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Exponentially fitted IMEX peer methods for an advection-diffusion problem

Dajana Conte, Leila Moradi*, and Beatrice Paternoster
University of Salerno, Fisciano, 84084, Salerno, Italy.

Abstract

In this paper, Implicit-Explicit (IMEX) Exponential Fitted (EF) peer methods are proposed for the numerical solution of an advection-diffusion problem exhibiting an oscillatory solution. Adapted numerical methods both in space and in time are constructed. The spatial semi-discretization of the problem is based on finite differences, adapted to both the diffusion and advection terms, while the time discretization employs EF IMEX peer methods. The accuracy and stability features of the proposed methods are analytically and numerically analyzed.

Keywords. Advection-diffusion problems, EF IMEX peer methods, Boussinesq equation, Finite differences.
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1. Introduction

Advection-diffusion equations are used to model a wide range of engineering and industrial applications [57], as well as many problems in physics, chemistry, and other areas of science. For example, they are used to model the dispersion of solutes in the liquid flowing through a tube [1], the dispersion of detectors in a porous media [30], the dispersion of groundwater soluble salts [35], the heat transfer in a discharge film [39], the transfer of water in the soil [47], the dispersion of pollutants in shallow lakes [51], and the long-term transfer of pollutants into the atmosphere [70].

IMEX methods are widely used strategies for problems characterized by the sum of two terms: a stiff term and a non stiff one. IMEX methods aim to treat the stiff part by implicit methods, so that the stepsize is not constrained by stability requirements and the non-stiff part by explicit methods, due to their low cost per step. The literature is rich in contributions to the field of IMEX numerical methods, see [2, 6, 33, 34] for IMEX Runge-Kutta methods, [3, 29, 31, 36, 50] for IMEX linear multistep methods [8, 9, 14, 42, 68] for IMEX general linear methods, [15, 69] for IMEX two-step Runge-Kutta methods and [64] for IMEX galerkin methods.

In the paper [59], IMEX peer methods based on implicit peer methods for the stiff part [60] and explicit peer methods for the non-stiff part [65], were derived. For more knowledge on the properties of implicit and explicit peer methods refer to [4, 5, 32, 49, 54, 55] and [65, 66], respectively.

When the solution of advection-diffusion problems has a high oscillatory behavior both in space and in time, classical methods can require a very small stepsize to accurately follow the oscillating behavior of the exact solution, because they are based on general purpose formulas constructed to be exact on polynomials to a certain degree (within round off error). As we concentrate on systems with an oscillating exact solution, fitting formulae developed to be accurate on other functions than polynomials can be used more conveniently: this technique recently is known as exponential fitting (see [40, 48]), and the basic functions usually belong to a finite dimensional fitting space.

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* Corresponding author. Email: lmoradi@unisa.it .

The fitting space is chosen on the basis of the apriori known exact solution information and, as a direct result of that choice, the basic functions typically rely on solution related parameters (e.g. oscillation frequency for oscillatory problems). As a consequence, selecting an appropriate fitting space and correctly estimating the unknown parameters are the key challenges associated with an appropriate use of EF methods [21, 22, 24]. EF numerical methods for several problems have been derived in the literature: see for example [17, 20, 41, 43, 44, 62] for numerical differentiation and quadrature, [16, 18, 19, 23, 27, 56, 63] for ordinary differential equations, [25, 26] for partial differential equations, [11, 12] for integral equations.

It is the purpose of this work, first of all, to derive EF IMEX peer method for the numerical solution of Ordinary Differential Equations (ODEs) exhibiting oscillatory solution, based on implicit EF peer method [19] for the stiff part and explicit EF peer method [16] for the non-stiff part. These methods will then be employed for the numerical solution of advection-diffusion problems [10, 45, 46, 67] after a spatial semi-discretization based on EF finite differences.

The remainder of the paper is organized as follows. In section 2, first of all s-stage partitioned peer methods and related order conditions are recalled. Then, IMEX EF peer methods for ODEs are introduced. Section 3 is devoted to the description of an advection-diffusion model, whose discretization by means IMEX EF peer methods, adapted both in space and in time, is derived in section 4. Section 5 is devoted to the stability properties of the proposed methods while numerical experiments are presented in Section 6. Section 7 is devoted to conclusions.

2. EF IMEX peer methods for ODEs

In many engineering and science problems, the right side is naturally split into two parts, one non-stiff and one stiff. For such systems IMEX methods involves implicit methods for the stiff part and explicit methods for the non-stiff part [2, 3, 64].

It is possible to write such systems in the following form:

$$y'(t) = f(t, y(t)) + g(t, y(t)), \qquad y(t_0) = y_0 \in \mathbb{R}^d, \qquad t \in [t_0, T], \tag{2.1}$$

where $f, g : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ are sufficiently smooth to ensure the solution exists and is unique. In addition, $f(t, y(t))$ depicts the stiff process, for diffusion term and $g(t, y(t))$ specifies the non-stiff advection term process.

2.1. **Partitioned peer methods.** System (2.1) can be transformed into a partitioned system of the form [59]:

$$z^{'} = \begin{pmatrix} u^{'} \\ v^{'} \end{pmatrix} = \begin{pmatrix} \tilde{f}(t, u, v) \\ \tilde{g}(t, u, v) \end{pmatrix}, \tag{2.2}$$

by setting $y = u + v,\ u' = \tilde{f}(t, u, v) = f(t, u + v),\ v' = \tilde{g}(t, u, v) = g(t, u + v)$ and $z = \begin{pmatrix} u \\ v \end{pmatrix}$.
Here $u^{'} = f$ is the stiff part and will be treated by implicit peer methods and $v^{'} = g$ shows the non-stiff part and will be treated by explicit peer methods.

We assume that for each stepsize $h > 0$ there exists a starting procedure for approximating the solution at the grid points the internal $t_{0,i} = t_0 + c_i h,\ i = 1, \ldots, s$. The following expression is used for a s-stage two step partitioned peer method with fixed step-size h:

$$\begin{aligned} U_{ni} &= \textstyle\sum_{j=1}^{s} b_{ij}\, U_{n-1,j} + h \sum_{j=1}^{s} a_{ij}\ \tilde{f}(t_{n-1,j}, U_{n-1,j}, V_{n-1,j}) + h \sum_{j=1}^{i} r_{ij}\ \tilde{f}(t_{nj}, U_{nj}, V_{nj}), \\ V_{ni} &= \textstyle\sum_{j=1}^{s} \hat{b}_{ij}\ V_{n-1,j} + h \sum_{j=1}^{s} \hat{a}_{ij}\ \tilde{g}(t_{n-1,j}, U_{n-1,j}, V_{n-1,j}) + h \sum_{j=1}^{i-1} \hat{r}_{ij}\ \tilde{g}(t_{nj}, U_{nj}, V_{nj}), \end{aligned} \tag{2.3}$$

where

$$U_{ni} \approx u(t_{ni}), \qquad V_{ni} \approx v(t_{ni}), \qquad t_{ni} = t_n + c_i h, \qquad i = 1, \ldots, s.$$

No extraordinary numerical solution is determined with distinct features: we assume that $c_s = 1$ and select the other nodes such that $c_i < 1$ for $i = 1, \dots, s-1$.

We define the vectors and matrices

$$U_n = [U_{ni}]_{i=1}^{s}, \; F(U_n, V_n) = \left[\tilde{f}(t_{ni}, U_{ni}, V_{ni})\right]_{i=1}^{s}, \; A = [a_{ij}]_{i,j=1}^{s},$$

$$B = [b_{ij}]_{i,j=1}^{s}, \; R = [r_{ij}]_{i,j=1}^{s},$$

$$V_n = [V_{ni}]_{i=1}^{s}, \; G(U_n, V_n) = [\tilde{g}(t_{ni}, U_{ni}, V_{ni})]_{i=1}^{s}, \; \hat{A} = [\hat{a}_{ij}]_{i,j=1}^{s},$$
$$\hat{B} = \left[\hat{b}_{ij}\right]_{i,j=1}^{s}, \; \hat{R} = [\hat{r}_{ij}]_{i,j=1}^{s},$$

where $A, \hat{A}, B$, and $\hat{B}$ are full matrices, R is a lower triangular matrix and $\hat{R}$ is a strictly lower triangular matrix. Method (2.3) can be then written in the compact form

$$\begin{aligned} U_n &= (B \otimes \mathbb{I}_d)\, U_{n-1} + h\,(A \otimes \mathbb{I}_d)\, F(U_{n-1}, V_{n-1}) + h\,(R \otimes \mathbb{I}_d)\, F(U_n, V_n), \\ V_n &= (\hat{B} \otimes \mathbb{I}_d)\, V_{n-1} + h\,(\hat{A} \otimes \mathbb{I}_d)\, G(U_{n-1}, V_{n-1}) + h\,(\hat{R} \otimes \mathbb{I}_d)\, G(U_n, V_n), \end{aligned} \tag{2.4}$$

where $\mathbb{I}_d$ is the identity matrix of dimension d. The coefficient matrices A, B, R, $\hat{A}$, $\hat{B}$, and $\hat{R}$ are determined to achieve high order (uniform for all components U_n and V_n) together with good stability properties.

We recall that the method (2.4) has consistency order p if $\Delta_n = \begin{pmatrix} \mathcal{O}(h^{p+1}) \\ \mathcal{O}(h^{p+1}) \end{pmatrix}$, where Δ_n denotes the residuals of the stiff and of the non-stiff part which are obtained by inserting the exact solutions in the numerical method (2.4). The following Theorem summarizes the order conditions.

Theorem 2.1. *[59] If the coeffiicients of the partitioned peer method (2.4) satisfy the conditions*

$$\mathrm{AB_i}(m) = \hat{\mathrm{AB}}_\mathrm{i}(m) = 0, \quad m = 0, \dots, p, \; i = 1, \dots, s,$$

with

$$\mathrm{AB_i}(m) = c_i^m - \sum_{j=1}^{s} b_{ij}\,(c_j - 1)^m - m \sum_{j=1}^{s} a_{ij}\,(c_j - 1)^{m-1} - m \sum_{j=1}^{i} r_{ij}\, c_j^{m-1}, \tag{2.5}$$

$$\hat{\mathrm{AB}}_\mathrm{i}(m) = c_i^m - \sum_{j=1}^{s} \hat{b}_{ij}\,(c_j - 1)^m - m \sum_{j=1}^{s} \hat{a}_{ij}\,(c_j - 1)^{m-1} - m \sum_{j=1}^{i-1} \hat{r}_{ij}\, c_j^{m-1}, \tag{2.6}$$

then the s-stage partitioned peer method (2.4) has order of consistency p.

Corollary 2.2. *The partitioned peer method (2.4) has order $p \geq s$ if*

$$B\,\mathbf{1} = \hat{B}\,\mathbf{1} = \mathbf{1}, \tag{2.7a}$$
$$AV_1D = CV_0 - B\,(C - \mathbb{I}_s)V_1 - RV_0D, \tag{2.7b}$$
$$\hat{A}V_1D = CV_0 - \hat{B}\,(C - \mathbb{I}_s)V_1 - \hat{R}V_0D, \tag{2.7c}$$

where $\mathbf{1} = [1, 1, ..., 1]^T$, $C = diag(c_1, \dots, c_s)$, $D = diag(1, \dots, s)$ *and*

$$V_0 = \begin{bmatrix} 1 & c_1 & \dots & c_1^{s-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & c_s & \dots & c_s^{s-1} \end{bmatrix}, \quad V_1 = \begin{bmatrix} 1 & (c_1 - 1) & \dots & (c_1 - 1)^{s-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & (c_s - 1) & \dots & (c_s - 1)^{s-1} \end{bmatrix}. \tag{2.8}$$

2.2. **EF IMEX peer methods.** In order to construct EF IMEX peer methods let us to introduce the linear difference operators

$$\begin{aligned}\mathcal{L}_i[h,\mathbf{w}]\,u(t) &= u(t+c_i\,h) - \sum_{j=1}^{s} b_{ij}\,u(t+(c_j-1)\,h) - h\sum_{j=1}^{s} a_{ij}\,u'(t+(c_j-1)\,h)\\ &\quad - h\sum_{j=1}^{i} r_{ij}\,u'(t+c_j\,h), \quad i=1,\dots,s,\end{aligned} \tag{2.9}$$

$$\begin{aligned}\mathcal{L}_i[h,\hat{\mathbf{w}}]\,v(t) &= v(t+c_i\,h) - \sum_{j=1}^{s} \hat{b}_{ij}\,v(t+(c_j-1)\,h) - h\sum_{j=1}^{s} \hat{a}_{ij}\,v'(t+(c_j-1)\,h)\\ &\quad - h\sum_{j=1}^{i-1} \hat{r}_{ij}\,v'(t+c_j\,h), \quad i=1,\dots,s,\end{aligned} \tag{2.10}$$

where the vectors $\mathbf{w}$ and $\hat{\mathbf{w}}$ contain all the coefficients of the method (2.4) and the u and v functions belong to the fitting space as follows:

$$\mathcal{F} = \{1, t, t^2, \dots, t^K, e^{\pm\mu t}, t\,e^{\pm\mu t}, t^2 e^{\pm\mu t}, \dots, t^P e^{\pm\mu t}\}, \tag{2.11}$$

with $\mu = i\omega$, where $\omega \in \mathbb{R}$ is problem's oscillating frequency. The constants K and P are related by $K+1 = s-1-2P$, and we will consider the choices for classical (CL) and EF peer methods summarized in Table 1.

Table 1. Choices for K and P in the fitting space (2.11).

Method	K	P
CL peer	s	-1
EF peer, with s even	0	$\frac{s}{2}-1$
EF peer, with s odd	-1	$\frac{s-1}{2}$

By using the linear difference operators (2.9)-(2.10), the six-step algorithm presented in [40] and by following the idea introduced in [16, 19] for the construction of EF peer method, order conditions for the coefficient matrices $A, \hat{A}, B, \hat{B}, R$ and $\hat{R}$ of partitioned EF peer methods of the form (2.4) are derived, as summarized in the following theorem.

This theorem makes use of the η-functions introduced in [13, 40]:

$$\eta_{-1}(Z) = \begin{cases} \cos(|Z|^{1/2}) & \text{if} Z \le 0, \\ \cosh(Z^{1/2}) & \text{if} Z > 0, \end{cases}, \qquad \eta_0(Z) = \begin{cases} \sin(|Z|^{1/2})/|Z|^{1/2} & \text{if } Z < 0, \\ 1 & \text{if } Z = 0, \\ \sinh(Z^{1/2})/Z^{1/2} & \text{if } Z > 0, \end{cases}, \tag{2.12}$$

$$\eta_m(Z) = \frac{1}{Z}[\eta_{m-2}(Z) - (2m-1)\eta_{m-1}(Z)], \quad Z \neq 0, \quad \eta_m(0) = \frac{1}{(2m+1)!!} \quad m \ge 1, \tag{2.13}$$

where $Z = \mu^2 h^2$. Moreover, for a vector v of dimension s, we define

$$\theta_{\sigma,v} = \left[\eta_\sigma\,(v_1^2\,Z), \dots, \eta_\sigma\,(v_s^2\,Z)\right]. \tag{2.14}$$

Theorem 2.3. *For any fixed matrices $\bar{B}$, $\hat{\bar{B}}$, lower triangular R and strictly lower triangular $\hat{R} \in \mathbb{R}^{s\times s}$ such that $\bar{B}\,\mathbf{1} = \hat{\bar{B}}\,\mathbf{1} = \mathbf{1}$, the peer method (2.4) has order $p = s$ and is adapted to the fitting space (2.11) with K and P given in Table 1, if the coefficient matrices $A, \hat{A}$ and $B, \hat{B}$ are calculated as*

$$A = (E_1 - \bar{B}E_2 - RE_4)E_3^{-1}, \tag{2.15a}$$
$$B = \bar{B} + \mathcal{H}_1 - ZA\mathcal{H}_2 - ZR\mathcal{H}_3, \tag{2.15b}$$

$$\hat{A} = (E_1 - \hat{\bar{B}}E_2 - \hat{R}E_4)E_3^{-1}, \tag{2.16a}$$
$$\hat{B} = \hat{\bar{B}} + \hat{\mathcal{H}}_1 - Z\hat{A}\mathcal{H}_2 - Z\hat{R}\mathcal{H}_3, \tag{2.16b}$$

where $Z = \mu^2 h^2$ *the matrices* E_i, $i = 1,2,3,4$ *are listed in Table 2 and the matrices* $\mathcal{H}_j$, $j = 1,2,3$ *and* $\hat{\mathcal{H}}_1$ *are listed in Table 3. In Tables 2-3, the matrices* V_0, V_1, C *and* D *are defined in Corollary 2.2, the vector* $\theta_{\sigma,v}$ *is defined in* (2.14)*, matrices* D_k, $k = 1,2,3,4$ *are reported in Table 4 and the matrices* F_k, $k = 1,2,3,4$ *are obtaind by deleting the first column from the corresponding matrix* D_k.

Table 2. Matrices E_i in order conditions (2.15a)-(2.16b).

Method	E_1	E_2	E_3	E_4
CL peer	CV_0	$(C - \mathbb{I}_s)V_1$	$V_1 D$	$V_0 D$
EF peer, with s even	D_1	D_2	D_3	D_4
EF peer, with s odd	F_1	F_2	F_3	F_4

Table 3. Matrices $\mathcal{H}_j$ in order conditions (2.15a)-(2.16b).

Method	$\mathcal{H}_1$	$\hat{\mathcal{H}}_1$	$\mathcal{H}_2$	$\mathcal{H}_3$
CL peer	0	0	0	0
EF peer, s even	0	0	0	0
EF peer, s odd	$(\mathbf{0} \,\vert\, \theta_{-1,\,c} - \bar{B}\theta_{-1,\,c-\mathbf{1}})$	$(\mathbf{0} \,\vert\, \theta_{-1,\,c} - \hat{\bar{B}}\theta_{-1,\,c-\mathbf{1}})$	$(\mathbf{0} \,\vert\, (C - \mathbb{I}_s)\,\theta_{0,\,c-\mathbf{1}})$	$(\mathbf{0} \,\vert\, C\,\theta_{0,\,c})$

Table 4. Elements of the matrices $D_k = D_k(c, Z)$, $k = 1,2,3,4$, for $i = 1, \ldots, s$ and $j = 1, \ldots, s$ if s is even while $j = 1, \ldots, s+1$ if s is odd.

Matrix	j odd, $k_j = \frac{j-1}{2}$	j even, $k_j = \frac{j-2}{2}$
$(D_1)_{i,j}$	$\frac{1}{2^{k_j}}\, c_i^{2k_j}\, \eta_{k_j-1}\left(c_i^2\, Z\right)$	$\frac{1}{2^{k_j}}\, c_i^{2k_j+1}\, \eta_{k_j}\left(c_i^2\, Z\right)$
$(D_2)_{i,j}$	$\frac{1}{2^{k_j}}\, \hat{c}_i^{2k_j}\, \eta_{k_j-1}\left(\hat{c}_i^2\, Z\right)$	$\frac{1}{2^{k_j}}\, \hat{c}_i^{2k_j+1}\, \eta_{k_j}\left(\hat{c}_i^2\, Z\right)$
$(D_3)_{i,j}$	$\frac{k_j}{2^{k_j-1}}\, \hat{c}_i^{2k_j-1}\, \eta_{k_j-1}\left(\hat{c}_i^2\, Z\right) + \frac{1}{2^{k_j}}\, \hat{c}_i^{2k_j+1}\, Z\eta_{k_j}\left(\hat{c}_i^2\, Z\right)$	$\frac{1}{2^{k_j}}\, \hat{c}_i^{2k_j}\, \eta_{k_j-1}\left(\hat{c}_i^2\, Z\right)$
$(D_4)_{i,j}$	$\frac{k_j}{2^{k_j-1}}\, c_i^{2k_j-1}\, \eta_{k_j-1}\left(c_i^2\, Z\right) + \frac{1}{2^{k_j}}\, c_i^{2k_j+1}\, Z\eta_{k_j}\left(c_i^2\, Z\right)$	$\frac{1}{2^{k_j}}\, c_i^{2k_j}\, \eta_{k_j-1}\left(c_i^2\, Z\right)$

Now, in order to derive EF IMEX peer methods, we determine the coefficients $A = A(Z)$, $B = B(Z)$, $R = R(Z)$, $\hat{A} = \hat{A}(Z)$, $\hat{B} = \hat{B}(Z)$ and $\hat{R} = \hat{R}(Z)$ by satisfying the order conditions of the Theorem 2.3. In the following, by using the idea of Soleimani et. al in [59], we describe the derivation of EF IMEX peer method.
Consider the system of ODEs (2.1). Using the framework presented in [2, 3, 64], system (2.1) can be converted into a system partitioned by components as follows:

$$\begin{aligned} y &= u + v, \\ u' &= \tilde{f}(u, v) = f(u + v), \\ v' &= \tilde{g}(u, v) = g(u + v). \end{aligned} \tag{2.17}$$

In order to define an EF IMEX peer methods, we refer to Theorem 2.3 and choose an even number s of stages. Then, by fixing $\bar{B} = \hat{\bar{B}}$, according to Table 3, we have $B(Z) = \hat{B}(Z) = \bar{B}$ and the method (2.4) assumes the form:

$$\begin{aligned} U_n &= (\bar{B} \otimes \mathbb{I}_d)\, U_{n-1} + h\,(A(Z) \otimes \mathbb{I}_d)\, F(U_{n-1} + V_{n-1}) + h\,(R(Z) \otimes \mathbb{I}_d)\, F(U_n + V_n), \\ V_n &= (\bar{B} \otimes \mathbb{I}_d)\, V_{n-1} + h\,(\hat{A}(Z) \otimes \mathbb{I}_d)\, G(U_{n-1} + V_{n-1}) + h\,(\hat{R}(Z) \otimes \mathbb{I}_d)\, G(U_n + V_n). \end{aligned} \tag{2.18}$$

Adding the Equations (2.18), by (2.17), it follows:

$$Y_n =(\bar{B} \otimes \mathbb{I}_d)\, Y_{n-1} + h\,(A(Z) \otimes \mathbb{I}_d)\, F(Y_{n-1}) + h\,(R(Z) \otimes \mathbb{I}_d)\, F(Y_n) \tag{2.19}$$

$$+ h\,(\hat{A}(Z) \otimes \mathbb{I}_d)\, G(Y_{n-1}) + h\,(\hat{R}(Z) \otimes \mathbb{I}_d)\, G(Y_n). \tag{2.20}$$

Method (2.19) is called EF IMEX peer method. We observe that $\bar{B}$, $A(Z)$ and $R(Z)$ are cofficients of an implicit EF peer method of order s while $\bar{B}$, $\hat{A}(Z)$ and $\hat{R}(Z)$ are cofficients of an explicit EF peer method of order s. The order conditions follow directly from Theorem 2.3. Moreover, when $Z \longrightarrow 0$, (2.19) tends to CL IMEX peer methods [59].

3. An Advection-Diffusion model

Dynamic interactions between aquifers and the sea in coastal regions can be modeled by the Boussinesq equation, which can be written in the following form [46, 67]

$$\frac{\partial \phi}{\partial t} = \frac{K}{S}(\phi \frac{\partial^2 \phi}{\partial x^2} + (\frac{\partial \phi}{\partial x})^2 - \vartheta \frac{\partial \phi}{\partial x}),$$

where S is the drainable porosity, K is the hydraulic conductivity and ϑ is impermeable base slope. If $\phi = \phi(X,t)$ shows a slight deviation from the depth of weight, by setting $\gamma = \frac{T}{S}$, $\nu = K\frac{\vartheta}{S}$, where $T = K\phi$ is referred to as transmissivity in groundwater hydrology, the model can be written as:

$$\frac{\partial \phi}{\partial t} = (\gamma \frac{\partial^2 \phi}{\partial x^2} - \nu \frac{\partial \phi}{\partial x}),$$

where $(x,t) \in [0,+\infty) \times [0,+\infty)$ and equipping it with the following initial and boundary conditions

$$\phi(x,0) = \phi_0(x), \quad \phi(0,t) = \phi(X(t),t) = f(t),$$

where $X(t) = cot(\gamma) f(t)$ is the moving boundary depending on time of the parametric formulation [61]. Therefore we consider the linear advection-diffusion problem

$$\begin{array}{ll} \phi_t(x,t) = \gamma \phi_{xx}(x,t) - \nu \phi_x(x,t), & (x,t) \in (0,X) \times (0,T), \\ \phi(x,0) = \phi_0(x), & x \in [0,X], \\ \phi(0,t) = \phi(X,t) = f(t), & t \in [0,T]. \end{array} \tag{3.1}$$

with an arbitrary periodic boundary condition

$$f(t) = \exp(i\omega t). \tag{3.2}$$

Logan and Zlotnik have shown in [45], that the problem described by (3.1)-(3.2) exhibits a solution of the form

$$\phi(x,t) = \exp(\alpha x + i(\beta x + \omega t)) = \exp((\alpha + i\beta)x) \cdot \exp(i\omega t), \tag{3.3}$$

where i is the imaginary unit and

$$\alpha = \frac{\nu}{2\gamma} - \mu, \quad \beta = -\rho, \tag{3.4}$$

with

$$\mu = \frac{1}{2\gamma}\sqrt{2\gamma\sqrt{\omega^2 + \frac{\nu^4}{16\gamma^2}} + \frac{\nu^2}{2\gamma}}, \qquad \rho = \frac{1}{2\gamma}\sqrt{2\gamma\sqrt{\omega^2 + \frac{\nu^4}{16\gamma^2}} - \frac{\nu^2}{2\gamma}}.$$

4. EF IMEX peer methods for Boussinesq equation

In this section, we construct EF IMEX peer methods for problem (3.1)-(3.2), whose solution, by (3.3), oscillates both in space and in time. Following the method of lines [38, 52, 53], we spatially discretize the domain D in

$$D_{\Delta x} = \{(x_n, t) : x_n = n\Delta x,\, n = 0, ..., N-1, \Delta x = \frac{X}{N-1}\},$$

where Δx is the spatial integration step. The resulting semi-discrete system (3.1) assumes the form

$$\begin{aligned} &\phi_0^{'}(t) = f^{'}(t), \\ &\phi_n^{'}(t) = \gamma\theta_{2,n} - \nu\theta_{1,n}, \qquad 1 \le n \le N-2, \\ &\phi_{N-1}^{'}(t) = f^{'}(t), \\ &\phi_n(0) = \phi_0(x_n), \qquad 0 \le n \le N-1, \end{aligned} \tag{4.1}$$

where $h_n(t) \simeq h(x_n, t)$, while $\theta_{1,n}$ and $\theta_{2,n}$ are finite differences approximating the first and second spatial derivatives in (3.1), respectively:

$$\theta_{1,n} = \frac{b_0\phi(x_{n-1}, t) + b_1\phi(x_n, t)}{\Delta x}, \quad \theta_{2,n} = \frac{a_0\phi(x_{n-1}, t) + a_1\phi(x_n, t) + a_2\phi(x_{n+1}, t)}{\Delta x^2}, \tag{4.2}$$

The coefficients a_0, a_1, a_2, b_0 and b_1 will be derived by the EF procedure [26, 28] by considering the following fitting spaces $\mathcal{G}$ and $\mathcal{F}$ for the first and second spatial derivatives, respectively.

$$\begin{aligned} &\mathcal{G} = \{1,\, exp(\zeta x)\}, \\ &\mathcal{F} = \{1,\, exp(\zeta x), xexp(\zeta x)\}, \end{aligned} \tag{4.3}$$

where $\zeta = \alpha + i\beta \in \mathbb{C}$, $z = \zeta\Delta x$.

The choice of fitting spaces (4.3) is motivated by (3.3).

4.1. **Discretization of the diffusion terms.** Using the fitting space $\mathcal{F}$ for the second order spatial derivative, we provide the approximation $\theta_{2,n}$ and evaluate a_0, a_1 and a_2. In summary, we have:

$$\phi_{xx}(x_n, t) \simeq \theta_{2,n} = \frac{a_0\phi(x_{n-1}, t) + a_1\phi(x_n, t) + a_2\phi(x_{n+1}, t)}{\Delta x^2}. \tag{4.4}$$

For computation of coefficients a_0, a_1 and a_2, we consider the following linear difference operator

$$\mathcal{L}[\Delta x]\, \phi(x, t) = \phi_{xx}(x, t) - \frac{a_0\phi(x-\Delta x, t) + a_1\phi(x, t) + a_2\phi(x+\Delta x, t)}{\Delta x^2}. \tag{4.5}$$

Enforcing the exactness of (4.4) on functions of the fitting space $\mathcal{F}$ in (4.3) is equivalent to annihilating the linear difference operator (4.5) on such functions. According to [40], it is enough to annihilate them for $x = 0$:

$$\begin{aligned} &\mathcal{L}[\Delta x]\, 1_{|x=0} = a_0 + a_1 + a_2 = 0, \\ &\mathcal{L}[\Delta x]\, \exp(\zeta x)_{|x=0} = z^2 - a_0\exp(-z) - a_1 - a_2\exp(z) = 0, \\ &\mathcal{L}[\Delta x]\, x\exp(\zeta x)_{|x=0} = 2z + a_0\exp(-z) - a_2\exp(z) = 0. \end{aligned} \tag{4.6}$$

Then, the coefficients are

$$\begin{aligned} a_0 &= -\frac{zexp(z)(2 - 2exp(z) + zexp(z))}{(exp(z) - 1)^2}, \\ a_1 &= \frac{z(2 - 2exp(2z) + z + zexp(2z))}{(exp(z) - 1)^2}, \\ a_2 &= -\frac{z(2 - 2exp(z) + z)}{(exp(z) - 1)^2}. \end{aligned} \tag{4.7}$$

These coefficients are functions of z, where $z = \zeta\Delta x = (\alpha + i\beta)\Delta x$.

Generally, $z \neq 0$ since Δx and ζ are non-zero. Moreover, as z tends to 0, the coefficients tend to the classic finite difference values:

$$a_0 = a_2 = 1,\ a_1 = -2. \tag{4.8}$$

Also EF finite differences preserve the accuracy of classical finite differences, which is equal to 2.

4.2. **Discretization of the advection terms.** Now we discretize the advection term by using the fitting space $\mathcal{G}$ for the first order spatial derivative, we approximate $\theta_{1,n}$ and compute the b_0 and b_1. In this case, we have:

$$\phi_x(x_n, t) \simeq \theta_{1,n} = \frac{b_0\phi(x_{n-1}, t) + b_1\phi(x_n, t)}{\Delta x}, \tag{4.9}$$

We use the following linear difference operator for computation of coefficients b_0 and b_1,

$$\mathcal{M}[\Delta x]\,\phi(x,t) = \phi_x(x,t) - \frac{b_0\phi(x-\Delta x, t) + b_1\phi(x,t)}{\Delta x}. \tag{4.10}$$

By imposing the exactness of (4.9) on functions of the fitting space $\mathcal{G}$ (4.3), i.e. by annihilating the linear difference operator (4.10) on such functions, we obtain

$$\begin{aligned} &\mathcal{M}[\Delta x]\,1_{|x=0} = b_0 + b_1 = 0,\\ &\mathcal{M}[\Delta x]\exp(\zeta x)_{|x=0} = z - b_0\exp(-z) - b_1 = 0. \end{aligned} \tag{4.11}$$

Then, the coefficients are

$$b_0 = \frac{z}{(exp(-z)-1)},\ b_1 = -\frac{z}{(exp(-z)-1)}. \tag{4.12}$$

These coefficients are functions of z, where $z = \zeta\Delta x = (\alpha + i\beta)\Delta x$.

Also in this case, the obtaining coefficients, when z tends to 0, follow the classic finite difference values

$$b_0 = -1,\ \ b_1 = 1. \tag{4.13}$$

Also EF finite differences preserve the accuracy of classical finite differences, which is equal to 1.

4.3. **The EF IMEX peer methods.** Now, we concentrate on time integration of the spatially semidiscretized system (4.1), which assumes the compact form

$$\boldsymbol{\phi}'(t) = \mathcal{A}(z)\boldsymbol{\phi}(t) + \mathcal{B}(z)\boldsymbol{\phi}(t) + g(z,t), \tag{4.14}$$

where

- $x_n = n\Delta x$, $n = 0, \dots, N-1$, $x_0 = 0$, $x_{N-1} = X$,
- $z = (\alpha + i\beta)\Delta x$,
- $\boldsymbol{\phi}(t) = [\phi(x_n, t)]_{n=1}^d$, $d = N-2$,
- $\mathcal{A}(z) = \dfrac{\gamma}{\Delta x^2}\mathtt{diag}(a_0, a_1, a_2), \mathcal{B}(z) = -\dfrac{\nu}{\Delta x}\mathtt{diag}(b_0, b_1, 0)$, tridiagonal matrices of dimension d,
- $g(z,t) = \left(\dfrac{\gamma a_0}{\Delta x^2} - \dfrac{\nu b_0}{\Delta x}\right) f(t)\mathbf{e_1} + \dfrac{\gamma a_2}{\Delta x^2} f(t)\mathbf{e_d}$, with $\mathbf{e_1} = (1, 0, ..., 0)^T \in \mathbb{R}^d$ and $\mathbf{e_d} = (0, ..., 0, 1)^T \in \mathbb{R}^d$.

The vector field of the system of ODEs (4.14) derives from processes of the advection and diffusion. The first summand is diffusion term that is typically stiff and depends on matrix $\mathcal{A}(z)$, and implicit methods have to be used. The part depending on matrix $\mathcal{B}(z)$, advection term, is non-stiff and can be treated by explicit methods [3, 37]. Indeed, IMEX methods, which implicitly integrate only the stiff constituents and explicitly integrate the others, can achieve benefits in stability and efficiency [3, 6, 7, 37].

We consider the fully discretized domain

$$D_{\Delta x,\Delta t} = \{(x_n, t_j) : x_n = n\Delta x,\ t_j = j\Delta t,\ n = 0, ..., N-1,\ j = 0, ..., M-1\},$$

being $\Delta x = \frac{X}{N-1}$, $\Delta t = \frac{T}{M-1}$. As the exact solution of the problem (3.1)-(3.2) has the form (3.3), we consider the time discretization by the adapted s-stage EF IMEX peer method (2.19) with $h = \Delta t$ and $Z = \mu^2(\Delta t)^2 = -\omega^2(\Delta t)^2$. Therefore, by applying to (4.14) the adapted s-stage EF IMEX peer method (2.19), we have:

$$\begin{aligned}\Phi_{j+1} =& (\bar{B} \otimes \mathbb{I}_d)\, \Phi_j + \Delta t\, (A(Z) \otimes \mathbb{I}_d)\, F(\Phi_j) + \Delta t\, (R(Z) \otimes \mathbb{I}_d)\, F(\Phi_{j+1}) + \\ & \Delta t\, (\hat{A}(Z) \otimes \mathbb{I}_d)\, G(\Phi_j) + \Delta t\, (\hat{R}(Z) \otimes \mathbb{I}_d)\, G(\Phi_{j+1}),\end{aligned} \tag{4.15}$$

where

$$\begin{aligned}F(\Phi_j) &= (\mathbb{I}_s \otimes \mathcal{A}(z))\, \Phi_j, \\ G(\Phi_j) &= (\mathbb{I}_s \otimes \mathcal{B}(z))\, \Phi_j + g(z, t_j + c\Delta t),\end{aligned} \tag{4.16}$$

where $c = (c_1, ..., c_s)^T$ and

$$\Phi_j \simeq \begin{pmatrix} \phi(x_1,\ t_j + c\Delta t) \\ \vdots \ \vdots \qquad \vdots \ \vdots \\ \phi(x_{N-2}, t_j + c\Delta t) \end{pmatrix} \ \in \ R^{sd}, \ \ d = N-2,$$

with

$$\phi(x_n,\ t_j + c\Delta t) = [\ \phi(x_n,\ t_j + c_1\Delta t), ...,\ \phi(x_n,\ t_j + c_s\Delta t)]^T \ \in\ R^s,\ n = 1, ..., N-2.$$

Remark 4.1. Observe that fully implicit peer methods derived in [19], applied to system (4.14) assume the form (4.15) with $\hat{A}(Z) = A(Z)$ and $\hat{R}(Z) = R(Z)$.

5. Stability analysis

We now analyze the stability properties of the proposed numerical method. According to the framework of [58], our goal is to verify stability by controlling the propagation of the error caused by an incoming perturbation. The solution of (4.15) Φ_j, $j = 0, ..., M-1$ is then perturbed, as follows:

$$\tilde{\Phi}_j = \Phi_j + \delta^j,$$

and we analyze the behavior of the error

$$E_j = \Phi_j - \tilde{\Phi}_j. \tag{5.1}$$

We have the following stability theorem.

Theorem 5.1. *For the EF IMEX peer methods* (4.15) *applied to the semidiscrete problem* (2.19)*, we obtain the following stability inequality*

$$\|E_{j+1}\|_\infty \leq \ \|\mathcal{M}\|_\infty \, \|E_j\|_\infty\,,$$

where

$$\mathcal{M} = \Omega^{-1}\,\Lambda, \tag{5.2}$$

being

$$\Omega = (\mathbb{I}_{d.s} - \Delta t\, (R(Z) \otimes \mathbb{I}_d)\, (\mathbb{I}_s \otimes \mathcal{A}(z)) - \Delta t\, (\hat{R}(Z) \otimes \mathbb{I}_d)\, (\mathbb{I}_s \otimes \mathcal{B}(z))), \tag{5.3}$$

and

$$\Lambda = (\bar{B} \otimes \mathbb{I}_d + \Delta t\, (A(Z) \otimes \mathbb{I}_d)\, (\mathbb{I}_s \otimes \mathcal{A}(z)) + \Delta t\, (\hat{A}(Z) \otimes \mathbb{I}_d)\, (\mathbb{I}_s \otimes \mathcal{B}(z))). \tag{5.4}$$

Proof. By the discretization error in a fixed time grid point (5.1) and applying the IMEX EF peer method (4.15), we have:

$$E_{j+1} = (\bar{B} \otimes \mathbb{I}_d)\, E_j + \Delta t\, (A(Z) \otimes \mathbb{I}_d)\, (F(\Phi_j) - F(\tilde{\Phi}_j)) + \Delta t\, (R(Z) \otimes \mathbb{I}_d)\, (F(\Phi_{j+1}) \tag{5.5}$$

$$- F(\tilde{\Phi}_{j+1})) + \Delta t(\hat{A}(Z) \otimes \mathbb{I}_d)(G(\Phi_j) - G(\tilde{\Phi}_j)) + \Delta t(\hat{R}(Z) \otimes \mathbb{I}_d)(G(\Phi_{j+1}) - G(\tilde{\Phi}_{j+1})), \tag{5.6}$$

Applying (4.16) and taking into (5.5), we have:

$$E_{j+1} = \Omega^{-1}\,\Lambda\, E_j,$$

where Ω and Λ are given by (5.3) and (5.4), respectively. The tesis immediately follows. $\square$

According to Theorem 5.1, stability is ensured if $\|\mathcal{M}\|_\infty < 1$, where $\mathcal{M}$ given by the (5.2).

Since infinity norm of

$$\|\mathcal{B}(z)\|_\infty = \frac{(|b_0| + |b_1|)|\nu|}{\Delta x},$$

and

$$\|\mathcal{A}(z)\|_\infty = \frac{(|a_0| + |a_1| + |a_2|)|\gamma|}{\Delta x^2},$$

then,

$$\|\mathcal{M}\|_\infty \leq \left\|\Omega^{-1}\right\|_\infty \Big(\left\|\bar{B}\right\|_\infty + \Delta t \left\|A(Z)\right\|_\infty \big(\frac{3(|z(2-2exp(2z)+z+zexp(z))|)|\gamma|}{\Delta x^2 |(exp(z)-1)^2|}\big) + \Delta t \left\|\hat{A}(Z)\right\|_\infty \big(\frac{2(|z|)|\nu|}{\Delta x|(exp(-z)-1)|}\big)\Big).$$

By setting: $\varepsilon_1 = \|A(Z)\|_\infty$, $\hat{\varepsilon_1} = \left\|\hat{A}(Z)\right\|_\infty$, $\varphi(z) = \dfrac{(|z(2-2exp(2z)+z+zexp(z))|)}{|(exp(z)-1)^2|}$ and $\hat{\varphi}(z) = \dfrac{(|z|)}{|(exp(-z)-1)|}$, the stability condition $\|\mathcal{M}\|_\infty < 1$ reduces to:

$$\left\|\Omega^{-1}\right\|_\infty \Big(\left\|\bar{B}\right\|_\infty + 6\frac{\Delta t}{\Delta x^2}|\gamma|\varepsilon_1\varphi(z) + 2\frac{\Delta t}{\Delta x}|\nu|\hat{\varepsilon_1}\hat{\varphi}(z)\Big) < 1.$$

For the classical case, when z tend to zero, $\lim\limits_{z\to 0}\ \varphi(z) = 2$, $\lim\limits_{z\to 0}\ \hat{\varphi}(z) = 1$, then for stability condition it is enough to ensure that:

$$\left\|\Omega^{-1}\right\|_\infty \Big(\left\|\bar{B}\right\|_\infty + 12\frac{\Delta t}{\Delta x^2}|\gamma|\varepsilon_1 + 2\frac{\Delta t}{\Delta x}|\nu|\hat{\varepsilon_1}\Big) < 1.$$

6. Numerical experiments

In this section, we present the numerical results obtained by applying the IMEX EF peer method developed in the previous section to Boussinesq equation (3.1)-(3.2). We report in the tables 5-8, the error calculated as the infinite norm of the difference at the end point between the numerical solution and the exact solution. Moreover, in the figures, we represent the profile of real part of numerical solutions computed by different solvers and compare them based on stability behavior.

Example 6.1. We consider the Boussinesq equation (3.1) with $X = 10$, $T = 10$, and the periodic boundary condition$f(t) = \exp(i\omega t)$ and $\phi_0(x) = e^{\alpha x + i\beta x}$ where α and β given by (3.4).

Consider $s = 2$. By according to section 2.2, in this case $K = 0$ and $P = 0$. We fix $c_1 = 0$, $c_2 = 1$, the corresponding EF IMEX peer method is

$$
\begin{aligned}
&\bar{B}=\begin{bmatrix}0&1\\0&1\end{bmatrix},\ R(Z)=\begin{bmatrix}1&0\\0&1\end{bmatrix},\ \hat{R}(Z)=\begin{bmatrix}0&0\\0&0\end{bmatrix},\\
&\hat{A}(Z)=\begin{bmatrix}0&0\\ \frac{1-\eta_{-1}(Z)}{Z\eta_0(Z)}&-\eta_{-1}(Z)\frac{1-\eta_{-1}(Z)}{Z\eta_0(Z)}+\eta_0(Z)\end{bmatrix},\\
&A(Z)=\begin{bmatrix}0&-1\\ \frac{1-\eta_{-1}(Z)}{Z\eta_0(Z)}+1&\frac{\eta_{-1}(Z)}{Z\eta_0(Z)}(\eta_0(Z)-1-(Z\eta_0(Z)-\eta_{-1}(Z)))+\eta_0(Z)-1\end{bmatrix},
\end{aligned}
\tag{6.1}
$$

by refering to Theorem 2.3.

The corresponding classical IMEX peer method is obtained in the limit when $Z\to 0$ and has coefficients:

$$
\begin{aligned}
&c=\begin{bmatrix}0\\1\end{bmatrix},\quad \bar{B}=\begin{bmatrix}0&1\\0&1\end{bmatrix},\quad R=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad \hat{R}=\begin{bmatrix}0&0\\0&0\end{bmatrix},\\
&A=\begin{bmatrix}0&-1\\1&0\end{bmatrix},\quad \hat{A}=\begin{bmatrix}0&0\\ \frac{1}{2}&\frac{3}{2}\end{bmatrix}.
\end{aligned}
\tag{6.2}
$$

The EF IMEX peer method introduced in Section 4 for the numerical solution of (3.1) in based on two levels of adaptation: in space by means of EF finite differences, and in time by means of EF peer methods. We will use the following notations to indicate the usage of classical or adapted numerical methods:

- **CL FD:** Spatial semidiscretization based on classical finite difference (4.14) with coefficients (4.8) and (4.13),
- **EF FD:** Spatial semidiscretization based on EF finite difference (4.14) with coefficients (4.7) and (4.12),
- **CL IMEX P2:** Classical IMEX peer time integration of order 2 with coefficients (6.2),
- **EF IMEX P2:** EF IMEX peer time integration of order 2 with coefficients (6.1),
- **CL IM P2:** Classical implicit peer time integration of order 2 with coefficients $\bar{B}$, A, R (6.2) and $\hat{A}=A$, $\hat{R}=R$ according to Remark 4.1,
- **EF IM P2:** EF implicit peer time integration of order 2 with coefficients $\bar{B}$, $A(Z)$, $R(Z)$ (6.1) and $\hat{A(Z)}=A(Z)$, $\hat{R(Z)}=R(Z)$ according to Remark 4.1.

The obtained results confirm the effectiveness of the proposed EF IMEX method. In Tables 5 and 7, we compare fully implicit peer methods [19] and IMEX peer methods for system (4.14). We observe that EF explicit peer methods reported in [16, 18] are unstable because of the presence of stiff part. From Tables 5 and 7, we observe that implicit and IMEX peer methods have the same behavior in accuracy but the IMEX methods have smaller computational cost. Also the results listed in Tables 5 and 7 show that the EF peer methods produce smaller errors with respect to their classic counterparts and the best results are obtained by adapting the method both in space and time.

Tables 6 and 8 show the behavior of the methods when the parameters ω, α and β characterizing the exact solution are not known exactly. By denoting with δ the relative error in the parameters, we apply EF FD combined with EF IM and EF IMEX peer methods whose coefficients are calculated in correspondence of $z=(\alpha(1+\delta)+i\beta(1+\delta))\Delta x$, $Z=-\omega^2(1+\delta)^2(\Delta t)^2$. Observe that the error of EF peer methods keeps smaller than the corresponding classic counterparts and when δ increases it approaches the result of classic methods.

Figure 1 represents the profile of real part of numerical solution computed by both classical methods in space and time, while in Figure 6 we employed EF methods both in space and time. We observe that an unstable behavior of CL FD+CL IMEX P2 solver is clearly visible, while EF FD+EF IMEX P2 solver is able to correctly reproduce the profile of the solution.

Table 5. Errors with parameter values $\gamma = 5, \nu = 2, \omega = 2, \Delta x = \Delta t = 0.1$.

Space/Time	CL IM P2	CL IMEX P2	EF IM P2	EF IMEX P2
CL FD	$6.05e-03$	$2.59e-02$	$5.04e-03$	$5.04e-03$
EF FD	$9.84e-03$	$2.71e-02$	$6.67e-14$	$6.57e-14$

Table 6. Errors with parameter values $\gamma = 5, \nu = 2, \omega = 2, \Delta x = \Delta t = 0.1$ when the coefficients of EF FD and EF P2 are computed in correspondence of $z = (\alpha(1+\delta) + i\beta(1+\delta))\Delta x$, $Z = -\omega^2(1+\delta)^2(\Delta t)^2$.

	EF	EF $\delta = 10^{-7}$	EF $\delta = 10^{-1}$	CL
FD + IM P2	$6.67e-14$	$2.40e-09$	$2.30e-03$	$6.05e-03$
FD + IMEX P2	$6.57e-14$	$5.39e-09$	$7.76e-03$	$2.59e-02$

Table 7. Errors with parameter values $\gamma = 5, \nu = 2, \omega = 20, \Delta x = \Delta t = 0.1$.

Space/Time	CL IM P2	CL IMEX P2	EF IM P2	EF IMEX P2
CL FD	$1.03e-01$	$1.53e+00$	$3.85e-03$	$3.85e-03$
EF FD	$3.98e-01$	$1.53e+00$	$5.36e-15$	$4.00e-15$

Table 8. Errors with parameter values $\gamma = 5, \nu = 2, \omega = 20, \Delta x = \Delta t = 0.1$ when the coefficients of EF FD and EF P2 are computed in correspondence of $z = (\alpha(1+\delta) + i\beta(1+\delta))\Delta x$, $Z = -\omega^2(1+\delta)^2(\Delta t)^2$.

	EF	EF $\delta = 10^{-7}$	EF $\delta = 10^{-1}$	CL
FD + IM P2	$5.36e-15$	$4.31e-08$	$3.37e-02$	$4.03e-01$
FD + IMEX P2	$4.00e-15$	$2.57e-07$	$2.12e-01$	$1.53e+00$

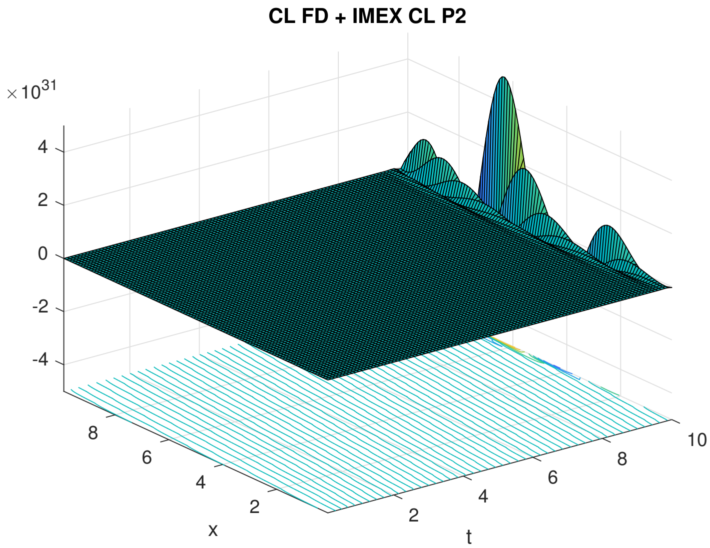

Figure 1. Profile of real part of numerical solution computed by CL FD+CL IMEX P2 solver for $\gamma = 5, \nu = 20, \omega = 20, \Delta x = \Delta t = 0.1$.

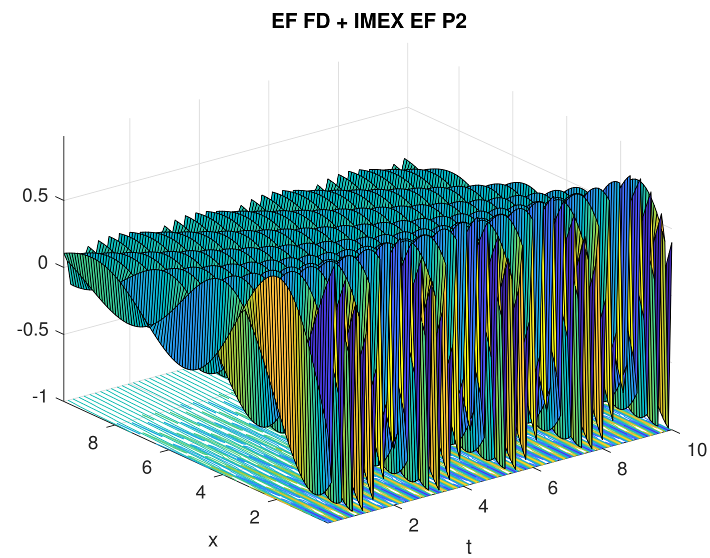

Figure 2. Profile of real part of numerical solution computed by EF FD+EF IMEX P2 solver for $\gamma = 5, \nu = 20, \omega = 20, \Delta x = \Delta t = 0.1$.

7. Conclusions

We have developed a novel IMEX method for numerical solution of advection-diffusion problems with oscillatory solutions. In fact, we have proposed an adapted numerical method both in space and in time. The spatial semidiscretization of the problem is based on finite differences, adapted to both the diffusion and advection terms while the time discretization employed EF implicit-explicit peer methods. Numerical experiments have shown the convenience of the new method with respect to classical, fully implicit, and IMEX methods.

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