kernel method (RKM) for solving nonlinear forced Duffing equations with integral boundary conditions is developed. The proposed collocation technique is based on the idea of RKM and the orthonormal Bernstein polynomials (OBPs) approximation together with the quasi-linearization method. In our method, contrary to the classical RKM, there is no need to use the Gram-Schmidt orthogonalization procedure and only a few nodes are used to obtain efficient numerical results. Three numerical examples are included to show the applicability and efficiency of the suggested method. Also, the obtained numerical results are compared with some results in the literature.

Keywords. Duffing equations, Integral boundary conditions, Reproducing kernel method, Bernstein polynomials, Quasi-linearization method.
2010 Mathematics Subject Classification. 65L60, 34B15, 46E22.

1. Introduction

Many problems in different areas of science and engineering such as underground water flow, heat, population dynamics, and thermoelasticity lead to integral boundary value problems (see say [3, 13] and references therein). In this work, we focus on the following forced Duffing equation (FDE)

$$u''(t) + \sigma u'(t) + f(t, u(t), u'(t)) = 0, \quad t \in (0, 1), \sigma \in \mathbb{R} - \{0\}, \tag{1.1}$$

with integral boundary conditions (IBCs):

$$u(0) - \mu_1 u'(0) = \int_0^1 h_1(s)u(s)\, ds, \qquad u(1) + \mu_2 u'(1) = \int_0^1 h_2(s)u(s)\, ds. \tag{1.2}$$

Here, $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ and h_1, h_2 are given continuous functions. Also, μ_1 and μ_2 are nonnegative constants. The study of the Duffing equation or Duffing oscillator has its origins in the early twentieth century. George Duffing (1861-1944) can be considered as the first to study the classical nonlinear Duffing equation [29]. This equation has a main role in modeling many applications such as brain modeling, biological systems, disease prediction, orbit extraction, and so on [8, 10, 28]. The existence and uniqueness of the solutions for the FDE with IBCs are discussed in [2, 7]. While many numerical and analytical methods have been proposed to solve the Duffing equation with two-point boundary conditions [13, 27], there has been less research on the Duffing equations with IBCs such as Eqs. (1.1), and (1.2). For instance, the reproducing kernel space method [9, 14], the homotopy perturbation method, and the reproducing kernel Hilbert space method [16], and the Legendre multiwavelets method [23] are used for solving Eqs. (1.1), and (1.2) where $f = f(t, u)$. Also, hybrid functions approaches based on the combination of block-pulse functions with Bernoulli polynomials [22] or Legendre polynomials [13] are employed for solving Eqs. (1.1), and (1.2).
In recent two decades, the RKMs have been widely used for solving various kinds of problems including, two-point boundary value problems [18, 24, 30], Duffing equations [9], integro-differential equations [31], nonlinear integral

Received: 05 July 2023 ; Accepted: 18 September 2023.
∗ Corresponding author. Email: saadatmandi@kashanu.ac.ir, a.saadatmandi@gmail.com.

equations [15] and partial differential equations [1]. For some historical remarks and for more applications of the RKM, the interested reader can see [11]. However, there are some disadvantages to using approximations based on the classical RKMs. In the classical RKM, a Gram-Schmidt orthogonalization process and a dense sequence of nodal points are used to obtain a set of orthonormal basis functions. As said in [9], because of the randomness of Gram-Schmidt's orthogonalization coefficients, the convergence order of this method is not high. Also, for applying the classical RKM, the Gram-Schmidt orthogonalization procedure is unstable numerically, and much more time-consuming is needed [15, 24]. In order to solve the disadvantages mentioned above, some efforts have been made in recent years (see e.g., [9, 15, 30]).
In this research, based on the work of [30], we improve the classical RKM to obtain an efficient and accurate method for the numerical solution of Eqs. (1.1), and (1.2). In fact, we combine the idea of RKM and the orthonormal Bernstein polynomials approximation, which we call the Bernstein-reproducing kernel method (BRKM), to avoid the Gram-Schmidt orthogonalization procedure. Also, in BRKM, contrary to the classical RKM, only a few nodes are used to obtain efficient numerical results.

In the next section, a reproducing kernel space with OBPs will be introduced. Also, in this section a new set of linearly independent bases in $W_n[0,1]$ for solving Eqs. (1.1), and (1.2) will be constructed. In section 3, we present three numerical examples to show the effectiveness of the BRKM. Conclusions are given in section 4.

2. Description of the method

In the BRKM, we must first construct the reproducing kernel functions with OBPs in the reproducing space. In the next step, we create a set of bases that are linearly independent in $W_n[0,1]$. We use these new bases to approximate the solution of FDE ((1.1)) and obtain an approximate solution with excellent accuracy.

2.1. A new reproducing kernel space based on Bernstein polynomials.

In recent decades, orthogonal functions have been widely used to approximate functions(see e.g., [4, 17, 20]). In this paper we use OBPs.
For $i = 0, ..., n$, the classical Bernstein polynomials of degree n are defined on $[0,1]$ by

$$\beta_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}.$$

These polynomials form a complete basis over $[0,1]$. Unfortunately, the classical Bernstein polynomials do not admit the orthogonality [6]. In 2014, Bellucci [6] presented an explicit representation of the OBPs by

$$b_i^n(t) = \sqrt{2n+1-2i}(1-t)^{n-i} \sum_{\ell=0}^{i} (-1)^\ell \binom{2n-\ell+1}{i-\ell} \binom{i}{\ell} t^{i-\ell}.$$

Also, applying binomial expansion of $(1-t)^{n-i}$, $b_i^n(t)$ can be written as [26]

$$b_i^n(t) = \sqrt{2n+1-2i} \sum_{j=0}^{n} \left(\sum_{\lambda=\max\{0,j-n+i\}}^{\min\{i,j\}} \theta_{i,j-\lambda}\, \vartheta_{i,\lambda} \right) t^j, \qquad i = 0, 1, \ldots, n, \tag{2.1}$$

where $\theta_{i,r}$, $(r = 0, 1, \ldots, n-i)$ and $\vartheta_{i,\lambda}$, $(\lambda = 0, 1, \ldots, i)$ are given as

$$\theta_{i,r} = \big(-1\big)^r \binom{n-i}{r}, \qquad \vartheta_{i,\lambda} = \big(-1\big)^{i-\lambda} \binom{2n+1+\lambda-i}{\lambda} \binom{i}{i-\lambda}.$$

Now let $W_n[0,1]$ be the inner product space of polynomials of degree at most n on $[0,1]$. The inner product and the norm of $W_n[0,1]$ are defined as

$$\langle u, v \rangle_{W_n} = \int_0^1 uv\, dt, \qquad \|u\|_{W_n} = \sqrt{\langle u, u \rangle_{W_n}}, \qquad u, v \in W_n[0,1].$$

Note that $W_n[0,1]$ is a closed finite dimensional subspace of $L^2[0,1]$, therefore $W_n[0,1]$ is a $(n+1)$-dimensional Hilbert space [12], $\langle u, v \rangle_{W_n} = \langle u, v \rangle_{L^2}$ and $\|u\|_{W_n} = \|u\|_{L^2}$. Here, $\{b_0^n(t), b_1^n(t), ..., b_n^n(t)\}$ are orthonormal basis functions of $W_n[0,1]$. So, according to [11, page 4], one obtains:

Lemma 2.1. *$W_n[0,1]$ is a reproducing kernel space and its polynomial reproducing kernel function is*

$$\mathcal{R}(s,t) = \sum_{i=0}^{n} b_i^n(t) b_i^n(s). \tag{2.2}$$

Hence, by the definition of the kernel function, for any function $u(s) \in W_n[0,1]$ we have [11]

$$u(t) = \langle u(s), \mathcal{R}(s,t) \rangle_{W_n}. \tag{2.3}$$

2.2. **Construction of a new basis for $W_n[0,1]$ to approximate solutions of Eqs. (1.1), and (1.2):** In this part, we first use the quasi-linearization method (QLM) [5, 19, 21, 25] to linearize the non-linear Eq. (1.1), then we form a new base for $W_n[0,1]$. The quasi-linearized form of the FDE (1.1) is given by (see say [25])

$$u''_{k+1}(t) + (\sigma + q_k(t))\, u'_{k+1}(t) + p_k(t) u_{k+1}(t) = g_k(t), \qquad k = 0, 1, 2, \cdots \tag{2.4}$$

with boundary conditions

$$u_{k+1}(0) - \mu_1 u'_{k+1}(0) = \int_0^1 h_1(s) u_{k+1}(s)\, ds, \quad u_{k+1}(1) + \mu_2 u'_{k+1}(1) = \int_0^1 h_2(s) u_{k+1}(s)\, ds, \tag{2.5}$$

in which:

$$q_k(t) = f_{u'}\big(t, u_k, u'_k\big), \qquad p_k(t) = f_u\big(t, u_k, u'_k\big),$$

and

$$g_k(t) = u_k f_u\big(t, u_k, u'_k\big) + u'_k f_{u'}\big(t, u_k, u'_k\big) - f\big(t, u_k, u'_k\big).$$

Here, functions $f_u = \partial f / \partial u$ and $f_{u'} = \partial f / \partial u'$ are functional derivatives of $f(t, u, u')$. Note that, the QLM needs initial approximation $u_0(t)$ and can be chosen using mathematical considerations or the boundary conditions. The convergence of QLM has been proved in [5, 21]. They have shown that the order of convergence of this technique is 2. Also, QLM is appropriate for computer programming and has several modifications enabling one to improve the convergence for non-linear problems [19].

Now, we introduce an operator $\mathbb{L} : W_n[0,1] \to L^2[0,1]$ defined by

$$\mathbb{L} u_{k+1} = u''_{k+1} + (\sigma + q_k(t))\, u'_{k+1} + p_k(t) u_{k+1}. \tag{2.6}$$

Lemma 2.2. *Suppose $p_k(t)$ and $q_k(t)$ are continuous functions on $[0,1]$, then $\mathbb{L}$ is a bounded linear operator.*

Proof. It is clear that $\mathbb{L}$ is a linear operator. Since $\mathcal{R}(s,t)$ is the reproducing kernel function of $W_n[0,1]$, we can infer that [11, 30] both $\frac{\partial \mathcal{R}(s,t)}{\partial t}$ and $\frac{\partial^2 \mathcal{R}(s,t)}{\partial t^2}$ are continuous functions on $[0,1]$. Thus,

$$\Big\| \frac{\partial \mathcal{R}(s,t)}{\partial t} \Big\|_{W_n} \le \mathcal{N}_1, \quad \text{and} \quad \Big\| \frac{\partial^2 \mathcal{R}(s,t)}{\partial t^2} \Big\|_{W_n} \le \mathcal{N}_2,$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are positive real numbers. Therefore, for any $u_{k+1} \in W_n[0,1]$, we obtain

$$\begin{aligned} |u'_{k+1}(t)| &= \left| \left\langle u_{k+1}(s), \frac{\partial \mathcal{R}(s,t)}{\partial t} \right\rangle_{W_n} \right| \\ &\le \big\| u_{k+1}(s) \big\|_{W_n} \Big\| \frac{\partial \mathcal{R}(s,t)}{\partial t} \Big\|_{W_n} \le \mathcal{N}_1 \big\| u_{k+1}(s) \big\|_{W_n}, \end{aligned} \tag{2.7}$$

and

$$|u''_{k+1}(t)| \le \big\| u_{k+1}(s) \big\|_{W_n} \Big\| \frac{\partial^2 \mathcal{R}(s,t)}{\partial t^2} \Big\|_{W_n} \le \mathcal{N}_2 \big\| u_{k+1}(s) \big\|_{W_n}. \tag{2.8}$$

Using Eq. (2.6) and the triangle inequality we get

$$\|\mathbb{L} u_{k+1}\|_{L^2} \le \|u''_{k+1}\|_{L^2} + \big\| (\sigma + q_k(t))\, u'_{k+1} \big\|_{L^2} + \|p_k(t) u_{k+1}\|_{L^2}.$$

Due to Eqs. (2.7), and (2.8) and the fact that $p_k(t)$ and $q_k(t)$ are bounded functions on $[0,1]$, say $|q_k(t)| \leq \mathcal{V}_1$ and $|p_k(t)| \leq \mathcal{V}_2$, we have

$$\begin{aligned}\| (\sigma + q_k(t))\, u'_{k+1}\|^2_{L^2} &\leq \sigma^2 \int_0^1 |u'_{k+1}|^2\, dt + 2\sigma \int_0^1 |q_k(t)||u'_{k+1}|^2\, dt + \int_0^1 |q_k(t)|^2 |u'_{k+1}|^2\, dt \\ &\leq \sigma^2 \mathcal{N}_1^2 \|u_{k+1}\|^2_{W_n} + 2\sigma \mathcal{V}_1 \mathcal{N}_1^2 \|u_{k+1}\|^2_{W_n} + \mathcal{V}_1^2 \mathcal{N}_1^2 \|u_{k+1}\|^2_{W_n} \\ &= \underbrace{\left(\sigma^2 + 2\sigma\mathcal{V}_1 + \mathcal{V}_1^2\right)}_{\mathcal{M}^2} \mathcal{N}_1^2 \|u_{k+1}\|^2_{W_n},\end{aligned}$$

and

$$\|p_k(t) u_{k+1}\|^2_{L^2} \leq \int_0^1 |p_k(t)|^2 |u_{k+1}|^2\, dt \leq \mathcal{V}_2^2 \mathcal{N}^2 \|u_{k+1}\|^2_{W_n}.$$

We thus conclude that

$$\|\mathbb{L} u_{k+1}\|_{L^2} \leq \mathcal{N}_2 \|u_{k+1}\|_{W_n} + \mathcal{M}\mathcal{N}_1 \|u_{k+1}\|_{W_n} + \mathcal{V}_2 \mathcal{N} \|u_{k+1}\|_{W_n}.$$

Therefore, $\|\mathbb{L} u_{k+1}\|_{L^2} \leq \mathcal{T} \|u_{k+1}\|_{W_n} = \mathcal{T} \|u_{k+1}\|_{L^2}$, where $\mathcal{T} = \mathcal{N}_2 + \mathcal{M}\mathcal{N}_1 + \mathcal{V}_2 \mathcal{N}$. □

Now, let us suppose that $\{t_i\}_{i=1}^{n-1}$ be $(n-1)$ distinct points in the interval $(0,1)$ and let $r(s,t)$ be the kernel of $L^2[0,1]$. Also, let $\mathbb{L}^*$ be the conjugate operator of $\mathbb{L}$. For $i = 1, 2, \cdots, n-1$, setting $\ell_i(t) = \mathbb{L}_s^* r(s,t)|_{s=t_i}$. Here, subscript s in $\mathbb{L}^*$ indicates that $\mathbb{L}^*$ applies to the function of s.

Lemma 2.3. *For $i = 1, 2, \cdots, n-1$, we have $\ell_i(t) = \mathbb{L}_s \mathcal{R}(s,t)|_{s=t_i}$.*

Proof. Using reproducing kernel properties in kernel reproducing space [11, 24] we have

$$\ell_i(t) = \langle \mathbb{L}_s^* r(s,y)|_{s=t_i}, \mathcal{R}(y,t)\rangle_{W_n} = \langle r(s,y), \mathbb{L}_y \mathcal{R}(y,t)|_{y=t_i}\rangle_{L^2} = \mathbb{L}_s \mathcal{R}(s,t)|_{s=t_i}.$$

□

Also, based on the boundary conditions given in Eq. (1.2), we define

$$B_1(t) = \mathcal{R}(s,t)|_{s=0} - \mu_1 \frac{\partial}{\partial s} \mathcal{R}(s,t)|_{s=0}, \qquad B_2(t) = \mathcal{R}(s,t)|_{s=1} + \mu_2 \frac{\partial}{\partial s} \mathcal{R}(s,t)|_{s=1}.$$

By similar processes as in [24, Theorem 2.1], we obtain

Theorem 2.4. *$\{\ell_1(t), \ell_2(t), \cdots, \ell_{n-1}(t), B_1(t), B_2(t)\}$ are linearly independent in $W_n[0,1]$.*

Finally, since $\dim(W_n[0,1]) = n+1$, it is clear that $\{\ell_1(t), \ell_2(t), \cdots, \ell_{n-1}(t), B_1(t), B_2(t)\}$ is a new basis of $W_n[0,1]$.

2.3. **Implementation of the BRKM for Eqs. (1.1), and (1.2).** In this section, we apply the BRKM together with the collocation approach for solving the linear differential equations in Eq. (2.4) with its boundary conditions given by Eq. (2.5). First of all, we approximate $u_{k+1}(t)$, by the new basis of $W_n[0,1]$, as

$$u_{n,k+1}(t) = \sum_{i=1}^{n-1} c_{i,k+1} \ell_i(t) + a_{1,k+1} B_1(t) + a_{2,k+1} B_2(t), \qquad k = 0, 1, \cdots, MAX, \tag{2.9}$$

where $c_{1,k+1}, c_{2,k+1}, \cdots, c_{n-1,k+1}, a_{1,k+1}, a_{2,k+1}$ are unknown coefficients and MAX is the maximum number of iterations. Now, the residual error function for Eq. (2.4), at each iteration, is constructed by replacing $u_{n,k+1}(t) \in W_n[0,1]$ instead of $u_{k+1}(t)$ as

$$\mathcal{RES}_{k+1}(t) = u''_{n,k+1}(t) + (\sigma + q_{n,k}(t))\, u'_{n,k+1}(t) + p_{n,k}(t) u_{n,k+1}(t) - g_{n,k}(t), \tag{2.10}$$

in which:

$$q_{n,k}(t) = f_{u'}\left(t, u_{n,k}, u'_{n,k}\right), \qquad p_{n,k}(t) = f_u\left(t, u_{n,k}, u'_{n,k}\right),$$

and

$$g_{n,k}(t) = u_{n,k} f_u\left(t, u_{n,k}, u'_{n,k}\right) + u'_{n,k} f_{u'}\left(t, u_{n,k}, u'_{n,k}\right) - f\left(t, u_{n,k}, u'_{n,k}\right).$$

Let us consider a set of $n-1$ collocation points

$$t_i = \frac{1}{2}\left(\cos\left(\frac{i\pi}{n}\right)+1\right), \qquad i=1,\cdots,n-1. \tag{2.11}$$

In every step of iteration $k=0,1,\cdots,MAX$, by collocating Eq. (2.10) in $n-1$ points $t_i,\ i=1,\cdots,n-1$, we have

$$\mathcal{RES}_{k+1}(t_i)=0, \quad i=1,\cdots,n-1. \tag{2.12}$$

Also, by substituting $u_{n,k+1}(t)$ in boundary conditions, we obtain

$$u_{n,k+1}(0)-\mu_1 u'_{n,k+1}(0)=\int_0^1 h_1(s)u_{n,k+1}(s)\,ds, \tag{2.13}$$

$$u_{n,k+1}(1)+\mu_2 u'_{n,k+1}(1)=\int_0^1 h_2(s)u_{n,k+1}(s)\,ds. \tag{2.14}$$

Hence, Eqs. (2.12), (2.14) generates a set of $(n+1)$ linear algebraic equations. By solving this system, the approximate solution $u_{n,k+1}(t)$ can be found via Eq. (2.9). Throughout this paper, we use Maple's *fsolve* command for solving this system.

It is worth mentioning here that, in the classical RKM we need a dense sequence of nodal points while in the BRKM presented above we need only a finite sequence of collocation points.

3. Numerical examples

In this section, we provide three examples to illustrate the validity and accuracy of the BRKM. We also compare the obtained results with the exact solution and other methods. In all examples, we choose the initial guess, required in QLM, so that it satisfies the boundary conditions. For this purpose, the initial guess is taken as $u_0(t)=\alpha_0+\alpha_1 t+\alpha_2 t^2$, where the unknown coefficients α_0,α_1 and α_2 are obtained using the boundary conditions. Also, in Examples 3.1 and 3.2, we put $MAX=4$.

Example 3.1. At first, we consider the following FDE [9, 13, 16, 22]

$$u''(t)+u'(t)+(t-t^2)u^3(t)=f(t), \quad t\in(0,1),$$

with IBCs

$$u(0)-\frac{2}{\pi^2}u'(0)=\int_0^1 -u(s)\,ds, \qquad u(1)+\frac{1}{\pi^2}u'(1)=\int_0^1 -su(s)\,ds,$$

where $f(t)=-\sin(\pi t)\left(\pi^2+(t^2-t)\sin^2(\pi t)\right)+\pi\cos(\pi t)$. The analytical solution of this equation is $u(t)=\sin(\pi t)$. For solving this example by BRKM, we put $u_0(t)=-1.007+t+0.019t^2$. In Table 1, the relative errors obtained by the present method with $n=16,20$ are compared with those obtained by the combination of the homotopy perturbation method and the reproducing kernel Hilbert space method [16], hybrid functions approach [22], improving the RKM [9] and a method based on the hybrid Legendre Block-pulse functions [13]. From Table 1, we see that our method was clearly reliable if compared with the numerical results given in [9, 13, 16, 22]. Note that, in [9], to achieve the desired accuracy 10^{-14}, the number of bases is $n=100$, while in our method, we have reached higher accuracy by using a much smaller number of basis functions. Also, Figure 1 shows the absolute error function corresponding to our method with $n=25$. Moreover, we plot the logarithmic graph of the maximum absolute error for $n=4,8,12,16,20$ in Figure 2. These figures show that we can get a very good result by using BRKM.

TABLE 1. Comparison of relative errors for Example 3.1.

t	Method of [16] $n = 100$, $m = 5$	Method of [22] $N = 6$, $M = 6$	Method of [13] $P = 4$, $Q = 8$	Method of [9] $n = 100$	Present method $n = 16$	$n = 20$
0.01	7.49E-5	6.73 E-7	2.02E-08	7.31E-14	1.92E-15	5.00E-21
0.08	8.17E-5	2.07 E-8	3.29E-09	5.66E-14	2.48E-16	5.77E-22
0.16	8.11E-5	2.68 E-8	2.12E-10	5.53E-14	1.31E-16	2.93E-22
0.32	8.09E-5	2.25 E-8	1.29E-10	5.47E-14	6.60E-17	1.56E-22
0.48	8.08E-5	1.04 E-8	7.04E-10	5.44E-14	4.87E-17	1.31E-22
0.64	8.03E-5	1.24 E-8	8.12E-10	5.35E-14	5.80E-17	1.41E-22
0.80	7.97E-5	6.22 E-8	9.83E-10	5.21E-14	8.73E-17	1.82E-22
0.96	7.58E-5	4.29 E-7	3.58E-10	4.85E-14	3.31E-16	9.54E-22

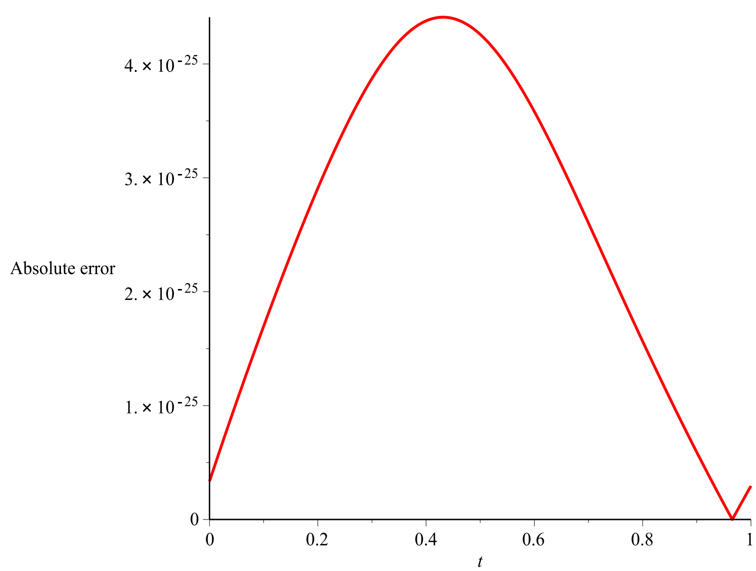

FIGURE 1. Absolute error function with $n = 25$ for Example 3.1.

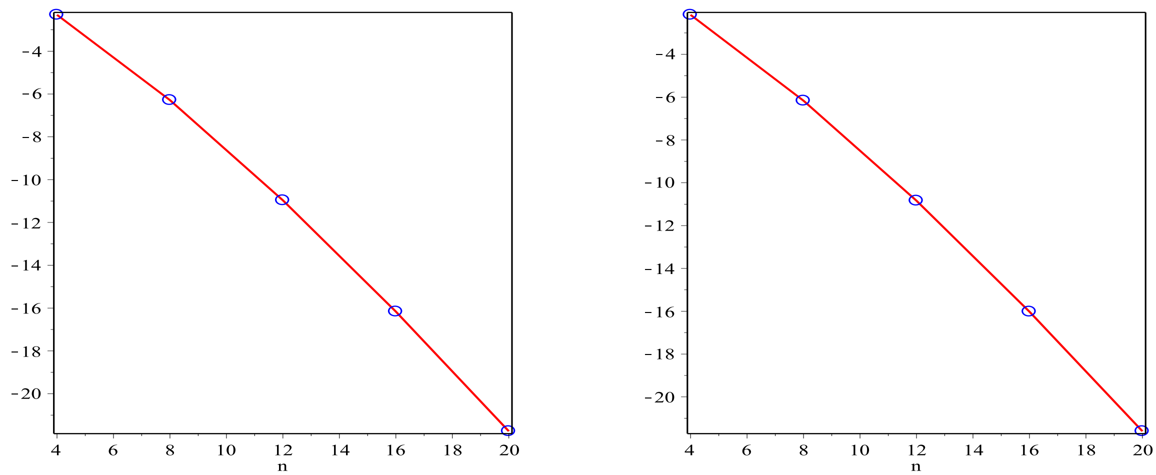

FIGURE 2. The logarithmic graph of the maximum absolute error for Example 3.1 (left) and Example 3.2 (right).

Example 3.2. Consider the following FDE [9, 13]

$$u''(t) - u'(t) - 2u(t) + \sin(u(t)) = f(t), \quad t \in (0,1),$$

with IBCs

$$u(0) - \frac{4}{3\pi^2}u'(0) = \int_0^1 -\cos\left(\frac{\pi s}{2}\right) u(s)\, ds, \qquad u(1) + \frac{6}{\pi^2}u'(1) = \int_0^1 -2\,(s+1)\, u(s)\, ds,$$

where $f(t) = -(2+\pi^2)\sin(\pi t) + \sin(\sin(\pi t)) - \pi\cos(\pi t)$. The analytical solution of this equation is $u(t) = \sin(\pi t)$. Taking $u_0(t) = -1.016 - 0.014t + t^2$. In Table 2, the absolute errors of the presented method are compared with those obtained by RKM [9] and the hybrid functions approach [13]. Also, Figure 2 demonstrates the logarithmic graph of the maximum absolute error for $n = 4, 8, 12, 16, 20$. As we see, by increasing n the maximum absolute errors are decreasing quickly.

TABLE 2. Comparison of absolute errors for Example 3.2.

t	Method of [13] $P=6, Q=6$	RKM [9] $n=50$	Present method $n=15$	$n=19$
0.2	1.12E-05	7.89E-13	1.32E-14	3.09E-20
0.4	3.82E-05	1.21E-12	1.47E-14	5.43E-20
0.6	4.31E-05	1.15E-12	1.61E-14	6.12E-20
0.8	2.03E-05	6.97E-13	1.80E-14	5.21E-20
1	1.20E-05	2.42E-14	1.91E-14	7.93E-20

Example 3.3. As the third example, let us consider the following FDE [14]

$$u''(t) + u'(t) + u(t)\cos(u(t)) = f(t), \quad t \in (0,1),$$

with IBCs

$$u(0) - 2u'(0) = \int_0^1 -su(s)\, ds, \qquad u(1) + \frac{12}{25}u'(1) = \int_0^1 2\,(s+1)\, u(s)\, ds,$$

where $f(t) = 1 + \left(1 + t + t^4\right) \cos\left(1 + t + t^4\right) + 12t^2 + 4t^3$. The analytical solution of this equation is $u(t) = t^4 + t + 1$. We solved this problem by choosing $u_0(t) = 1 + 1.325t + 2.833t^2$. The maximum absolute errors for $n = 4$ and $MAX = 3, 4, 5, 6$ are shown in Table 3. This problem is solved in [14] by traditional RKM. In [14] to achieve an absolute error of about 10^{-4}, the number of bases is $n = 100$ (see Figure 3 in [14]), while in our method, we have reached much higher accuracy by using $n = 4$.

Table 3. Maximum absolute error with $n = 4$ for Example 3.3.

$MAX = 3$	$MAX = 4$	$MAX = 5$	$MAX = 6$
1.00E-03	4.10E-08	2.54E-17	1.87E-35

4. Conclusion

In this research, we presented the higher-order BRKM method for solving the nonlinear FDE (1.1) with its boundary conditions given in Eq. (1.2). Two important advantages of our method compared to other classical RKMs are (i) we used very few nodes but we obtained very accurate numerical results. (ii) In the presented method, we do not use the Gram-Schmidt orthogonalization process. Numerical results show the effectiveness and accuracy of the BRKM method. In the future, we intend to develop the technique presented in this paper to solve partial differential equations and fractional differential equations.

Acknowledgment

The authors sincerely acknowledge and thank the referees and the editor.

References

[1] S. Abbasbandy and H. R. Khodabandehlo, *Application of reproducing kernel Hilbert space method for generalized 1-D linear telegraph equation*, International Journal of Nonlinear Analysis and Applications, *13* (2022), 487–497.

[2] B. Ahmad and A. Alsaedi, *Existence of approximate solutions of the forced Duffing equation with discontinuous type integral boundary conditions*, Nonlinear Analysis: Real World Applications, *10* (2009), 358–367.

[3] B. Ahmad, S. K. Ntouyas, and A. Alsaedi, *On a coupled system of fractional differential equations with coupled nonlocal and integral boundary conditions*, Chaos Solitons and Fractals, *83* (2016), 234–241.

[4] A. Alipanah, M. Pendar, and K. Sadeghi, *Integrals involving product of polynomials and Daubechies scale functions*, Mathematics Interdisciplinary Research, *6* (2021), 275–291.

[5] R. E. Bellman and R. E. Kalaba, *Quasilinearization and Nonlinear Boundary-value Problems*, Elsevier publishing company, New York, 1965.

[6] M. A. Bellucci, *On the explicit representation of orthonormal Bernstein polynomials*, arXiv:1404.2293v2, (2014).

[7] A. Boucherif, *Second-order boundary value problems with integral boundary conditions*, Nonlinear Analysis: Theory, Methods and Applications, *70* (2009), 364–371.

[8] M. Chai and L. Ba, *Application of EEG signal recognition method based on Duffing equation in psychological stress analysis*, Advances in Mathematical Physics, *2021* (2021), article ID 1454547.

[9] Z. Chen, W. Jiang, and H. Du, *A new reproducing kernel method for Duffing equations*, International Journal of Computer Mathematics, *98* (2021), 2341–2354.

[10] A. A. Cherevko, E. E. Bord, A. K. Khe, V. A. Panarin, and K. J. Orlov, *The analysis of solutions behaviour of Van der Pol Duffing equation describing local brain hemodynamics*, Journal of Physics: Conference Series, *894* (2017), 012012.

[11] M. Cui and Y. Lin, *Nonlinear numerical analysis in the reproducing kernel space*, Nova Science Publishers, Inc., 2009.

[12] F. Deutsch , *Best Approximation in Inner Product Spaces*, Springer, New York, 2001.

[13] M. R. Doostdar, M. Kazemi, and A. Vahidi, *A numerical method for solving the Duffing equation involving both integral and non-integral forcing terms with separated and integral boundary conditions*, Computational Methods for Differential Equations, *11* (2023), 241–253.

[14] J. Du and M. Cui, *Solving the forced Duffing equation with integral boundary conditions in the reproducing kernel space*, International Journal of Computer Mathematics, *87* (2010), 2088–2100.

[15] S. Farzaneh Javan, S. Abbasbandy, and M. A. Fariborzi Araghi, *Application of reproducing kernel Hilbert space method for solving a class of nonlinear integral equations*, Mathematical Problems in Engineering, *2017* (2017), article ID 7498136.

[16] F. Z. Geng and M. Cui, *New method based on the HPM and RKHSM for solving forced Duffing equations with integral boundary conditions*, Journal of Computational and Applied Mathematics, *233* (2009), 165–172.

[17] H. Jafari and H. Tajadodi, *Electro-spunorganic nanofibers elaboration process investigations using BPs operational matrices*, Iranian Journal of Mathematical Chemistry, *7* (2016), 19-27.

[18] M. Khaleghi, M. T. Moghaddam, E. Babolian, and S. Abbasbandy, *Solving a class of singular two-point boundary value problems using new effective reproducing kernel technique*, Applied Mathematics and Computation, *331* (2018), 264–273.

[19] V. Lakshmikantham and A. S. Vatsala, *Generalized Quasilinearization for Nonlinear Problems*, MIA, Kluwer Academic Publishers, Dordrecht, 1998.

[20] M. Lotfi and A. Alipanah, *Legendre spectral element method for solving Volterra-integro differential equations*, Results in Applied Mathematics, *7* (2020), 100116.

[21] V. B. Mandelzweig and F. Tabakin, *Quasilinearization approach to nonlinear problems in physics with application to nonlinear ODEs*, Computer Physics Communications, *141* (2001), 268–281.

[22] S. Mashayekhi, Y. Ordokhani, and M. Razzaghi, *A hybrid functions approach for the Duffing equation*, Physica Scripta, *88* (2013), 025002.

[23] R. Najafi and B. Nemati Saray, *Numerical solution of the forced Duffing equations using Legendre multiwavelets*, Computational Methods for Differential Equations, *5* (2017), 43–55.

[24] J. Niu, M. Xu, Y. Lin, and Q. Xue, *Numerical solution of nonlinear singular boundary value problems*, Journal of Computational and Applied Mathematics, *331* (2018), 42–51.

[25] K. Parand, Y. Lotfi, and J. A. Rad, *An accurate numerical analysis of the laminar two-dimensional flow of an incompressible Eyring-Powell fluid over a linear stretching sheet*, The European Physical Journal Plus, *132* (2017), 1–21.

[26] A. Saadatmandi, A. Ghasemi-Nasrabady, and A. Eftekhari, *Numerical study of singular fractional Lane-Emden type equations arising in astrophysics*, Journal of Astrophysics and Astronomy, *40* (2019), 1–12.

[27] A. Saadatmandi and S. Yeganeh, *New approach for the Duffing equation involving both integral and non-integral forcing terms*, UPB Scientific Bulletin, Series A: Applied Mathematics and Physics, *79* (2017), 46–52.

[28] R. Srebro, *The Duffing oscillator: a model for the dynamics of the neuronal groups comprising the transient evoked potential*, Electroencephalography and clinical Neurophysiology, *96* (1995), 561–573.

[29] J. J. Stokes, *Nonlinear Vibrations, Intersciences*, New York, 1950.

[30] M. Xu and E. Tohidi, *A Legendre reproducing kernel method with higher convergence order for a class of singular two-point boundary value problems*, Journal of Applied Mathematics and Computing, *67* (2021), 405–421.

[31] W. Yulan, T. Chaolu, and P. Jing, *New algorithm for second-order boundary value problems of integro-differential equation*, Journal of Computational and Applied Mathematics, *229* (2009), 1–6.