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Finite element solution of a class of parabolic integro-differential equations with nonhomogeneous jump conditions using FreeFEM++

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Abstract

The finite element solution of a class of parabolic integro–partial differential equations with interfaces is presented. The spatial discretization is based on the triangular element while a two-step implicit scheme together with the trapezoidal method is employed for time discretization. For the spatial discretization, the elements in the neighborhood of the interface are more refined such that the interface is at σ-distance from the approximate interface. The convergence rate of optimal order in L^2-norm is analyzed with the assumption that the interface is arbitrary but smooth. Examples are given to support the theoretical findings with implementation on FreeFEM++.

Keywords. Optimal convergence, Integro–partial differential equations, Trapezoidal, Interface.
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1. Introduction

We consider a class of parabolic integro-differential equations of the form

$$u_t(x,t)-\nabla\cdot(a(x)\nabla u(x,t))+b(x)u(x,t)-\int_0^t \nabla\cdot(\alpha(x)\nabla u(x,s))-\beta(x)u(x,s)\,ds = f(x,t), \quad \text{in } \Omega\times(0,T], \tag{1.1}$$

with initial and boundary conditions

$$\begin{cases} u(x,0) &= u_0(x), \quad \Omega,\\ u(x,t) &= 0, \qquad \partial\Omega\times[0,T], \end{cases} \tag{1.2}$$

and interface conditions

$$\begin{cases} [u]_\Gamma &= 0,\\ \left[a(x)\dfrac{\partial u(x,t)}{\partial n}+\alpha(x)\displaystyle\int_0^t \dfrac{\partial u(x,s)}{\partial n}ds\right]_\Gamma &= g(x,t), \end{cases} \tag{1.3}$$

where $0<T<\infty$, Ω is a bounded convex domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and partitioned into two subdomains (Ω_1 and Ω_2) by arbitrary but smooth interface Γ (See Figure 1). $[u]$ represents the jump of u across the interface Γ while n is the unit outward normal to the boundary $\partial\Omega_1$. The interface conditions are defined as the difference of the limiting values from each side of the interface. The input function $f(x,t)$ is assumed continuous over each domain but discontinuous across the interface for $t\in[0,T]$. For simplicity of exposition, $a(x)$ is assumed positive and piecewise constant on Ω (i.e. $a(x)=a^i\in\mathbb{R}^+$, $x\in\Omega_i$, $i=1,2$), while $b(x)$, $\alpha(x)$ and $\beta(x)$ are assumed non-negative and piecewise constant on Ω.

The integro-differential equations of the form (1.1) are often referred to as parabolic partial differential equation with memory. Equations of the form (1.1)−(1.3) arise in many applications, for example, heat conduction in material with memory [13, 22], nuclear reactor dynamics [10, 28], compression of poro-viscoelasticity media [23], and the epidemic

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phenomena in biology [9]. In a more general setting, Ω represents a domain consisting of more than one material medium with different properties such as diffusion constants, conductivities, etc. Thus (1.1) becomes an interface problem with interface condition (1.3). Solutions of interface problems are less regular in the entire physical domain than in each individual material region [7, 29].

In the absence of memory term in (1.1), convergence analysis of finite element method (FEM) for parabolic interface problem has been studied by several authors, see [2, 4, 5, 20, 21, 32, 36, 45] for recent works. Solutions of parabolic integro-differential equations without interfaces using standard finite element methods have been presented in [8, 10, 30, 34, 40, 41, 43, 46, 49]. Several other techniques have been studied, including mixed finite element methods [18, 27, 31, 33, 44], domain decomposition method [37], two-grid finite element method [11, 26], discontinuous Galerkin method [48], hp-local discontinuous Galerkin methods [35], mixed covolume methods [50, 52], least-squares finite element methods [19] and the weak Galerkin method [51]. The finite element solution of time fractional gas dynamics model using cubic Hermit element was discussed in [42]. However discussion on the finite element solution of parabolic integro-differential equations with interfaces is still rare in literature.

Analysis of finite element solution of parabolic integro-differential equations with interfaces was carried out by Deka and Deka [15] and later in [16]. In both articles, the special case of homogeneous interface condition was considered. Using finite element space consisting of piecewise linear functions on a mesh that does not necessarily resolve the interface, they established convergence rates of optimal order for semi discretization and full discretization. Their time discretization, based on backward difference scheme, was shown to be first-order accurate. In [16] where the memory term is a second-order partial differential operator, the authors did not include the contribution of the memory term to the flux across the interface. This could have adverse effect on the convergence of the method. Finite element solution of a second-order hyperbolic integro-differential equation with interface has been presented in [3]. Ritz-Volterra operator and some auxiliary error estimates in the neighborhood of the interface were used to obtain convergence estimates. The scheme was implemented using MATLAB. Recently, [38] proved posteriori error estimates for linear non-interface parabolic integro-differential equation on finite element with a two-step backward time discretization formula. The second-order accuracy in time was achieved.

In this work, we include the contribution of the memory term to the flux across the interface in our model problem (1.1)$-$(1.3). The interface is first approximated by piecewise continuous straight lines and the mesh is fitted to this approximation. Sufficient conditions that guarantee the existence of a unique solution are given. Under these assumptions, the convergence rate of optimal order in $L^2(\Omega)$ norm is proved. In the work of [15, 16], the Euler scheme (which is only first-order accurate) was used for the time discretization and the memory term was approximated by a rectangle rule (which is also first-order accurate in time). However in this work, use is made of 2-step implicit scheme and the use of the trapezoidal method is proposed for the approximation of the memory term. In [39], it was reported that there is no generally available software that is able to solve the parabolic integro-differential equation numerically. We further demonstrate, in this work, that finite element solution of parabolic integro-differential equation can be implemented using FreeFEM++ and convergence of optimal order will still be obtained if certain conditions are met.

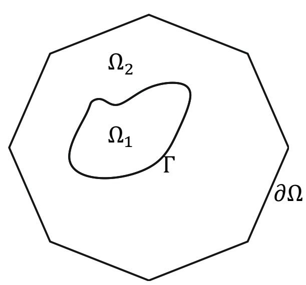

Figure 1. A convex polygonal domain $\Omega = \Omega_1 \cup \Omega_2$ with interface Γ

This paper is organized as follows. In section 2, we define the notations used in this work and describe a finite element discretization of the problem. In section 3, we give the fully discrete version of (2.1) and analyze convergence rate of optimal order in $L^2(\Omega)$-norm. We confirm our theoretical analysis with examples in section 4. Throughout this paper, C is a generic positive constant (which is independent of the mesh parameter h and the time step size k) and may take on different values at different occurrences.

2. Preliminaries

In this section, the regularity of the solution under the appropriate regularity of u_0 and f as well as finite element discretization is considered.

2.1. **Notation and Regularity.** The definitions and notations for Sobolev spaces and norms in [1] are used in this work. For

$$\aleph(x,t) = \begin{cases} \aleph_1(x,t) & \text{in} \quad \Omega_1 \times (0,T], \\ \aleph_2(x,t) & \text{in} \quad \Omega_2 \times (0,T], \end{cases}$$

with $\aleph_1(x,t) \in H^1(\Omega_1)$ and $\aleph_2(x,t) \in H^1(\Omega_2)$, we define

$$\|\aleph(x,t)\|_{H^1(\Omega)} = \|\aleph_1(x,t)\|_{H^1(\Omega_1)} + \|\aleph_2(x,t)\|_{H^1(\Omega_2)}, \quad t \in (0,T].$$

Let v_i be the restriction of v to Ω_i, $i = 1,2$. The following spaces will be required for our analysis

$$X = \left\{v : v \in H^1(\Omega), v_i \in H^2(\Omega_i)\right\}, \qquad Y = \left\{v : v \in L^2(\Omega), v_i \in H^1(\Omega_i)\right\},$$

equipped with the norms

$$\begin{aligned} \|v\|_X &= \|v\|_{H^1(\Omega)} + \|v_1\|_{H^2(\Omega_1)} + \|v_2\|_{H^2(\Omega_2)} \quad \forall\, v \in X, \\ \|v\|_Y &= \|v\|_{L^2(\Omega)} + \|v_1\|_{H^1(\Omega_1)} + \|v_2\|_{H^1(\Omega_2)} \quad \forall\, v \in Y. \end{aligned}$$

We shall use the following norm for our error estimation.

$$\|v\|_Z = \|v_1\|_{H^2(\Omega_1)} + \|v_2\|_{H^2(\Omega_2)} \quad \forall\, v \in Z.$$

Let $A(.,.)$ and $B(t,s;.,.)$ be bilinear forms on $H^1(\Omega) \times H^1(\Omega)$ defined by

$$A(\phi,\psi) = \int_\Omega a(x)\nabla\phi \cdot \nabla\psi + b(x)\phi\psi \, dx,$$

$$B(t,s;\phi,\psi) = \int_\Omega \alpha(x)\nabla\phi \cdot \nabla\psi + \beta(x)\phi\psi \, dx.$$

The weak form of (1.1)–(1.3) is to find $u(t) \in H_0^1(\Omega)$, $t \in (0,T]$ such that

$$(u_t, v) + A(u,v) + \int_0^t B(t,s;u,v)\, ds = (f,v) + \langle g, v\rangle_\Gamma \quad \forall\, v \in H_0^1(\Omega),\ t \in (0,T], \tag{2.1}$$

where

$$(\phi,\psi) = \int_\Omega \phi\psi \, dx, \qquad \langle \phi,\psi\rangle_\Gamma = \int_\Gamma \phi\psi \, ds.$$

We have the following regularity result:

Theorem 2.1. *Assume $a(x)$ is positive and $b(x)$ is nonnegative for $x \in \Omega$. Let $f \in H^1(0,T;L^2(\Omega))$, $g \in H^1(0,T;H^{1/2}(\Gamma))$ and $u_0 \in H_0^1(\Omega)$. Then problem (1.1)–(1.3) has a unique solution*

$$u \in L^2(0,T; X \cap H_0^1(\Omega)) \cap H^1(0,T;Y).$$

Proof. Using the regularity of result of the parabolic interface problems [12],

$$u \in L^2(0,T; X \cap H_0^1(\Omega)) \cap H^1(0,T;Y).$$

Now, let $\tilde{u}$ be another solution of (1.1)–(1.3), then

$$(\tilde{u}_t, v) + A(\tilde{u},v) + \int_0^t B(t,s;\tilde{u},v)\, ds = (f,v) + \langle g, v\rangle_\Gamma \quad \forall\, v \in H_0^1(\Omega),\ t \in (0,T]. \tag{2.2}$$

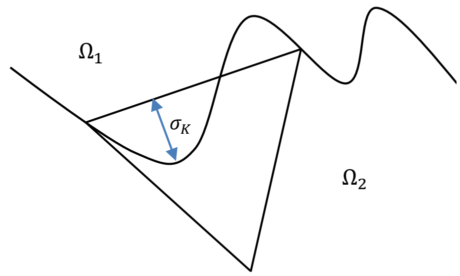

FIGURE 2. A typical interface element

Subtract (2.2) from (2.1), we have

$$(u_t - \tilde{u}_t, v) + A(u_t - \tilde{u}, v) + \int_0^t B(t, s; u_t - \tilde{u}, v)\; ds \;=\; 0 \quad \forall\, v \in H_0^1(\Omega),\; t \in (0, T].$$

Let $v = u - \tilde{u}$, then we obtain

$$\frac{1}{2}\frac{d}{dt}\|u - \tilde{u}\|_{L^2(\Omega)} + a(x)\|\nabla u - \nabla\tilde{u}\|_{L^2(\Omega)} + b(x)\|u - \tilde{u}\|_{L^2(\Omega)} + \min\{\alpha(x), \beta(x)\}\int_0^t \|u - \tilde{u}\|_{H^1(\Omega)}\; ds \;\le\; 0, \quad \forall\, t \in (0, T].$$

This implies

$$\frac{1}{2}\frac{d}{dt}\|u - \tilde{u}\|_{L^2(\Omega)} + a(x)\|\nabla u - \nabla\tilde{u}\|_{L^2(\Omega)} + b(x)\|u - \tilde{u}\|_{L^2(\Omega)} + \min\{\alpha(x), \beta(x)\}t\|u - \tilde{u}\|_{L^\infty(0,t;H^1(\Omega))} \;\le\; 0, \quad \forall\, t \in (0, T].$$

Integrating both sides from 0 to t and using the fact that $u(x, 0) = \tilde{u}(x, 0) = u_0(x)$, we have

$$\|u - \tilde{u}\|_{L^2(\Omega)} + a(x)\int_0^t \|\nabla u - \nabla\tilde{u}\|_{L^2(\Omega)}\; d\tau + b(x)\int_0^t \|u - \tilde{u}\|_{L^2(\Omega)}\; d\tau + \min\{\alpha(x), \beta(x)\}\int_0^t \tau\|u - \tilde{u}\|_{L^\infty(0,\tau;H^1(\Omega))}\; d\tau \;\le\; 0, \quad \forall\, t \in (0, T].$$

This implies $u(x, t) = \tilde{u}(x, t)$ $a.e.(x, t) \in \Omega \times [0, T)$. □

2.2. **Finite Element Discretization.** We adopt the finite element discretization used in [3]. $\mathcal{T}_h$ denotes a conforming triangulation of Ω. Let h_K be the diameter of an element $K \in \mathcal{T}_h$ and $h = \max_{K\in\mathcal{T}_h} h_K$, $0 < h < 1$. Let $\mathcal{T}_h^\star$ denote the set of all elements that are intersected by the interface Γ;

$$\mathcal{T}_h^\star = \{K \in \mathcal{T}_h : K \cap \Gamma \neq \emptyset\},$$

$K \in \mathcal{T}_h^\star$ is called an interface element and we write $\Omega_h^\star = \bigcup_{K\in\mathcal{T}_h^\star} K$.

The domain Ω_1 is approximated by a polygonal domain Ω_1^h with boundary Γ_h whose vertices all lie on the interface Γ. Ω_2^h represents the domain with Γ_h and $\partial\Omega$ as its interior and exterior boundaries respectively. For each $K \in \mathcal{T}_h^\star$, let σ_K be the maximum distance between Γ and Γ_h (see Figure 2) and let $\sigma = \max_{K\in\mathcal{T}_h} \sigma_K$. The triangulation $\mathcal{T}_h$ of the domain Ω is fitted to Ω_1^h such that $\sigma = O(h^2)$ and satisfies the conditions stated in [3].

Let $S_h \subset H_0^1(\Omega)$ represent the space of continuous piecewise linear functions on $\mathcal{T}_h$ vanishing on $\partial\Omega$. The FE solution $u_h(x, t) \in S_h$ is represented as

$$u_h(x, t) = \sum_{j=1}^{N_h} \alpha_j(t)\phi_j(x)\;,$$

where each basis function ϕ_j, $(j = 1, 2, \ldots, N_h)$ is a pyramid function with unit height.

The approximation g_h of g is given as

$$g_h(x,t) = \sum_{j=1}^{n_h} \beta_j(t)\psi_j(x) \,,$$

where $\{z_j\}_{j=1}^{n_h}$ is the set of all nodes of the triangulation $\mathcal{T}_h$ that lie on the interface Γ and $\{\psi_j\}_{j=1}^{n_h}$ is the hat functions corresponding to $\{z_j\}_{j=1}^{n_h}$ in the space S_h. We have the following ([12]).

Lemma 2.2. *Assume that $g \in H^2(\Gamma)$. Then we have*

$$|\langle g, v_h\rangle_\Gamma - \langle g_h, v_h\rangle_{\Gamma_h}| \leq Ch^{3/2}\|g\|_{H^2(\Gamma)}\|v_h\|_{H^1(\Omega_h^\star)} \qquad \forall\, v_h \in S_h.$$

The result below is useful in our analysis. See [47, Chapter 6] for proofs.

Lemma 2.3. *Let $f \in H^2(\Omega)$ for $t \in [0,T]$, we have*

$$|(f,v) - (f,v)_h| \quad \leq \quad Ch^2\|f\|_{H^2(\Omega)}\|v\|_{H^1(\Omega)}.$$

3. Error Estimate

We discuss a fully discrete scheme based on two-step backward difference approximation. Convergence rate of optimal order in $L^2(\Omega)$-norm is analyzed. The interval $[0,T]$ is divided into M equally spaced (for simplicity) subintervals:

$$0 = t_0 < t_1 < \ldots < t_M = T,$$

with $t_n = nk$, $k = T/M$ being the time step. Let

$$u^n = u(x,t_n), \quad f^n = f(x,t_n), \quad \text{and} \quad g^n = g(x,t_n).$$

For a given sequence $\{w_n\}_{n=0}^M \subset L^2(\Omega)$, we have the backward differentiation formula of order two defined by

$$\partial w^n = \frac{3w^n - 4w^{n-1} + w^{n-2}}{2k}, \qquad n = 2,3,\ldots,M.$$

Let z be any of a, b, α, β. For each triangle $K \in \mathcal{T}_h$, let $z_K = z_i$ if $K \subset \Omega_i^h$, $i = 1,2$. Then the approximation z_h of z is defined as

$$z_h = z_K \qquad \forall\, K \in \mathcal{T}_h.$$

The fully discrete finite element approximation to (2.1) is defined as follows:
Let $U_h^0 = \pi_h u_0$, find $U_h^n \in S_h$, such that

$$(\partial U_h^n, v_h)_h + A_h(U_h^n, v_h) + \frac{k}{2}\sum_{j=1}^{n}\Big(B_h(t_n,t_j;U_h^{j-1},v_h) + B_h(t_n,t_j;U_h^j,v_h)\Big)$$
$$= (f^n, v_h)_h + \langle g_h^n, v_h\rangle_{\Gamma_h} \qquad \forall\, v_h \in S_h \quad n = 2,3,\ldots,M, \quad (3.1)$$

where $(\psi,\phi)_h : H^1(\Omega)\times H^1(\Omega) \to \mathbb{R}$, $A_h(\phi,\psi) : H^1(\Omega)\times H^1(\Omega) \to \mathbb{R}$ and $\langle g(x,t), v_h\rangle_{\Gamma_h} : H^{1/2}(\Gamma)\times H^1(\Omega) \to \mathbb{R}$ are defined as

$$(\psi,\phi)_h = \sum_{K\in\mathcal{T}_h}\int_K \psi\phi\, dx\,, \quad A_h(\phi,\psi) = \sum_{K\in\mathcal{T}_h}\int_K [a_h\nabla\phi\cdot\nabla\psi + b_h\phi\psi]\; dx\,,$$

$$B_h(t_n,t_j;\phi,\psi) = \sum_{K\in\mathcal{T}_h}\int_K [\alpha_h\nabla\phi\cdot\nabla\psi + \beta_h\phi\psi]\; dx, \quad \langle g(x,t), v_h\rangle_{\Gamma_h} = \int_{\Gamma_h} g(x,t)\phi\, dx,$$

$\forall\ \phi,\psi \in H^1(\Omega)$, $g \in H^{1/2}(\Gamma)$, $t \in [0,T]$. $(\psi,\phi)_h : H^1(\Omega)\times H^1(\Omega) \to \mathbb{R}$, $A_h(\phi,\psi) : H^1(\Omega)\times H^1(\Omega) \to \mathbb{R}$ and $\langle g(x,t), v_h\rangle_{\Gamma_h} : H^{1/2}(\Gamma)\times H^1(\Omega) \to \mathbb{R}$ are the discrete versions of $(\psi,\phi) : H^1(\Omega)\times H^1(\Omega) \to \mathbb{R}$, $A(\phi,\psi) : H^1(\Omega)\times H^1(\Omega) \to \mathbb{R}$ and $\langle g(x,t), v_h\rangle_\Gamma : H^{1/2}(\Gamma)\times H^1(\Omega) \to \mathbb{R}$ respectively and are obtained numerically using quadrature schemes. See [47] and the reference therein, for more information on numerical integration in FEM.

Let

$$\mathcal{A}(u,v) = A(u,v) + \int_0^t B(t,s;u,v)\ ds \quad \text{for } u,v \in H^1(\Omega),$$

and $\mathcal{A}_h(u,v)$ be the semi-discrete version of $\mathcal{A}(u,v)$ defined as

$$\mathcal{A}_h(u,v) = A_h(u,v) + \int_0^t B_h(t,s;u,v)\ ds \quad \text{for } u,v \in S_h.$$

Let $R_h : X \cap H_0^1(\Omega) \to S_h$ be the Ritz-Volterra projection (which was first presented in [30]) of the exact solution u in S_h defined by

$$\mathcal{A}_h(R_h u, \phi) = \mathcal{A}(u,\phi) \quad \forall\, \phi \in S_h,\ t \in [0,T]. \tag{3.2}$$

For this projection, we have

Lemma 3.1. *Let $a_i(x)$, $b_i(x)$, $\alpha_i(x)$, $\beta_i(x)$ be continuous on $\Omega_i \times (0,T]$, $i = 1,2$. Assume that $u \in Z \cap H_0^1$ and let $R_h u$ be defined as in (3.2), then*

$$\|R_h u - u\|^2_{H^1(\Omega)} \;\leq\; Ch^2 \left(\|u\|_Z^2 + \int_0^t \|u(s)\|_Z^2 ds \right), \tag{3.3}$$

$$\|R_h u - u\|^2_{L^2(\Omega)} \;\leq\; Ch^4 \left(\|u\|_Z^2 + \int_0^t \|u(s)\|_Z^2 ds \right), \tag{3.4}$$

$$\|(R_h u - u)_t\|_{H^1(\Omega)} \;\leq\; Ch^2 \left(\|u\|_Z^2 + \|u_t\|_Z^2 + \int_0^t \|u(s)\|_Z^2 + \|u_s(s)\|_Z^2\ ds \right), \tag{3.5}$$

$$\|(R_h u - u)_t\|^2_{L^2(\Omega)} \;\leq\; Ch^4 \left(\|u\|_Z^2 + \|u_t\|_Z^2 + \int_0^t \|u(s)\|_Z^2 + \|u_s(s)\|_Z^2\ ds \right). \tag{3.6}$$

Proof. It follows from [3, Lemma 4.3] with sufficient refinement in the neighbourhood of the interface such that $\sigma = O(h^2)$. □

Below is our main result:

Theorem 3.2. *Let u^n and U_h^n be the solutions of (2.1) and (3.1) respectively. Suppose $a_i(x,t)$, $b_i(x,t)$, $\alpha_i(x,t)$, $\beta_i(x,t)$ and $f_i(x,t)$ are continuous on $\Omega_i \times (0,T]$, $i = 1,2$ and $g(x,t) \in L^2(0,T; H^{1/2}(\Gamma) \cap H^2(\Gamma))$, $u \in H^3(0,T; L^2(\Omega)) \cap L^2(0,T;Z)$. Let $\mu_1 = \min\{a_1, a_2\}$. For $0 < k < \frac{1}{1+\mu_1}$, there exists a positive constant C independent of h and k such that*

$$\|u^n - U_h^n\|_{L^2(\Omega)} \;\leq\; \left[k^2 + h^2\right] C(u, u_t, u_{ttt}, f, g). \tag{3.7}$$

Proof. We split the error

$$u^n - U_h^n = u^n - R_h u^n + R_h u^n - U_h^n = \theta^n + z^n.$$

The term θ^n can be bounded using Lemma 3.1. In the sequel, we find an estimate for z^n.

Letting $z^n = R_h u^n - U_h^n$ in (3.1) and using (3.2), we have

$$(\partial z^n, v_h)_h + A_h(z^n, v_h) \;=\; B_1 + B_2 + B_3, \tag{3.8}$$

where

$$\begin{aligned}
B_1 &= (\partial(R_h u^n - u^n), v_h)_h + (\partial u^n - u_t^n, v_h) + (\partial u^n, v_h)_h - (\partial u^n, v_h), \\
B_2 &= (f^n, v_h) - (f^n, v_h)_h + \langle g^n, v_h\rangle_\Gamma - \langle g_h^n, v_h\rangle_{\Gamma_h}, \\
B_3 &= \frac{k}{2}\sum_{j=1}^n \Big(B_h(t_n, t_j; U_h^{j-1}, v_h) + B_h(t_n, t_j; U_h^j, v_h) \Big) \int_0^{t_n} B_h(t,s; R_h u, v_h) ds,
\end{aligned}$$

with $v_h = z^n$. The Lemma 2.3 is used to obtain

$$B_1 \leq 2\|\partial(R_h u^n - u^n)\|^2_{L^2(\Omega)} + \frac{1}{4}\|z^n\|^2_{L^2(\Omega)} + 2\|\partial u^n - u^n_t\|^2_{L^2(\Omega)} + \gamma C h^4\|\partial u^n\|^2_X + \frac{1}{4\gamma}\|z^n\|^2_{H^1(\Omega)} \tag{3.9}$$

and Lemmas 2.2, 2.3 are used to obtain

$$\begin{aligned} B_2 &\leq Ch^2\|f^n\|_{H^2(\Omega)}\|z^n\|_{H^1(\Omega)} + Ch^2\|g^n\|_{H^2(\Gamma)}\|z^n\|_{H^1(\Omega)}, \\ &\leq C(\gamma)h^4\left(\|f^n\|^2_{H^2(\Omega)} + \|g^n\|^2_{H^2(\Gamma)}\right) + \frac{1}{2\gamma}\|z^n\|^2_{H^1(\Omega)}. \end{aligned} \tag{3.10}$$

For B_3, we have

$$\begin{aligned} B_3 &= \frac{k}{2}\sum_{j=1}^{n}\left(B_h(t_n,t_j;R_h u^{j-1},z^n) + B_h(t_n,t_j;R_h u^j,z^n)\right) \\ &\quad - \int_0^{t_n} B_h(t,s;R_h u,z^n)\,ds - \frac{k}{2}\sum_{j=1}^{n}\left(B_h(t_n,t_j;z^{j-1},z^n) + B_h(t_n,t_j;z^j,z^n)\right). \end{aligned}$$

Using [17, Section 5.2],

$$\begin{aligned} B_3 &\leq Ck^2\|R_h u^n_t\|_{H^1(\Omega)}\|z^n\|_{H^1(\Omega)} + Ck^2\|R_h u^0_t\|_{H^1(\Omega)}\|z^n\|_{H^1(\Omega)} \\ &\quad + k\mu_3\|z^n\|_{H^1(\Omega)}\sum_{j=0}^{n-1}\|z^j\|_{H^1(\Omega)}, \\ &\leq Ck^4\left(\|R_h u^n_t\|^2_{H^1(\Omega)} + \|R_h u^0_t\|^2_{H^1(\Omega)}\right) + \frac{3}{4\gamma}\|z^n\|^2_{H^1(\Omega)} + \gamma\mu_3^2 t_n k\sum_{j=0}^{n-1}\|z^j\|^2_{H^1(\Omega)}. \end{aligned} \tag{3.11}$$

Substituting (3.9)–(3.11) into (3.8), we have

$$\begin{aligned} \frac{1}{k}\|z^n\|^2_{L^2(\Omega)} + \mu_1\|z^n\|^2_{H^1(\Omega)} &\leq \frac{C}{k}\left(\|z^n\|_{L^2(\Omega)}\|z^{n-1}\|_{L^2(\Omega)} + \|z^n\|_{L^2(\Omega)}\|z^{n-2}\|_{L^2(\Omega)}\right) + \frac{1}{4}\|z^n\|^2_{L^2(\Omega)} \\ &\quad + 2\|\partial(R_h u^n - u^n)\|^2_{L^2(\Omega)} + 2\|\partial u^n - u^n_t\|^2_{L^2(\Omega)} + Ch^4\|\partial u^n\|^2_X \\ &\quad + \frac{3}{2\gamma}\|z^n\|^2_{H^1(\Omega)} + Ck^4\left(\|R_h u^n_t\|^2_{H^1(\Omega)} + \|R_h u^0_t\|^2_{H^1(\Omega)}\right) \\ &\quad + Ch^4\left(\|f^n\|^2_{H^2(\Omega)} + \|g^n\|^2_{H^2(\Gamma)}\right) + \gamma\mu_3^2 t_n\sum_{j=0}^{n-1}\|z^j\|^2_{H^1(\Omega)} + \mu_1\|z^n\|^2_{L^2(\Omega)}. \end{aligned}$$

Using Young's inequality and the discrete version of Grönwall's inequality taking $\gamma = \dfrac{3}{2\mu_1}$, we obtain

$$\begin{aligned} \left(1 - \frac{1}{2}k - 2k\mu_1\right)\|z^n\|^2_{L^2(\Omega)} &\leq C\left(\|z^{n-1}\|^2_{L^2(\Omega)} + \|z^{n-2}\|^2_{L^2(\Omega)}\right) + Ck\Big[\|\partial(R_h u^n - u^n)\|^2_{L^2(\Omega)} \\ &\quad + \|\partial u^n - u^n_t\|^2_{L^2(\Omega)} + h^4\|\partial u^n\|^2_X + h^4\left(\|f^n\|^2_{H^2(\Omega)} + \|g^n\|^2_{H^2(\Gamma)}\right) \\ &\quad + k^4\left(\|R_h u^n_t\|^2_{H^1(\Omega)} + \|R_h u^0_t\|^2_{H^1(\Omega)}\right)\Big]. \end{aligned} \tag{3.12}$$

For $0 < k < \frac{1}{1+4\mu_1}$, $\left(1 - \frac{1}{2}k - 2k\mu_1\right)^{-1} < 2$, therefore (3.12) becomes

$$\begin{aligned} \|z^n\|^2_{L^2(\Omega)} &\leq C\left(\|z^{n-1}\|^2_{L^2(\Omega)} + \|z^{n-2}\|^2_{L^2(\Omega)}\right) + Ck\Big[\|\partial(R_h u^n - u^n)\|^2_{L^2(\Omega)} + \|\partial u^n - u^n_t\|^2_{L^2(\Omega)} \\ &\quad + h^4\|\partial u^n\|^2_X + h^4\left(\|f^n\|^2_{H^2(\Omega)} + \|g^n\|^2_{H^2(\Gamma)}\right) + k^4\left(\|R_h u^n_t\|^2_{H^1(\Omega)} + \|R_h u^0_t\|^2_{H^1(\Omega)}\right)\Big], \end{aligned}$$

for $n = 2, \dots, M$. By iteration on n, we have

$$\begin{aligned}
\|z^n\|^2_{L^2(\Omega)} \;\; \leq \;\; & C\left[\|z^0\|^2_{L^2(\Omega)} + \|z^1\|^2_{L^2(\Omega)}\right] + Ck\sum_{j=2}^{n} \|\partial(R_h u^j - u^j)\|^2_{L^2(\Omega)} \\
& + Ch^4 k \sum_{j=2}^{n}\left(\|f^j\|^2_{H^2(\Omega)} + \|g^j\|^2_{H^2(\Gamma)}\right) + Ck\sum_{j=2}^{n}\|\partial u^j - u^j_t\|^2_{L^2(\Omega)} \\
& + Ch^4 k\sum_{j=2}^{n}\|\partial u^j\|^2_X + Ck^5\sum_{j=2}^{n}\left(\|R_h u^j_t\|^2_{H^1(\Omega)} + \|R_h u^0_t\|^2_{H^1(\Omega)}\right).
\end{aligned}$$

A simple calculation shows that

$$\begin{aligned}
\|z^n\|^2_{L^2(\Omega)} \;\; \leq \;\; & C\left(\|z^0\|^2_{L^2(\Omega)} + \|z^1\|^2_{L^2(\Omega)}\right) + C\int_0^{t_n} \|(R_h u - u)_t\|^2_{L^2(\Omega)}\, dt \\
& + Ck^4\int_0^{t_n}\|u_{ttt}\|^2_{L^2(\Omega)}\, dt + Ch^4\int_0^{t_n}\left(\|u_t\|^2_X + \|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)}\right)\, dt \\
& + Ck^4\int_0^{t_n}\|R_h u_t\|^2_{H^1(\Omega)}\, dt + Ck^4\|R_h u^0_t\|^2_{H^1(\Omega)}, \\
\leq \;\; & C\left(\|z^0\|^2_{L^2(\Omega)} + \|z^1\|^2_{L^2(\Omega)}\right) + Ch^4\int_0^{t_n}\left(\|u\|^2_Z + \|u_t\|^2_Z + \|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)}\right)\, dt \\
& + Ck^4\int_0^{t_n}\|u_{ttt}\|^2_{L^2(\Omega)}\, dt + C\left(h^4 + k^4\right)\int_0^{t_n}\int_0^{\tau}\left(\|u(s)\|^2_Z + \|u_t(s)\|^2_Z\right)\, ds\, d\tau \\
& + Ck^4\left(\|u_0\|^2_Z + \|u_t(0)\|^2_Z\right).
\end{aligned}$$

where use is made of (3.5) to obtain the above inequality. We take $U^0_h = \pi_h u_0$, $U^1_h = U^0_h + k\pi_h[\nabla\cdot(a\nabla u_0) - bu_0 + f(x,0)]$ and use triangle inequality to obtain

$$\begin{aligned}
\|u^n - U^n_h\|^2_{L^2(\Omega)} \;\; \leq \;\; & Ch^4\int_0^{t_n}\left(\|u\|^2_Z + \|u_t\|^2_Z + \|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)}\right)\, dt \\
& + Ck^4\left(\int_0^{t_n}\|u_{ttt}\|^2_{L^2(\Omega)}\, dt + \|u_0\|^2_Z + \|u_t(0)\|^2_Z\right)\, dt \\
& + C\left(h^4 + k^4\right)\int_0^{t_n}\int_0^{\tau}\left(\|u(s)\|^2_Z + \|u_t(s)\|^2_Z\right)\, ds\, d\tau.
\end{aligned}$$

Equation (3.7) follows immediately. □

4. Numerical Experiments

We give examples to verify our main result. The mesh generation and computation are done with FreeFEM++ [25].

Example 4.1. Consider the computational domain $\Omega = (-1,1)\times(-1,1)$ where the interface Γ is a circle centered at $(0,0)$ with radius 0.5. $\Omega_1 = \{(x,y) : x^2 + y^2 < 0.25\}$, $\Omega_2 = \Omega \setminus \overline{\Omega}_1$.
On $\Omega \times (0,T]$, $0 < T < \infty$, we consider the problem (1.1)−(1.3) whose exact solution, is

$$u = \begin{cases} 0.5(0.25 - x^2 - y^2)\sin t, & \text{in} \quad \Omega_1 \times (0,T], \\ (0.25 - x^2 - y^2)(1 - x^2)(1 - y^2)t\exp(-2t), & \text{in} \quad \Omega_2 \times (0,T], \end{cases}$$

The source function f, interface function g and the initial data u_0 are determined from the choice of u with

$$a = \begin{cases} 1 & \text{in} \quad \Omega_1 \times (0,T], \\ 3 & \text{in} \quad \Omega_2 \times (0,T], \end{cases} \qquad b = \begin{cases} 0.5 & \text{in} \quad \Omega_1 \times (0,T], \\ 1 & \text{in} \quad \Omega_2 \times (0,T], \end{cases}$$

$$\alpha = \begin{cases} 2 & \text{in} \quad \Omega_1 \times (0,T], \\ 1 & \text{in} \quad \Omega_2 \times (0,T], \end{cases} \qquad \beta = \begin{cases} 0 & \text{in} \quad \Omega_1 \times (0,T], \\ 0 & \text{in} \quad \Omega_2 \times (0,T]. \end{cases}$$

The finite element solution is presented in Figure 3. Errors in L^2-norm at $t = 1$ for various step size h time step k are presented in Table 1. To achieve this, we choose $k \leq Ch$. Agreement of the numerical experiment with the theoretical result is verified using

$$\text{order of convergence} = \frac{\log(e_{i+1}/e_i)}{\log(h_{i+1}/h_i)},$$

where e_i is the error at the i-th iteration corresponding to the mesh size h_i.

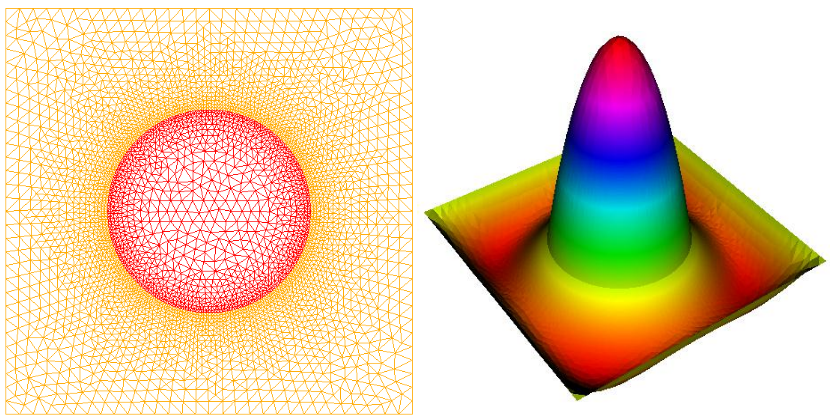

FIGURE 3. Computational domain and Finite element solution of Example 4.1 with $h = 0.101379$, $k = 0.02$.

TABLE 1. Error estimates for Example 4.1 in L^2-norm.

k	h	$\|\text{Error}\|_{L^2(\Omega)}$	Order of convergence
0.040	0.2028	1.42653×10^{-3}	
0.020	0.1014	3.45921×10^{-4}	2.044
0.010	0.0507	8.43860×10^{-5}	2.035
0.005	0.0250	2.02345×10^{-5}	2.025

Although the analysis was done for parabolic integro-differential equations with the homogeneous Dirichlet condition, the next example shows that the error estimate (3.7) is valid for equations with the non-homogeneous Dirichlet condition.

Example 4.2. We consider problem (1.1)–(1.3) in $\Omega \times (0,T]$ where $T < \infty$ and $\Omega = (-1,1) \times (-1,1)$. $\Omega_1 = \{(x,y) \in \Omega : x^2 + y^2 < 0.25\}$, $\Omega_2 = \Omega \setminus \Omega_1$ and the interface Γ is a circle centered at $(0,0)$ with radius 0.5.
For the exact solution, we choose

$$u = \begin{cases} 10(0.25 - x^2 - y^2)t\exp(-t)\sin(t) + 0.75\sin(t), & \text{in} \quad \Omega_1 \times (0,T], \\ (1 - x^2 - y^2)\sin t, & \text{in} \quad \Omega_2 \times (0,T], \end{cases}$$

and

$$a = \begin{cases} 8 & \text{in } \Omega_1 \\ 1 & \text{in } \Omega_2 \end{cases}, \qquad b = \begin{cases} 1 & \text{in } \Omega_1 \\ 4 & \text{in } \Omega_2 \end{cases}, \qquad \alpha = \begin{cases} 1 & \text{in } \Omega_1 \\ 2 & \text{in } \Omega_2 \end{cases}, \qquad \beta = \begin{cases} 4 & \text{in } \Omega_1 \\ 1 & \text{in } \Omega_2 \end{cases}.$$

The source function f, interface function g, initial data u_0 and the boundary conditions are determined from the choice of u. Errors in L^2 norm at $t = 2$ for various step size h time step k are presented in Table 2.

TABLE 2. Error estimates for Example 4.2 in L^2-norm.

k	h	$\|\text{Error}\|_{L^2(\Omega)}$	Order of convergence
0.08	0.2028	7.50121×10^{-3}	
0.04	0.1006	1.84727×10^{-3}	1.999
0.02	0.0491	4.26196×10^{-4}	2.045
0.01	0.0247	1.06495×10^{-4}	2.018

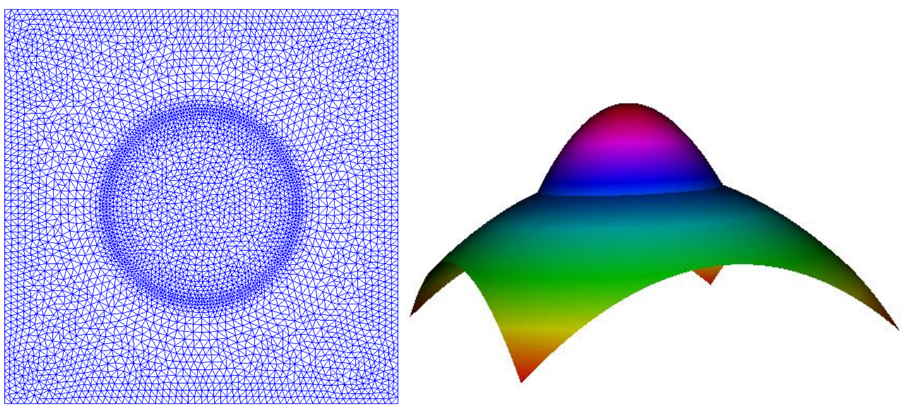

FIGURE 4. Computational domain and Finite element solution of Example 4.2 with $h = 0.0603449$, $k = 0.002$.

We have been able to show that the error estimate (3.7) is valid for the equations with non-homogeneous Dirichlet condition. Next we consider a model of real life situation.

Example 4.3. We consider a model describing the heat flow in materials with memory [6, 24]. Sometimes, the material may have two or more parts with different properties. Let the domain under consideration $\Omega = (-2, 2) \times (-2, 2)$ be of two materials Ω_1 and Ω_2. Let the interface Γ be an ellipse centered at $(1, 0)$ with described by $\Omega_1 = \{(x, y) : 4(x-1)^2 + y^2 < 1\}$, $\Omega_2 = \Omega \backslash \overline{\Omega}_1$. We also assume the material has an elliptical hole centered at $(-1, 0)$ $\{(x, y) : 4(x+1)^2 + y^2 < 1\}$ (kindly check the domain discretization in Figure 5).

The governing equation is

$$\begin{cases} w_t(x,t) - c\nabla^2 w(x,t) = f(x,t) + \displaystyle\int_0^t \left[H(t-s)\nabla^2 w(x,s)\right] \, ds, & \text{in } \Omega, \\ [w] = 0, \quad \left[c\nabla w \cdot \mathbf{n} + \displaystyle\int_0^t H(t-s)\nabla w(x,s) \cdot \mathbf{n} \, ds\right] = g, & \text{on } \Gamma, \\ w(x,t) = 0, & \text{on } \partial\Omega, \\ w(x,0) = w_0(x), & \text{for } x \in \Omega, \end{cases}$$

where $c > 0$ represents the thermal diffusivity of the material, $f(x,t)$ represents the distributed control in the domain Ω and $H(t-s)$ is the memory kernel. This kind of equation also describes the situation where a conservative tracer is transported by convection and dispersion under a steady, saturated, incompressible groundwater flow in a nondeformable porous medium of constant porosity [14].

For this experiment, we choose

$$c = \begin{cases} 1 & \text{in} \quad \Omega_1 \times (0,T] \\ 2 & \text{in} \quad \Omega_2 \times (0,T] \end{cases}, \qquad H(t-s) = \begin{cases} e^{-(t-s)} & \text{in} \quad \Omega_1 \times (0,T] \\ e^{-2(t-s)} & \text{in} \quad \Omega_2 \times (0,T] \end{cases}.$$

The initial data w_0, interface function g and the control function f are chosen corresponding to the exact solution

$$w = \begin{cases} \left(1 - 4(x-1)^2 - y^2\right) t \exp(-t), & \text{in} \quad \Omega_1 \times (0,T], \\ \frac{1}{1000}(4(x-1)^2 + y^2 - 1)(4-x^2)(4-y^2) \sin t, & \text{in} \quad \Omega_2 \times (0,T]. \end{cases}$$

Errors in L^2 norm at $t = 1$ for various step size h time step k are presented in Table 3.

TABLE 3. Error estimates for Example 4.3 in L^2-norm.

k	h	$\|\text{Error}\|_{L^2(\Omega)}$	Order of convergence
0.1000	0.5070	9.69262×10^{-3}	
0.0500	0.2534	2.47972×10^{-3}	1.966
0.0250	0.1288	6.46641×10^{-4}	1.987
0.0125	0.0651	1.62088×10^{-4}	2.025

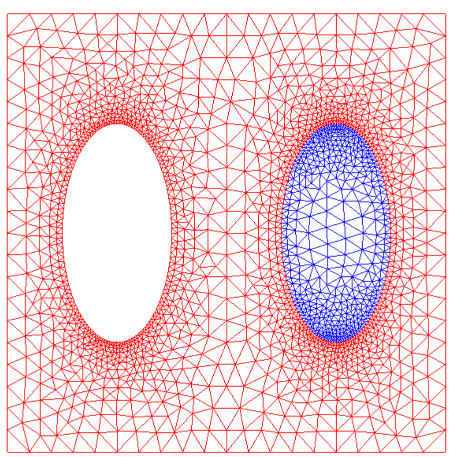

FIGURE 5. Computational domain of Example 4.3 with $h = 0.2828$.

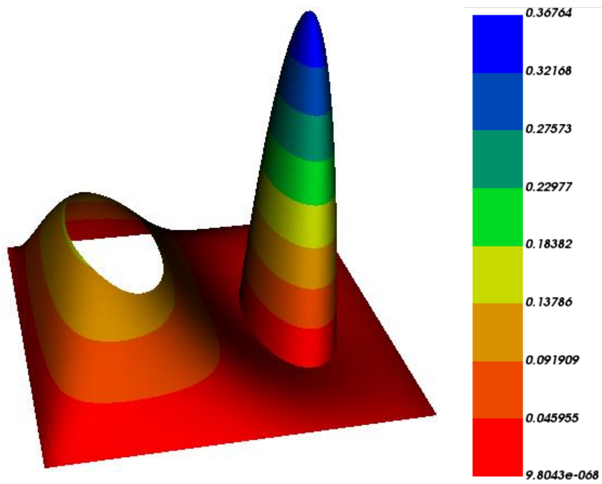

FIGURE 6. Finite element solution of Example 4.3 with $h = 0.0651$, $k = 0.0125$.

5. Concluding Remarks

In this work, a conforming linear triangular finite element method is proposed for the solution of parabolic integro-differential equations with memory on the domain with interfaces. The implicit difference scheme together with the trapezoidal rule was proposed for time discretization. The proposed scheme is shown to have optimal order of convergence in L^2 norm. To the best of the author's knowledge, this is the article that first presents the implementation of finite element solution of integro-differential equation on FreeFEM++. Although the analysis and computation in this work have been presented for interfaces that partitions the domain, extension of the method to open interfaces within the domain is straightforward.

The method presented in this work is the continuous Galerkin method on fitted mesh. Extension of this work to parabolic integro-differential equations with moving interfaces or evolving domains in the framework of extended finite element method is still open and may be considered in the future work.

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References

[1] R. A. Adams, *Sobolev spaces.* Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[2] M. O. Adewole, *Almost optimal convergence of FEM-FDM for a linear parabolic interface problem*, Electron. Trans. Numer. Anal., *46* (2017), 337–358.

[3] M. O. Adewole, *FEM-IDS for a second order strongly damped wave equation with memory*, Comput. Math. Appl., *80*(12) (2020), 2896–2914.

[4] M. O. Adewole, *Approximation of quasilinear hyperbolic problems with discontinuous coefficients: an optimal error estimate*, Bull. Iranian Math. Soc., *47*(2) (2021), 307–331.

[5] M. O. Adewole and V. F. Payne, *Linearized four-step implicit scheme for nonlinear parabolic interface problems*, Turkish J. Math., *42*(6) (2018), 3034–3049.

[6] G. Amendola, M. Fabrizio, and J. M. Golden, *Thermodynamics of materials with memory*, Springer, Cham, second edition, [2021] ©2021. Theory and applications.

[7] I. Babuška, *The finite element method for elliptic equations with discontinuous coefficients*, Computing (Arch. Elektron. Rechnen), *5* (1970), 207–213.

[8] J. R. Cannon and Y. P. Lin, *A priori L^2 error estimates for finite-element methods for nonlinear diffusion equations with memory*, SIAM J. Numer. Anal., *27*(3) (1990), 595–607.

[9] V. Capasso, *Asymptotic stability for an integro-differential reaction-diffusion system*, J. Math. Anal. Appl., *103*(2) (1984), 575–588.

[10] C. Chen and T. Shih, *Finite element methods for integrodifferential equations*, volume 9 of Series on Applied Mathematics, World Scientific Publishing Co., Inc., River Edge, NJ, 1998.

[11] C. Chen, X. Zhang, G. Zhang, and Y. Zhang, *A two-grid finite element method for nonlinear parabolic integro-differential equations*, Int. J. Comput. Math., *96*(10) (2019), 2010–2023.

[12] Z. Chen and J. Zou, *Finite element methods and their convergence for elliptic and parabolic interface problems*, Numer. Math., *79*(2) (1998), 175–202.

[13] B. D. Coleman and M. E. Gurtin, *Equipresence and constitutive equations for rigid heat conductors*, Z. Angew. Math. Phys., *18* (1967), 199–208.

[14] M. Dehghan and F. Shakeri, *Solution of parabolic integro-differential equations arising in heat conduction in materials with memory via He's variational iteration technique*, Int. J. Numer. Methods Biomed. Eng., *26*(6) (2010), 705–715.

[15] B. Deka and R. C. Deka, *Finite element method for a class of parabolic integro-differential equations with interfaces*, Indian J. Pure Appl. Math., *44*(6) (2013), 823–847.

[16] B. Deka and R. C. Deka, *A priori $L^\infty(L^2)$ error estimates for finite element approximations to parabolic integro-differential equations with discontinuous coefficients*, Proc. Indian Acad. Sci. Math. Sci., *129*(4) (2019), 49.

[17] J. F. Epperson, *An introduction to numerical methods and analysis*, John Wiley & Sons, Inc., Hoboken, NJ, second edition, 2013.

[18] D. Goswami, A. K. Pani, and S. Yadav, *Optimal error estimates of two mixed finite element methods for parabolic integro-differential equations with nonsmooth initial data*, J. Sci. Comput., *56*(1) (2013), 131–164.

[19] H. Guo and H. Rui, *Crank-Nicolson least-squares Galerkin procedures for parabolic integro-differential equations*, Appl. Math. Comput., *180*(2) (2006), 622–634.

[20] J. S. Gupta, R. K. Sinha, G. M. M. Reddy, and J. Jain, *A posteriori error analysis of two-step backward differentiation formula finite element approximation for parabolic interface problems*, J. Sci. Comput., *69*(1) (2016), 406–429.

[21] J. S. Gupta, R. K. Sinha, G. M. M. Reddy, and J. Jain, *New interpolation error estimates and a posteriori error analysis for linear parabolic interface problems*, Numer. Methods Partial Differential Equations, *33*(2) (2017), 570–598.

[22] M. E. Gurtin and A. C. Pipkin, *A general theory of heat conduction with finite wave speeds*, Arch. Rational Mech. Anal., *31*(2) (1968), 113–126.

[23] G. J. Habetler and R. L. Schiffman, *A finite difference method for analyzing the compression of poro-viscoelastic media*, Computing (Arch. Elektron. Rechnen), *6* (1970), 342–348.

[24] A. Halanay and L. Pandolfi, *Lack of controllability of the heat equation with memory*, Systems Control Lett., *61*(10) (2012), 999–1002.

[25] F. Hecht, *New development in freefem++*, J. Numer. Math., *20*(3-4) (2012), 251–265.

[26] T. Hou, L. Chen, and Y. Yang, *Two-grid methods for expanded mixed finite element approximations of semi-linear parabolic integro-differential equations*, Appl. Numer. Math., *132* (2018), 163–181.

[27] Z. Jiang, *$L^\infty(L^2)$ and $L^\infty(L^\infty)$ error estimates for mixed methods for integro-differential equations of parabolic type*, M2AN Math. Model. Numer. Anal., *33*(3) (1999), 531–546.

[28] W. E. Kastenberg and P. L. Chambré, *On the stability of nonlinear space dependent reactor kinetics*, Nucl. Sci. Eng., *31* (1968), 67–79.

[29] R. B. Kellogg, *Singularities in interface problems*, Numerical Solution of Partial Differential Equations, II (SYNSPADE 1970) (Proc. Sympos., Univ. of Maryland, College Park, Md., 1970), Academic Press, New York, 1971, 351–400.

[30] Y. P. Lin, V. Thomée, and L. B. Wahlbin, *Ritz-Volterra projections to finite-element spaces and applications to integrodifferential and related equations*, SIAM J. Numer. Anal., *28*(4) (1991), 1047–1070.

[31] Y. Liu, H. Li, J. Wang, and W. Gao, *A new positive definite expanded mixed finite element method for parabolic integrodifferential equations*, J. Appl. Math., *24* (2012), 391372.

[32] R. Mohammadali, M. Bayareh, and A. Ahmadi Nadooshan, *Numerical investigation on the effects of cell deformability and dld microfluidic device geometric parameters on the isolation of circulating tumor cells*, Iranian Journal of Chemistry and Chemical Engineering, (2023), in press.

[33] A. K. Pani and G. Fairweather, *H^1-Galerkin mixed finite element methods for parabolic partial integro-differential equations*, IMA J. Numer. Anal., *22*(2) (2002), 231–252.

[34] A. K. Pani, V. Thomée, and L. B. Wahlbin, *Numerical methods for hyperbolic and parabolic integro-differential equations*, J. Integral Equations Appl., *4*(4) (1992), 533–584.

[35] A. K. Pani and S. Yadav, *An hp-local discontinuous Galerkin method for parabolic integro-differential equations*, J. Sci. Comput., *46*(1) (2011), 71–99.

[36] A. Patel, S. K. Acharya, and A. K. Pani, *Stabilized Lagrange multiplier method for elliptic and parabolic interface problems*, Appl. Numer. Math., *120* (2017), 287–304.

[37] D. Pradhan, N. Nataraj, and A. K. Pani, *An explicit/implicit Galerkin domain decomposition procedure for parabolic integro-differential equations*, J. Appl. Math. Comput., *28*(1-2) (2008), 295–311.

[38] G. M. M. Reddy, *Fully discrete a posteriori error estimates for parabolic integro-differential equations using the two-step backward differentiation formula*, BIT Numerical Mathematics, *62*(1) (2022), 251–277.

[39] G. M. M. Reddy, A. B. Seitenfuss, D. d. O. Medeiros, L. Meacci, M. Assunção, and M. Vynnycky, *A compact FEM implementation for parabolic integro-differential equations in 2D*, Algorithms (Basel), *13*(10) (2020), 242.

[40] G. M. M. Reddy and R. K. Sinha, *On the Crank-Nicolson anisotropic a posteriori error analysis for parabolic integro-differential equations*, Math. Comp., *85*(301) (2016), 2365–2390.

[41] G. M. M. Reddy, R. K. Sinha, and J. A. Cuminato, *A posteriori error analysis of the Crank-Nicolson finite element method for parabolic integro-differential equations*, J. Sci. Comput., *79*(1) (2019), 414–441.

[42] M. A. Shallal, A. H. Taqi, H. N. Jabbar, H. Rezazadeh, B. F. Jumaa, A. Korkmaz, and A. Bekir, *A numerical technique of the time fractional gas dynamics equation using finite element approach with cubic Hermit element*, Appl. Comput. Math., *21*(3) (2022), 269–278.

[43] N. Sharma and K. K. Sharma, *Finite element method for a nonlinear parabolic integro-differential equation in higher spatial dimensions*, Appl. Math. Model., *39*(23-24) (2015), 7338–7350.

[44] R. K. Sinha, R. E. Ewing, and R. D. Lazarov, *Mixed finite element approximations of parabolic integro-differential equations with nonsmooth initial data*, SIAM J. Numer. Anal., *47*(5) (2009) 3269–3292.

[45] L. Song and C. Yang, *Convergence of a second-order linearized BDF-IPDG for nonlinear parabolic equations with discontinuous coefficients*, J. Sci. Comput., *70*(2) (2017), 662–685.

[46] V. Thomée and N. Y. Zhang, *Error estimates for semidiscrete finite element methods for parabolic integro-differential equations*, Math. Comp., *53*(187) (1989), 121–139.

[47] R. Wait and A. R. Mitchell, *Finite element analysis and applications*, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1985.

[48] H. Q. Wang, H. Li, S. He, and Y. Liu, *Error estimate for a space-time discontinuous finite element method for a nonlinear parabolic integro-differential equation*, Numer. Math. J. Chinese Univ., *35*(2) (2013), 128–142.

[49] E. G. Yanik and G. Fairweather, *Finite element methods for parabolic and hyperbolic partial integro-differential equations*, Nonlinear Anal., *12*(8) (1988), 785–809.

[50] A. Zhu, *Discontinuous mixed covolume methods for linear parabolic integrodifferential problems*, J. Appl. Math., *8* (2014), 649468.

[51] A. Zhu, T. Xu, and Q. Xu, *Weak Galerkin finite element methods for linear parabolic integro-differential equations*, Numer. Methods Partial Differential Equations, *32*(5) (2016), 1357–1377.

[52] A. L. Zhu, Z. W. Jiang, and Q. Xu, *Expanded mixed covolume method for a linear integro-differential equation of parabolic type*, Numer. Math. J. Chinese Univ., *31*(3) (2009), 193–205.