Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 12, No. 2, 2024, pp. 304-313
DOI:10.22034/CMDE.2023.56266.2352

Inverse coefficient problem in hyperbolic partial differential equations: An analytical and computational exploration

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Abstract

This investigation centers on the analysis of an inverse hyperbolic partial differential equation, specifically addressing a coefficient inverse problem that emerges under the imposition of an over-determination condition. In order to address this challenging problem, we employ the well-established homotopy analysis technique, which has proven to be an effective and reliable approach in similar contexts. By utilizing this technique, our primary objective is to achieve an efficient and accurate solution to the inverse problem at hand. To substantiate the effectiveness and reliability of the proposed method, we present a numerical example as a practical illustration, demonstrating its applicability in real-world scenarios.

Keywords. Hyperbolic partial differential equation, Coefficient inverse problem, Homotopy analysis method, Convergence analysis, Numerical simulations.

2010 Mathematics Subject Classification. 35Lxx, 35L05, 65M32, 35C10, 35Axx.

1. Introduction

The application of ordinary and partial differential equations plays a crucial role in various fields, serving as powerful tools for understanding and modeling complex phenomena. The field of applied ordinary and partial differential equations has witnessed significant advancements through numerous research papers, further enriching our knowledge and expanding our capabilities in these areas. For instance, Machado et al. [38] proposed a highly accurate scheme specifically designed for solving the Cauchy problem of the generalized Burgers-Huxley equation, enabling precise predictions and detailed analysis of the system's behavior. Mokhtary et al. [23] developed a computational approach tailored to address non-linear weakly singular Volterra integral equations with proportional delay, thereby extending the range of applications for these equations. Mostaghim et al. [24] introduced a computational technique for simulating variable-order fractional Heston models, which find practical utility in financial modeling, particularly in the context of the US stock market. They also ventured into the numerical simulation of fractional-order dynamical systems in noisy environments, effectively tackling challenges associated with real-world scenarios [25]. Moniri et al. [26] made significant contributions by devising an efficient and robust numerical solver for the impulsive control of fractional chaotic systems, opening up avenues for controlling and manipulating complex dynamical behaviors. Lastly, the numerical investigation conducted by Moghaddam et al. [28] centered on fractional dynamical systems with impulsive effects, offering significant understanding of the dynamics exhibited by these systems in different domains. Similarly, Sharifi et al. [32] presented an efficient numerical simulation approach for the fractional-order Van der Pol impulsive system.

In the realm of hyperbolic partial differential equations, the task of finding solutions and examining their well-posedness poses significant challenges. This difficulty is particularly pronounced when addressing inverse problems associated with hyperbolic partial differential equations. In the subsequent paragraphs, we will briefly introduce papers

Received: 21 April 2023 ; Accepted: 15 August 2023.
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in this context that have made notable contributions to the field. Colton and Monk [4] conducted a study comparing two approaches to inverse scattering for acoustic waves in nonhomogeneous media, analyzing the strengths and weaknesses of each method. Kedzierawski [13] employed the Colton-Monk approach to calculate the inverse scattering of acoustic waves and determine the complex refraction index. Sylvester and Uhlmann [37] explored research on anisotropic inverse hyperbolic problems, while Tsien and Chen [39] adopted the pulse-spectrum approach to address inverse problems in electromagnetic wave propagation. Chen and Liu [37] utilized this method to solve a two-dimensional inverse linear wave equation. These papers contribute to the understanding and resolution of inverse issues related to hyperbolic partial differential equations.

Yamamoto [41] obtained the stability, regularization, and reconstruction formula for an inverse hyperbolic problem using a control method. He also investigated ill-posedness and Tikhonov regularization for a multidimensional inverse hyperbolic problem [42]. Yamamoto and Zhang employed the Carleman estimate [43] to determine the global uniqueness and stability of the inverse source problem of the wave equation. In order to determine the coefficient and source function of the impulse type for a two-dimensional wave equation, Romanov [29] considered inverse difficulties.

Consider the following hyperbolic partial differential equation [10],

$$u_{tt} = u_{xx} + d(t)u + f(x,t), \qquad (x,t) \in (0,l) \times [0,T], \tag{1.1}$$

with initial and boundary conditions

$$u(x,0) = g(x),\, u_t(x,0) = h(x), \qquad x \in [0,l], \tag{1.2}$$

$$u(0,t) = \phi(t),\, u(l,t) = \psi(t), \qquad t \in [0,T]. \tag{1.3}$$

The following problem is referred to as a direct problem if all the data, including the coefficient $d(t)$, the source term $f(x,t)$, the beginning conditions $g(x)$, $h(x)$, and the boundary conditions $\phi(t)$, $\psi(t)$, are known and the goal is to identify the solution u from these data. Otherwise, the issue is one that is reversed.

For the aforementioned hyperbolic partial differential equation, we now explore an inverse problem. The objective is to use the previously provided known data to determine the solution u and coefficient d. An additional condition is needed in order to discover these unknowns:

$$u(x_0,t) = \chi(t), \qquad x_0 \in (0,l). \tag{1.4}$$

This type of problem is called coefficient inverse problem. For other kinds of inverse problems see [11] and references therein.

Using the homotopy analysis method (HAM), the inverse problem indicated will be resolved. Liao [15] proposed HAM in 1992 to solve nonlinear differential equations. HAM is independent of any physical parameter, in contrast to perturbation approaches [27, 40], which rely on physical parameters. Additionally, certain analytical approximation techniques are used in this method, including the Lyapunov artificial small parameter method [21], the Adomian decomposition method [1, 2], the δ-expansion method [12], the homotopy perturbation method, and others.The base function, starting guess, and equation type can all be chosen with significant freedom and flexibility in HAM. In contrast to other analytical approximation techniques, HAM provides a convergence control parameter that ensures the approach will converge. Later, numerous nonlinear issues in science and engineering are solved using this method; for example, see [14, 16, 19, 22, 30, 33, 35, 44]. This technique was applied by Shidfar et al. [34] to find the unknown source term in a parabolic partial differential equation. Shidfar and Molabahrami [36] also use HAM to tackle inverse heat conduction difficulties. In this study, the inverse issue for a hyperbolic partial differential equation is solved using a weighted method built from HAM, known as the weighted homotopy analysis method (WHAM).

2. Description of HAM

In this section, we will discuss the homotopy analysis approach, an analytical approximation technique that may successfully solve a given nonlinear differential equation. Take the nonlinear differential equation, for example

$$\mathcal{N}[u(x,t)] = 0, \qquad (x,t) \in \Omega \times (0,T), \tag{2.1}$$

$\mathcal{N}$ is a nonlinear differential operator, u is a function that needs to be discovered, and x and t, respectively, are spatial and temporal independent variables. Based on homotopy in topology, one can construct zeroth-order deformation equation as follows

$$(1-q)\mathcal{L}[\phi(x,t;q)-u_0(x,t)] = q\hbar H(x,t)\mathcal{N}[\phi(x,t;q)], \qquad q\in[0,1], \tag{2.2}$$

where q is a homotopy or embedding parameter when it increases from 0 to 1, $\phi(x,t;q)$ varies from $u_0(x,t)$ to the exact solution of the nonlinear differential equation, $\mathcal{L}$ is an auxiliary linear operator, u_0 is an initial guess of the exact solution that satisfies the initial and boundary conditions, $H(x,t)$ is an auxiliary function, and $\hbar$ is called the convergence-control parameter (2.1).

One needs to define homotopy derivative in order to obtain the high-order deformation equation. The m-th order homotopy derivative for ϕ as a function of the homotopy parameter q can be stated as

$$\mathcal{D}_m\phi(x,t;q) = \frac{1}{m!}\frac{d^m\phi}{dq^m}|_{q=0}, \tag{2.3}$$

and $\mathcal{D}_m$ is called m-th order homotopy-derivative operator. For properties of the homotopy derivative, one can see [20, Chapter 4]. Assume that

$$\phi(x,t;q) = \sum_{m=0}^{\infty} u_m(x,t)q^m, \tag{2.4}$$

is the homotopy-Macluarin series of ϕ, where $u_m = \mathcal{D}_m\phi(x,t;q)$. Then, applying the m-th order homotopy derivative $\mathcal{D}_m$ on both sides of zeroth-order deformation Eq. (2.2) and using the properties of homotopy derivative, one can get the high-order deformation equation

$$\mathcal{L}[u_m(x,t)-\chi_m u_{m-1}(x,t)] = \hbar H(x,t)\mathcal{D}_{m-1}[\mathcal{N}(\phi(x,t;q)], \tag{2.5}$$

where

$$\chi_m = \begin{cases} 0, & m\le 1, \\ 1, & m>1. \end{cases} \tag{2.6}$$

The high-order deformation equation is a linear differential equation, just like the zeroth-order deformation equation, and it can be solved using a computer algebra system like Mathematica or Maple. Be aware that the HAM's fundamental principle is to convert the original nonlinear problem into a series of linear problems. See [15, Chapter 4] for a more general form of the deformation equation. To express the solution of the nonlinear differential equation, we now require a collection of base functions (2.1). The high flexibility and freedom in setting the base function, initial estimate u_0, and auxiliary linear operator $\mathcal{L}$ is one of the benefits of HAM over other analytical approximation techniques. Assume that the expression of $u(x,t)$ is Assume that $u(x,t)$ is expressed by

$$u(x,t) = \sum_{k=1}^{\infty} a_k e_k(x,t), \tag{2.7}$$

where a_k is a coefficient and $e_k(x,t)$ is the base function. To choose the base functions, some rules are given in [20, Chapter 4]. Now, a proper initial approximation and an appropriate auxiliary linear operator must be chosen in such a way that the homotopy series $u_0+\sum_{m=1}^{\infty}u_m$ converge.

Convergence-control-parameter $\hbar$ is another benefit of HAM. By applying $\hbar$-curves to this parameter, one can determine the region where the homotopy series converges. Interested readers can refer to [20, Chapter 4] for more information on the choice of base functions, initial guess u_0, auxiliary linear operator $\mathcal{L}$, and convergence control-parameter.

3. Description of WHAM

To solve the coefficient inverse problem of the hyperbolic partial differential equation (1.1), we provide a weighted homotopy analysis method in this section (1.3). Using (1.4) and (1.1) with $x = x_0$ as the replacement, one can obtain

$$d(t) = \frac{\chi''(t) - u_{xx}(x_0, t) - f(x_0, t)}{\chi(t)}. \tag{3.1}$$

Once again, substituting this into relation (1.1), yields

$$u_{tt} = u_{xx} + \xi(t)u + \eta(t)u_{xx}(x_0, t)u + f(x, t), \tag{3.2}$$
$$u(x, 0) = g(x),\ u_t(x, 0) = h(x), \tag{3.3}$$
$$u(0, t) = \varphi(t),\ u(l, t) = \psi(t), \tag{3.4}$$

where

$$\alpha(t) = \frac{\chi''(t) - f(x_0, t)}{\chi(t)},\ \beta(t) = -\frac{1}{\chi(t)}.$$

A nonlinear hyperbolic partial differential equation is what we obtain as a result. The initial and boundary conditions allow for the definition of two subproblems. The subproblem I is

$$u_{tt} = u_{xx} + \xi(t)u + \eta(t)u_{xx}(x_0, t)u + f(x_0, t), \tag{3.5}$$
$$u(x, 0) = g(x),\ u_t(x, 0) = h(x), \tag{3.6}$$

and the subproblem II is defined by

$$u_{tt} = u_{xx} + \xi(t)u + \eta(t)u_{xx}(x_0, t)u + f(x_0, t), \tag{3.7}$$
$$u(0, t) = \varphi(t),\ u(l, t) = \psi(t). \tag{3.8}$$

We can solve the main problem (3.2) by using HAM to resolve these two subproblems (3.4). One should compute $\mathcal{D}_{m-1}\mathcal{N}[\phi(x, t; q)]$ to obtain the solution series for the aforementioned subproblems. First, let's express the subsequent theorem.

Theorem 3.1. *Let ϕ and ω be analytic in $[0, a)$ and assume that their homotopy-Maclaurin series are*

$$\phi = \sum_{k=0}^{\infty} u_k q^k,\ \omega = \sum_{k=0}^{\infty} w_k q^k, \tag{3.9}$$

then the following relations hold

(1) $\mathcal{D}_m\phi = u_m$,
(2) $\mathcal{D}_m(\phi\omega) = \sum_{k=0}^{m} \mathcal{D}_k\phi\mathcal{D}_{m-k}\omega = \sum_{k=0}^{m} u_m w_{m-k}$.

Proof. Refer to [20, Theorem 4.1]. □

At this moment, we need to define $\mathcal{N}[\phi(x, t; q)]$. Assume that

$$\mathcal{N}[\phi(x, t; q)] = \phi_{tt} - \phi_{xx} - \xi(t)\phi - \eta(t)\phi_{xx}(x_0, t; q)\phi - f(x, t). \tag{3.10}$$

On the other hand, homotopy derivative operator $\mathcal{D}_m$ is linear [20, Theorem 4.1]. Thus, using this fact and Theorem 3.1, we have for $m = 1$

$$\mathcal{D}_0\mathcal{N}[\phi(x, t; q)] = \frac{\partial^2 u_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial x^2} - \xi(t)u_0 - \eta(t)\frac{\partial^2 u_0}{\partial x^2}(x_0, t; q)u_0 - f(x, t), \tag{3.11}$$

and for $m \geq 2$

$$\mathcal{D}_{m-1}\mathcal{N}[\phi(x, t; q)] = \frac{\partial^2 u_{m-1}}{\partial t^2} - \frac{\partial^2 u_{m-1}}{\partial x^2} - \xi(t)u_{m-1} - \eta(t)\sum_{k=0}^{m-1}\frac{\partial^2 u_k}{\partial x^2}(x_0, t; q)u_{m-k-1}. \tag{3.12}$$

We have already made progress toward resolving subproblems I and II. We must select an initial guess, u_0, and an auxiliary linear operator $\mathcal{L}$, in order to solve any of the aforementioned subproblems. Allow these values to be provided by subproblem I.

$$\mathcal{L}\phi(x,t;q) = \frac{\partial^2 \phi}{\partial t^2}\phi(x,t;q), \tag{3.13}$$

$$\hat{u}_0(x,t) = (1+g_1(t))g(x) + g_2(t)h(x) + g_3(t), \tag{3.14}$$

where $g_1(0) = g_2(0) = g_3(0) = 0$, $g_1'(0) = g_3'(0) = 0$, $g_2'(0) = 1$. Now, assume that $\hat{u}_m^*$ is the special solution of

$$\mathcal{L}[\hat{u}_m^*] = \hbar H(x,t)\mathcal{D}_{m-1}\mathcal{N}[\phi(x,t;q)], \tag{3.15}$$

and $\sum_{m=0}^{\infty} \hat{u}_m$ is the solution series, obtained by HAM, of subproblem I. Therefore,

$$\hat{u}_m(x,t) = \chi_m \hat{u}_{m-1}(x,t) + \hat{u}_m^*(x,t) - t\frac{\partial \hat{u}_m^*}{\partial t}(x,0) - \hat{u}_m^*(x,0), \qquad m \geq 1. \tag{3.16}$$

Note that initial conditions for each $\hat{u}_m(x,t)$ are given by $\hat{u}_m(x,0) = 0$, $\frac{\partial}{\partial t}\hat{u}_m(x,0) = 0$, for $m \geq 1$. Similarly, for subproblem II , we choose

$$\mathcal{L}\phi(x,t;q) = \frac{\partial^2 \phi}{\partial x^2}\phi(x,t;q), \tag{3.17}$$

$$\tilde{u}_0(x,t) = u(0,t) + g_4(x)(u(l,t) - u(0,t)) + g_5(x), \tag{3.18}$$

where $g_4(0) = g_5(0) = 0$, $g_4(l) = 1$ and $g_5(l) = 0$. Assuming that $\tilde{u}_m^*$ is the special solution of

$$\mathcal{L}[\tilde{u}_m^*] = \hbar H(x,t)\mathcal{D}_{m-1}\mathcal{N}[\phi(x,t;q)], \tag{3.19}$$

the solution series of subproblem II are computed by,

$$\tilde{u}_m(x,t) = \chi_m \tilde{u}_{m-1}(x,t) + \tilde{u}_m^*(x,t) - \frac{\tilde{u}_m^*(l,t) - \tilde{u}_m^*(0,t)}{l} - \tilde{u}_m^*(0,t), \qquad m \geq 1, \tag{3.20}$$

and the boundary conditions for each $\tilde{u}_m(x,t)$ are given by $\tilde{u}_m(0,t) = 0$, $\tilde{u}_m(l,t) = 0$, for $m \geq 1$. Let

$$\hat{s}_m(x,t) = \sum_{n=0}^{m} \hat{u}_n(x,t), \;\; \tilde{s}_m(x,t) = \sum_{n=0}^{m} \tilde{u}_n(x,t). \tag{3.21}$$

Assume also that $u_m(x,t)$ be the terms of solution series of the original problem. Then, we can define

$$u_m(x,t) = \alpha_m \hat{s}_m(x,t) + (1-\alpha_m)\tilde{s}_m(x,t), \tag{3.22}$$

where the optimal value of α_m is given by the following theorem.

Theorem 3.2. *Assume that $L^2(0,T)$ and $L^2(0,l)$ are the sets of square integrable functions and suppose that $\varphi(t), \psi(t) \in L^2(0,T)$ and $g(x), h(x) \in L^2(0,l)$. Let $\beta_n = \|\hat{s}_n(0,t) - \varphi(t)\|_2$, $\gamma_n = \|\hat{s}_n(l,t) - \psi(t)\|_2$, $\delta_n = \|\tilde{s}_n(x,0) - g(x)\|_2$ and $\lambda_n = \left\|\frac{\partial}{\partial t}\tilde{s}_n(x,0) - h(x)\right\|_2$. Therefore, the optimal value of α_n is obtained by*

$$\alpha_n(\hbar) = \frac{\delta_n^2 + \lambda_n^2}{\beta_n^2 + \gamma_n^2 + \delta_n^2 + \lambda_n^2}, \qquad n \geq 0. \tag{3.23}$$

Proof. Assume that J_n is the residual functional and is defined by

$$J_n = \|u_n(0,t) - \varphi(t)\|_2^2 + \|u_n(l,t) - \psi(t)\|_2^2 + \|u_n(x,0) - g(x)\|_2^2 + \left\|\frac{\partial u_n}{\partial t}(x,0) - h(x)\right\|_2^2. \tag{3.24}$$

Noting that

$$\hat{s}_n(x,0) = g(x), \quad \frac{\partial}{\partial t}\hat{s}_n(x,0) = h(x), \tag{3.25}$$

$$\tilde{s}_n(0,t) = \varphi(t), \quad \tilde{s}_n(l,t) = \psi(t), \tag{3.26}$$

and plugging $u_n(x,t)$ from (3.22), one can get

$$\begin{aligned} J_n =&\alpha_n^2\|\hat{s}_n(0,t)-\varphi(t)\|_2^2+\alpha_n^2\|\hat{s}_n(l,t)-\psi(t)\|_2^2 \\ &+(1-\alpha_n)^2\|\tilde{s}_n(x,0)-g(x)\|_2^2+(1-\alpha_n)^2\left\|\frac{\partial\tilde{s}_n}{\partial t}(x,0)-h(x)\right\|_2^2 \\ =&\alpha_n^2\beta_n^2+\alpha_n^2\gamma_n^2+(1-\alpha_n)^2\delta_n^2+(1-\alpha_n)^2\lambda_n^2, \end{aligned} \tag{3.27}$$

where

$$\beta_n=\|\hat{s}_n(0,t)-\varphi(t)\|_2,\quad \gamma_n=\|\hat{s}_n(l,t)-\psi(t)\|_2, \tag{3.28}$$

$$\delta_n=\|\tilde{s}_n(x,0)-g(x)\|_2,\quad \lambda_n=\left\|\frac{\partial}{\partial t}\tilde{s}_n(x,0)-h(x)\right\|_2. \tag{3.29}$$

Please take note that J_n is a residual functional. Then, J_n must be minimized in relation to n. We now differentiate J_n in relation to n and set the outcome to zero. Therefore

$$(\beta_n^2+\gamma_n^2)\alpha_n-\delta_n^2-\lambda_n^2+(\delta_n^2+\lambda_n^2)\alpha_n=0, \tag{3.30}$$

and this, in turn, yields

$$\alpha_n=\frac{\delta_n^2+\lambda_n^2}{\beta_n^2+\gamma_n^2+\delta_n^2+\lambda_n^2},\qquad n\geq 0, \tag{3.31}$$

and the proof is complete. □

4. Numerical result

An illustration of the WHAM's dependability is presented in this section. Our presumption for this test problem is that $H(x,t)=1$. Comparing the estimated and exact results demonstrates the method's excellent accuracy.

Example 4.1. Take into account the subsequent hyperbolic partial differential equation

$$u_{tt}=u_{xx}+d(t)u+e^x,\qquad (x,t)\in(0,1)\times[0,1], \tag{4.1}$$

when beginning and boundary conditions are present

$$u(x,0)=e^x,\ u_t(x,0)=e^x,\qquad x\in[0,1], \tag{4.2}$$

$$u(0,t)=e^t,\ u(1,t)=e^{1+t},\qquad t\in[0,1], \tag{4.3}$$

and the over determination condition for this problem is

$$u\left(\frac{1}{2},t\right)=e^{\frac{1}{2}+t}. \tag{4.4}$$

Furthermore, the exact solution u and exact coefficient d of the inverse problem (4.1)-(4.4)

$$u(x,t)=e^{x+t},\qquad d(t)=-e^{-t}. \tag{4.5}$$

Substituting $x_0=\frac{1}{2}$ in (4.1), computing $d(t)$ and constructing the two subproblems accordingly, we get

$$u_{tt}=u_{xx}+\xi(t)u+\eta(t)u_{xx}(x_0,t)u+e^x,\qquad (x,t)\in(0,1)\times[0,1], \tag{4.6}$$

$$u(x,0)=e^x,\ u_t(x,0)=e^x,\qquad x\in[0,1], \tag{4.7}$$

as subproblem I and

$$u_{tt}=u_{xx}+\xi(t)u+\eta(t)u_{xx}(x_0,t)u+e^x,\qquad (x,t)\in(0,1)\times[0,1], \tag{4.8}$$

$$u(0,t)=e^t,\ u(1,t)=e^{1+t},\qquad t\in[0,1], \tag{4.9}$$

as subproblem II, where

$$\xi(t)=1-e^{x-t-\frac{1}{2}},\ \eta(t)=-e^{-t-\frac{1}{2}}.$$

According to (3.14) and (3.18), initial solutions for subproblems I and II are

$$\hat{u}_0(x,t) = e^x(1+t), \tag{4.10}$$

$$\tilde{u}_0(x,t) = e^t + x(e^{t+1} - e^t), \tag{4.11}$$

Respectively. Applying relations (3.16) and (3.20) for computing $\hat{u}_1$ and $\tilde{u}_1$ respectively, we obtain

$$\begin{aligned}\hat{u}_1(x,t) =& \hbar e^{-t}((t^2+6t+11)e^x + (-0.333333t^3 - 1.5t^2 + 5t - 11)e^{t+x} \\ &+ (0.606531t + 1.81959)e^{2x} - 1.81959e^{t+2x} + 1.21306te^{t+2x}),\end{aligned}$$

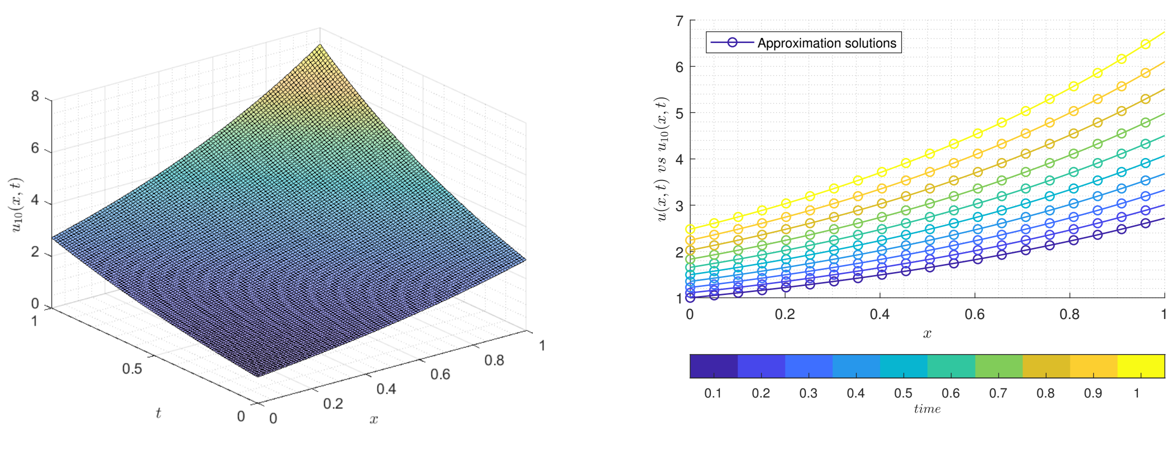
$$\tilde{u}_1(x,t) = \hbar(1.42468x + e^x(1.04219x - 2.47785) + 2.47785). \tag{4.12}$$

The convergence regions of the solution series $\{\hat{u}_n\}_{n\geq 0}$ and $\{\tilde{u}_n\}_{n\geq 0}$ can be determined by plotting their $\hbar$-curves. These curves illustrate how different parameters impact the convergence of the solution series. The $\hbar$-curve is a section that is nearly parallel to the horizontal axis. By computing more approximate solution series for $\{\hat{u}_n\}_{n\geq 0}$ and $\{\tilde{u}_n\}_{n\geq 0}$, a larger convergence region can be obtained for the control parameter $\hbar$. In this example, the $\hbar$-curves indicate that the solution series $\{\hat{u}_n\}_{n\geq 0}$ converges when $-3 < \hbar < 1$, and the series $\{\tilde{u}_n\}_{n\geq 0}$ converges when $-0.7 < \hbar < 1$. To calculate the values of α_n, the approximate solution $u_n(x,t)$, and the approximate coefficient $d_n(t)$, we set $\hbar = \frac{1}{10}$.

Using Equations (3.28)-(3.29), we compute β_n, γ_n, δ_n, and λ_n for $n \geq 0$. Then, we obtain the values of α_n using Equation (3.23) as follows:

$$\alpha_0 = 0.0586206, \qquad \alpha_1 = 0.127513. \tag{4.13}$$

By repeating this method and using Equations (3.16), (3.18), and (3.23) (specifically, equation (3.22)), one can obtain the approximate solution $\hat{u}_n(x,t)$, $\tilde{u}_n(x,t)$, and α_n. Figure 1 displays both the exact solution and the approximate solution $u_{10}(x,t)$ for the different values of t. Figure 2 illustrates the exact coefficient $d(t) = -\exp(-t)$ and the estimated coefficient $d_{10}(t)$ of the hyperbolic partial differential equation (1.1).



(a) (b)

FIGURE 1. Left: (a) Graph of the approximate solution $u_{10}(x,t)$. Right: (b) Comparison between analytical solution $u(x,t)$ and approximate solution $u_{10}(x,t)$ for different values of t.

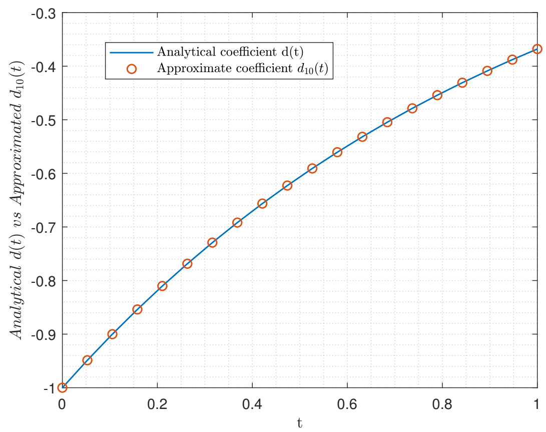


FIGURE 2. Comparison of the analytical coefficient $d(t) = -\exp(-t)$ and the approximate coefficient $d_{10}(t)$.

5. Conclusion

In conclusion, this study tackles inverse problems associated with hyperbolic partial differential equations using an analytical approach, specifically the Homotopy Analysis Method (HAM). The application of HAM has proven to be highly effective in solving numerous nonlinear differential equations, and in this study, it has been successfully applied to address the coefficient inverse problem of hyperbolic partial differential equations. The results demonstrate the accuracy and advantages of the HAM approach, particularly its ability to provide precise solutions without the need for discretization or numerical approximations.

Moving forward, there are several potential avenues for future research in this field. Firstly, further investigations can be conducted to explore the applicability of HAM to more complex and challenging inverse problems involving hyperbolic partial differential equations. Additionally, the development of hybrid numerical-analytical techniques that combine the strengths of HAM with other numerical methods can potentially enhance the efficiency and accuracy of solving inverse problems. Moreover, the extension of HAM to higher-dimensional hyperbolic partial differential equations and systems warrants exploration. Finally, the incorporation of additional constraints and data sources, such as noisy or incomplete measurements, can provide valuable insights into practical scenarios and contribute to the development of robust and reliable inverse problem-solving methodologies.

Acknowledgment

We would like to express our deep appreciation to the esteemed reviewers and the diligent editorial team for their thorough evaluation, valuable insights, and constructive suggestions on our manuscript.

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