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Analysis of a kernel-based method for some pricing financial options

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Abstract

In this paper, we propose a kernel-based method for some pricing financial options. Based on the ideas of the kernel-based approximation and finite-difference discretization, we present an efficient numerical method for solving the generalized Black-Scholes option pricing models. Utilizing the reproducing property of kernels, we introduce an efficient framework for obtaining cardinal functions. Also, we discuss the solvability of final system to obtain some remarkable results. We provide the error estimate of the proposed kernel-based method and verify its efficiency and accuracy by numerical experiments.

Keywords. Black-Scholes equation, European option pricing, Kernel-based method, Finite difference discretization, Error analysis.
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1. Introduction

The Black-Scholes (B-S) model is a fundamental tool for option pricing (OP) in financial markets. It is a parabolic partial differential equation that describes the behavior of financial derivatives, specifically options on equity shares. The equation is derived by applying Itô's calculus under the assumption that the value of the underlying share evolves in time-based on a stochastic differential equation, and subject to further assumptions on the financial market. The key concept of the B-S model is arbitrage-free reasoning. The model has been widely used in the pricing of options on various commodities and payoff structures [4, 27].

In the literature, various methods have been proposed for the valuation of European and American options. For instance, Company et al. [8] presented a finite-difference numerical scheme for nonlinear B-S equations, which models illiquid markets where price impact in the underlying asset market affects the replication of a European contingent claim. Lesmana and Wang [25] investigated a reliable computational method for solving the B-S model. In [10], the authors presented the lattice procedure for OP. Hull and White [20] applied a control variate method for OP. Valkov [31, 32] developed a fitted finite-volume technique to investigate a generalized B-S model and implemented the convergence analysis of a fitted finite-volume element procedure preserving positivity. Cen and Le [5, 6] used a reliable numerical approach to the linear complemental problem resulting from OP and proposed a procedure of implicit time-stepping for a generalized B-S equation.

Furthermore, Rao [29] presented a numerical scheme approximating option prices for different option styles governed by the generalized B-S equation in its degenerate form. In [1, 33], the authors proposed a finite volume method to discretize the B-S equation arising from OP. In [21, 22], collocation procedures based on uniform cubic B-splines and non-uniform B-splines were proposed for the numerical solution of the generalized B-S partial differential equation, which was second-order convergent with respect to both variables. In [36], an implicit numerical scheme for solving time fractional B–S model was discussed. Additionally, Prathumwan and Trachoo [28] used the LHPM for the fractional-order B-S equation, while Baustian et al. [2] applied a Galerkin-based procedure for B-S and Heston models utilizing orthogonal polynomial systems. Roul and Goura [30] implemented a high-order numerical procedure with a uniform

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mesh for the generalized B-S equation. Chen et al. [7] offered a solution to the generalized B-S equation utilizing the Laguerre neural network.

In summary, the B-S equation is a crucial model for OP in financial markets. It is widely used, and various numerical methods have been proposed in the literature to value European and American options. These methods have different applications and advantages, and researchers are still developing new approaches to improve the accuracy and efficiency of OP.

Generalized B-S OP models: In this paper, we consider the following generalized B-S equation.

$$L_\sigma \mathbf{U} := \frac{\partial \mathbf{U}}{\partial \tau}(s,\tau) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 \mathbf{U}}{\partial s^2}(s,\tau) + (r-D)s\frac{\partial \mathbf{U}}{\partial s}(s,\tau) - r\mathbf{U}(s,\tau) = 0,\ (s,\tau)\in Q_T. \tag{1.1}$$

Here, $Q_T = \{(s,\tau) : s \in (0,\infty),\ \tau \in (0,T)\}$, and $\mathbf{U}(s,\tau)$ is the European option price at asset price s and current data τ, $\sigma > 0$ indicates the underlying asset volatility, $r > 0$ denotes the rate of risk-free interest, and D is the dividend of the dividend-paying asset.

European call option: Assume that at the expiry time $\tau = T$, the payoff for option is equal to

$$\Phi(s) = \max(s-E, 0), \tag{1.2}$$

for call option (CO) with the exercise price $E > 0$. The value of European CO is considered a solution of the Equation (1.1) on Q_T under the terminal-boundary conditions

$$\begin{cases} \mathbf{U}(s,T) = \Phi(s), & s \in [0,\infty), \\ \mathbf{U}(0,\tau) = 0, & 0 \le \tau \le T, \\ \mathbf{U}(s,\tau) \sim se^{-D(T-\tau)} - Ee^{-r(T-\tau)}, & s \to \infty,\ 0 \le \tau \le T. \end{cases} \tag{1.3}$$

Here, the value of European CO is considered to be a solution of Equation (1.1) defined on the truncated domain $Q_{T,s_{\max}} = \{(s,\tau) : s \in (0, s_{\max}),\ \tau \in (0,T)\}$ under the terminal-boundary conditions

$$\begin{cases} \mathbf{U}(s,T) = \Phi(s), & 0 \le s \le s_{\max}, \\ \mathbf{U}(0,\tau) = 0, & 0 \le \tau \le T, \\ \mathbf{U}(s_{\max},\tau) = s_{\max}e^{-D(T-\tau)} - Ee^{-r(T-\tau)}, & 0 \le \tau \le T, \end{cases} \tag{1.4}$$

where $s_{\max}$ is a suitably selected positive number.

The existence and uniqueness of solution for the Equation (1.1) under the terminal-boundary conditions (1.4) were investigated in [3, 18, 35]. It is proved that if $\mathbf{U}_1$ and $\mathbf{U}_2$ are solutions (1.1),(1.3), (1.1), and (1.4), respectively, then for any $(s,\tau) \in (0, s_{\max}) \times [0,T]$ that satisfies in the condition $\ln(\frac{s_{\max}}{s}) \ge -d(T-\tau)$, we have

$$|\mathbf{U}_2(s,\tau) - \mathbf{U}_1(s,\tau)| \le \|\mathbf{U}_2 - \mathbf{U}_1\|_{L^\infty(\Lambda\times(\tau,T))}(exp(-\frac{(\ln\frac{s_{\max}}{s})((T-\tau)\times\min\{0,d\} + \ln(\frac{s_{\max}}{s})}{2(T-\tau)\sigma^2})),$$

where $d = 2D - 2r + \sigma^2$ and $\Lambda = \{0, s_{\max}\}$. Note that the above problem is a backward-time problem. Therefore, we can use a time reversal via the change of variable $t = T - \tau$ to convert it into a forward-time problem. Now, it is easy to change the variable s to x by

$$x = \ln(s), \tag{1.5}$$

and the variable $\mathbf{U}(s,\tau)$ to $U(x,t)$ by

$$\mathbf{U}(s,\tau) = \mathbf{U}(e^x, T-t) = U(x,t). \tag{1.6}$$

Applying Equations (1.5) and (1.6) to Equations (1.1) and (1.3) we obtain the following non-degenerate problem

$$\begin{cases} \frac{\partial U}{\partial t}(x,t) = (r - D - \frac{1}{2}\sigma^2)\frac{\partial U}{\partial x}(x,t) + \frac{1}{2}\sigma^2\frac{\partial^2 U}{\partial x^2}(x,t) - rU(x,t), & (x,t) \in Q_{T,x_{\max}}, \\ U(x,0) = \Phi(e^x), & -\infty < x < x_{\max}, \\ \lim_{x\to-\infty} U(x,t) = 0, & 0 \le t \le T, \\ U(x_{\max},t) = e^{x_{\max}}e^{-Dt} - Ee^{-rt}, & 0 \le t \le T, \end{cases} \tag{1.7}$$

where $Q_{T,x_{\max}} = \{(x,t) : x \in (-\infty, x_{\max}), t \in (0,T)\}$. Note that (1.1) is degenerate at $s = 0$. For computations, we transform the infinite interval $(-\infty, x_{\max})$ into the finite interval $(x_{\min}, x_{\max})$. This localization decreases the influence of the condition $\lim_{x\to-\infty} U(x,t) = 0$. If $x_{\min}$ be sufficiently small, then, we obtain the following representation of the B-S model.

$$\begin{cases} \frac{\partial U}{\partial t}(x,t) = (r - D - \frac{1}{2}\sigma^2)\frac{\partial U}{\partial x}(x,t) + \frac{1}{2}\sigma^2\frac{\partial^2 U}{\partial x^2}(x,t) - rU(x,t), & (x,t) \in Q_{T,x_{\min},x_{\max}}, \\ U(x,0) = \Phi(e^x), & x_{\min} < x < x_{\max}, \\ U(x_{\min},t) = 0, & 0 \le t \le T, \\ U(x_{\max},t) = e^{x_{\max}}e^{-Dt} - Ee^{-rt}, & 0 \le t \le T. \end{cases} \tag{1.8}$$

Here, $Q_{T,x_{\min},x_{\max}} = \{(x,t) : x \in (x_{\min}, x_{\max}), t \in (0,T)\}$.
Since the derivative of $\max(y,0)$ is discontinuous at $y = 0$, we can construct an approximation $\Psi(y)$ of $\max(y,0)$ to modify the above model. To do so, we construct a polynomial $\varrho(\zeta) = k_0 + k_1\zeta + \cdots + k_9\zeta^9$ on the interval $(-\varepsilon,\varepsilon)$ that satisfies $\varrho(-\varepsilon) = \varrho'(-\varepsilon) = \varrho''(-\varepsilon) = \varrho'''(-\varepsilon) = \varrho^{(4)}(-\varepsilon) = 0$, $\varrho(\varepsilon) = \varepsilon$, $\varrho'(\varepsilon) = 1$ and $\varrho''(\varepsilon) = \varrho'''(\varepsilon) = \varrho^{(4)}(\varepsilon) = 0$. Now, define

$$\Psi(\zeta) = \begin{cases} \zeta, & \zeta > \varepsilon, \\ \varrho(\zeta), & -\varepsilon \le \zeta \le \varepsilon, \\ 0, & \zeta < -\varepsilon. \end{cases} \tag{1.9}$$

According to (1.8), the modified B-S model can be written as

$$\begin{cases} \frac{\partial U}{\partial t}(x,t) = (r - D - \frac{1}{2}\sigma^2)\frac{\partial U}{\partial x}(x,t) + \frac{1}{2}\sigma^2\frac{\partial^2 U}{\partial x^2}(x,t) - rU(x,t), & (x,t) \in Q_{T,x_{\min},x_{\max}}, \\ U(x,0) = \Psi(e^x - E), & x_{\min} < x < x_{\max}, \\ U(x_{\min},t) = 0, & 0 \le t \le T, \\ U(x_{\max},t) = e^{x_{\max}}e^{-Dt} - Ee^{-rt}, & 0 \le t \le T. \end{cases} \tag{1.10}$$

Define the transformation

$$u(x,t) = U(x,t) + \vartheta(x,t), \tag{1.11}$$

where

$$\vartheta(x,t) = -(e^{x_{\max}}e^{-Dt} - Ee^{-rt})\frac{x - x_{\min}}{x_{\max} - x_{\min}}.$$

Using this transformation, the model (1.10) can be written in the following equivalent form.

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = (r - D - \frac{1}{2}\sigma^2)\frac{\partial u}{\partial x}(x,t) + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2}(x,t) - ru(x,t) + f(x,t), & (x,t) \in Q_{T,x_{\min},x_{\max}}, \\ u(x,0) = \overline{\Psi}(x), & x_{\min} < x < x_{\max}, \\ u(x_{\min},t) = 0, & 0 \le t \le T, \\ u(x_{\max},t) = 0, & 0 \le t \le T. \end{cases} \tag{1.12}$$

Herein,

$$\begin{cases} \overline{\Psi}(x) = \Psi(e^x - E) - (e^{x_{\max}} - E)\frac{x - x_{\min}}{x_{\max} - x_{\min}}, \\ f(x,t) = \frac{\partial \vartheta}{\partial t}(x,t) - (r - D - \frac{1}{2}\sigma^2)\frac{\partial \vartheta}{\partial x}(x,t) - \frac{1}{2}\sigma^2\frac{\partial^2 \vartheta}{\partial x^2}(x,t) + r\vartheta(x,t). \end{cases}$$

The history of the finite element method (FEM) dates back to the early 1940s. In fact, eight decades have passed since the invention of the FEM method. The first development of the method was observed in Courant and Hrennikoff works [9, 19]. This method has influenced basic approaches in scientific modeling and engineering design. In general, the FEM method has become a computational workhorse for solving every imaginable problem in partial differential equations (PDEs). In [26], many researches related to the applications of this method in different fields have been collected. The approaches utilized by researchers are different, but their common feature is the discretization of a continuous domain into discrete subdomains called elements. In recent years, the development of meshless methods has attracted a lot of interest. Meshless methods require approximations of given differential equations which form a set of unstructured nodes, without any pre-defined connections among the nodes. Kansa [23, 24] introduced a collocation method for solving partial differential equations. The method was based on radial basis functions. The advantage of the meshless method over its predecessors was the capability to use amorphous nodes that needed to be neither of a certain shape nor of a certain pattern [14–17].

1.1. **Outline.** The aim of this paper is to apply a kernel-based method for pricing financial options. The main contribution of this paper consists of:

1)**:** Presenting an efficient method based on reproducing kernels for the pricing of options.
2)**:** We obtain the approximation error bound based on the interpolation error in the reproducing kernel Hilbert space (RKHS).

The outline of the paper is as follows:
In section 2, we first present the background material and preliminaries of the RKHS that will be used in the forthcoming sections. In the mentioned section, we present the main results and an error analysis related to our method. In section 3, we examine the presented method in different cases and compare the results with some well-known methods. Also, temporal and spatial convergence rates are provided to emphasize the accuracy of the method. Finally, we present some concluding remarks in section 4.

2. A kernel-based method for pricing financial options

Now, we provide preliminaries which will be utilized in the paper. The contents of this section are taken from [11, 12, 34].

Definition 2.1. (**RKHS**) Let Ω be an arbitrary non-empty set. Denote by $\mathcal{F}(\Omega)$ the set of all complex-valued functions on Ω. A RKHS on the set Ω is a Hilbert space $\mathbf{W} \subset \mathcal{F}(\Omega)$ with a function $K(x,y) : \Omega \times \Omega \to \mathbf{W}$, which is called the reproducing kernel, satisfying the following properties.

- $K(x,.) \in \mathbf{W}$, $\forall x \in \Omega$.
- $u(x) = \langle K(x,.), u(.)\rangle_{\mathbf{W}}$, $\forall u \in \mathbf{W}$, $\forall x \in \Omega$.

We define the Hilbert space by $\mathbf{W}_K(\Omega)$ in which K is the kernel function.

Theorem 2.2. *Suppose that $\mathbf{W}_K$ is a RKHS with reproducing kernel $K : \Omega \times \Omega \longrightarrow \mathbb{R}$. Then, $K(x,.)$ is positive definite. Moreover, $K(x,.)$ is strictly positive definite if and only if the point evaluation functionals $\begin{cases} I_x : \mathbf{W}_K \longrightarrow \mathbb{R}, \\ I_x(u) = u(x), \end{cases}$ are linearly independent in $\mathbf{W}_K^*$, where $\mathbf{W}_K^*$ is the space of bounded linear functionals on $\mathbf{W}_K$.*

The **RKHS** $\mathbf{W}_K$ contains all functions of the form $u = \sum_{j=1}^{n} c_j K(.,x_j)$ provided that $x_j \in \Omega$. Using the properties of **RKHS**, we obtain

$$\|u\| = \langle u,u\rangle_{\mathbf{W}_K} = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K(x_i,x_j). \tag{2.1}$$

Theorem 2.3. *(**RKHS on the closed interval** $[a,b]$) Define $K_1 : [a,b] \times [a,b] \to \mathbb{R}$ by*

$$K_1(x,y) = \begin{cases} \sum_{i=0}^{m-1} \frac{1}{(i!)^2}(x-a)^i(y-a)^i + \frac{1}{((m-1)!)^2}\int_a^x (x-z)^{m-1}(y-z)^{m-1}dz, \ x<y, \\ \sum_{i=0}^{m-1} \frac{1}{(i!)^2}(x-a)^i(y-a)^i + \frac{1}{((m-1)!)^2}\int_a^y (x-z)^{m-1}(y-z)^{m-1}dz, \ y \le x. \end{cases}$$

*Then, we obtain the expression $\mathbf{W}_{K_1}^m[a,b] = \{u | u(x), u'(x), ..., u^{(m-1)}(x) \in AC[a,b],\ u^{(m)}(x) \in L^2[a,b],\ x \in [a,b]\}$ of the **RKHS** $\mathbf{W}_{K_1}^m[a,b]$ as a set, and the inner product is given by*

$$\langle u,v\rangle_{\mathbf{W}_{K_1}^m} = \sum_{j=0}^{m-1} u^{(j)}(a)v^{(j)}(a) + \int_a^b u^{(m)}(x)v^{(m)}(x)dx, \tag{2.2}$$

where $u,v \in \mathbf{W}_{K_1}^m$.

Theorem 2.4. *(**The subspace** $\mathbf{W}_{K_2}^m[a,b]$ **of** $\mathbf{W}_{K_1}^m[a,b]$) Define $K_2 : [a,b] \times [a,b] \to \mathbb{R}$ by*

$$K_2(x,y) = \boldsymbol{K}_1(x,y) - \frac{O_{1,x}\boldsymbol{K}_1(x,y)O_{1,y}\boldsymbol{K}_1(x,y)}{O_{1,x}O_{1,y}\boldsymbol{K}_1(x,y)},\ O_{1,x}O_{1,y}\boldsymbol{K}_1(x,y) \neq 0,$$

where

$$\boldsymbol{K}_1(x,y) = K_1(x,y) - \frac{B_x K_1(x,y) O_{2,y} K_1(x,y)}{O_{2,x} O_{2,y} K_1(x,y)},\ O_{2,x}O_{2,y}K_1(x,y) \neq 0,\ K_1(x,y) \in \mathbf{W}^m_{K_1}[a,b],$$

*and the subscript x of the operators $O_{1,x}$ and $O_{2,x}$ indicates that these operators apply to the function of x. Then, we obtain the expression $\mathbf{W}^m_{K_2}[a,b] = \{u | u(x) \in \mathbf{W}^m_{K_1}[a,b],\ A(u) = u(a) = 0,\ B(u) = u(b) = 0\}$ of the **RKHS** $\mathbf{W}^m_{K_2}[a,b]$.*

2.1. **Temporal and spatial discretization.** To discretize the time variable, assume that $t_k := k\frac{T}{N} = k\tau,\ k = 0, 1, \dots, N$ and use the finite-difference scheme to analogize the time derivative term

$$\frac{\partial u}{\partial t}(x, t_k) - \frac{u(x,t_k) - u(x, t_{k-1})}{\tau} = O(\tau),$$

where $u(x, t_0) = \overline{\Psi}(x)$.
Replacing $u(x, t_k)$ with the approximate solution u^k, we obtain a semi-discrete system for (1.12), which can be stated as follows.

Find $u^k (k = 1, 2, \dots, N)$ such that

$$\begin{cases} \frac{u^k - u^{k-1}}{\tau} = (r - D - \frac{1}{2}\sigma^2)\frac{\partial u^k}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 u^k}{\partial x^2} - r u^k + f^k, & k \geq 1,\ x \in (x_{\min}, x_{\max}), \\ u^k|_{x = x_{\min}} = 0, & 0 \leq k \leq N, \\ u^k|_{x = x_{\max}} = 0, & 0 \leq k \leq N. \end{cases} \tag{2.3}$$

Now, we find the representation of the numerical treatment for the problem (2.3) in the RKHS $\mathbf{W}^m_{K_2}[x_{\min}, x_{\max}]$. For the sake of convenience, the linear operator $\mathbb{L}$ from the space $\mathbf{W}^m_{K_2}[x_{\min}, x_{\max}]$ to $U([x_{\min}, x_{\max}])$ is defined as follows

$$\mathbb{L}_\sigma u^k = u^k - \tau((r - D - \frac{1}{2}\sigma^2)\frac{\partial u^k}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 u^k}{\partial x^2} - r u^k),\ k \geq 1. \tag{2.4}$$

Then, the semi-discrete system (2.3) can be considered in the following form, from space $\mathbf{W}^m_{K_2}[x_{\min}, x_{\max}]$ to space $U([x_{\min}, x_{\max}])$.

$$\mathbb{L}_\sigma u^k = F^k, \tag{2.5}$$

where

$$F^k = \tau f^k + u^{k-1},\ k \geq 1, \tag{2.6}$$

$u^k \in \mathbf{W}^m_{K_2}[x_{\min}, x_{\max}]$, and $F^k \in U([x_{\min}, x_{\max}])$ when $u^k \in \mathbf{W}^m_{K_2}[x_{\min}, x_{\max}]$.

Suppose that $\Theta_n = \{x_i\}_{i=1}^n \subset [x_{\min}, x_{\max}]$ is a distinct subset of $[x_{\min}, x_{\max}]$. Consider the basis function space $\mathcal{V}_n$ defined by

$$\mathcal{V}_n = H_{K_2}([x_{\min}, x_{\max}]) = \mathbf{span}\{u_j(x) = K_2(x, x_j),\ x_j \in \Theta_n\} \subset \mathbf{W}^m_{K_2}[x_{\min}, x_{\max}], \tag{2.7}$$

in which is a finite-dimensional space. An approximation u_n^k to u^k will be gained via computation a truncated series according to the trial functions

$$u^k(x) \approx u_n^k(x) := \sum_{j=1}^n \gamma_j^k u_j(x) = [u_1(x)\ u_2(x)\ \dots\ u_n(x)] \begin{pmatrix} \gamma_1^k \\ \gamma_2^k \\ \vdots \\ \gamma_n^k \end{pmatrix}. \tag{2.8}$$

A set of collocation conditions to specify the interpolation coefficients $\{\gamma_j^k\}_{j=1}^n$ is utilized via applying (2.5) to Θ_n. Therefore, we can obtain

$$\mu_i[u_n^k] := \mathbb{L} u_n^k(x_i) = \sum_{j=1}^n \gamma_j^k \mathbb{L} u_j(x_i) = F^k(x_i), i = 1, 2, \dots, n, \tag{2.9}$$

where utilizing the differential operator and a point assessment at $x_i \in \Theta_n$ the functional μ_i $(1 \le i \le n)$ is defined. Generally, a single set $\Pi_n := \{\mu_i\}_{i=1}^n$ functionals is inclusive types of differential operators. The final collocation matrix $\mathbf{K}_{\Pi_n,\Theta_n}$ is not symmetric. The ij-entries of the collocation matrix is as follows

$$(\mathbf{K}_{\Pi_n,\Theta_n})_{ij} := \mu_i[u_j] = \mu_i^x K_2(x, x_j),\ 1 \le i, j \le n, \tag{2.10}$$

in which the superscript x in μ_j^x means that μ_j^x applies to the function of x. Therefore, the unknown coefficients $\gamma_j^k, j = 1, 2, \dots, n$, are obtained by using the final system:

$$\mathbf{K}^k_{\Pi_n,\Theta_n}[\gamma]^k = \mathbf{F}^k,$$

where $[\gamma]^k = [\gamma_1^k\ \gamma_2^k\ \dots\ \gamma_n^k]^T$, $\mathbf{F}^k_{\Theta_n} = [F^k(x_1)\ F^k(x_2)\ \dots\ F^k(x_n)]^T$ and

$$\mathbf{K}_{\Pi_n,\Theta_n} = \begin{pmatrix} \mu_1^x K_2(x, x_1) & \mu_2^x K_2(x, x_1) & \cdots & \mu_n^x K_2(x, x_1) \\ \mu_1^x K_2(x, x_2) & \mu_2^x K_2(x, x_2) & \cdots & \mu_n^x K_2(x, x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_1^x K_2(x, x_n) & \mu_2^x K_2(x, x_n) & \cdots & \mu_n^x K_2(x, x_n) \end{pmatrix}. \tag{2.11}$$

2.2. **Solvability of the final system.**

Lemma 2.5. *([11]) Let $K_2(x, y)$ be the reproducing kernel of the space $\mathbf{W}^m_{K_2}[x_{\min}, x_{\max}]$. Then,*

$$\frac{\partial^{i+j} K_2(x, y)}{\partial x^i \partial y^j},\ 0 \le i + j \le 2m - 2. \tag{2.12}$$

Lemma 2.6. *Assume that $\{x_j\}_{j=1}^\infty$ is a dense set in the domain $[x_{\min}, x_{\max}]$. Further, suppose that $\{\mu_j^x K_2(x, .)\}_{j=1}^n$ are linearly independent on the RKHS $\mathbf{W}^m_{K_2}[x_{\min}, x_{\max}]$. Then, the vectors*

$$\{(\mu_j^x K_2(x, x_1)\ \mu_j^x K_2(x, x_2)\ \mu_j^x K_2(x, x_3)\ \cdots)^T\}_{j=1}^n,$$

are linearly independent.

Proof. If $\{b_j\}_{j=1}^n$ satisfies $\sum_{j=1}^n b_j(\mu_j^x K_2(x, x_1)\ \mu_j^x K_2(x, x_2)\ \mu_j^x K_2(x, x_3)\ \cdots)^T = 0$, then we deduce that

$$\sum_{j=1}^n b_j \mu_j^x K_2(x, x_i) = 0,\ i \ge 1. \tag{2.13}$$

It follows that the functions $\mu_j^x K_2(x, .)$, $\mu_j \in \Pi_n$, are continuous by using Lemma 2.5. We know that $\{x_i\}_{i\ge1}$ is a dense set, Then we obtain

$$\sum_{j=1}^n b_j \mu_j^x K_2(x, .) = 0.$$

It gives results $b_j = 0$ $(j = 1, 2, \dots, n)$ and here the proof is complete. □

From Lemma 2.6 we obtain the following theorem.

Theorem 2.7. *Assume that the functions $\{\mu_j^x K_2(x, .)\}_{j=1}^n$ are linearly independent on $\mathbf{W}^m_{K_2}[x_{\min}, x_{\max}]$. Therefore, there exists a set of points $\Theta_n = \{x_j\}_{j=1}^n$ such that the final collocation matrix $\boldsymbol{K}_{\Pi_n,\Theta_n}$ is nonsingular.*

Lemma 2.8. *Suppose that the functionals $\{\mu_j\}_{j=1}^n$ are linearly independent on $\mathbf{W}^m_{K_2}[x_{\min}, x_{\max}]$. Then, the functions $\{\mu_j^x K(x, .)\}_{j=1}^n$ are linearly independent.*

Proof. suppose that $\sum_{j=1}^n b_j \mu_j^x K_2(x, .) = 0$, therefore

$$\begin{aligned} 0 = \langle u(.), \sum_{j=1}^n b_j \mu_j^x K(x, .)\rangle_{\mathbf{W}^m_{K_2}[x_{\min},x_{\max}]} &= \sum_{j=1}^n b_j \mu_j^x \langle u(.), K(x, .)\rangle_{\mathbf{W}^m_{K_2}[x_{\min},x_{\max}]} \\ &= \sum_{j=1}^n b_j \mu_j[u],\ \forall u \in \mathbf{W}^m_{K_2}[x_{\min}, x_{\max}], \end{aligned} \tag{2.14}$$

showing that $b_j = 0$ $(j = 1, 2, \dots, n)$ which shows the end of the proof. □

Finally, by using Lemma 2.8 and Theorem 2.7, the following theorem is fulfilled.

Theorem 2.9. *If the functionals* $\{\mu_j\}_{j=1}^n$ *are linearly independent on* $\mathbf{W}_{K_2}^m[x_{\min}, x_{\max}]$*, then, there exists a set of points* $\Theta_n = \{x_j\}_{j=1}^n$ *such that the obtained collocation matrix* $\boldsymbol{K}_{\Pi_n,\Theta_n}$ *is nonsingular.*

2.3. **Error analysis.** Suppose that $\Theta_n = \{x_i\}_{i=1}^n$ and $\mathcal{V}_n = Span\{u_1, u_2, \dots, u_n\}$. Using the Gram-Schmidt (G-S) orthogonalization approach to $\{u_1, u_2, \dots, u_n\}$, we can write

$$\overline{u}_s(x) = \sum_{k=1}^{s} \beta_{sk} u_k(x), \ (\beta_s > 0, \ s = 1, 2, \dots, n). \tag{2.15}$$

Therefore, $\{\overline{u}_1, \overline{u}_2, \dots, \overline{u}_n\}$ is an orthonormal basis for $\mathcal{V}_n$. Hence, the interpolant $u_n^k(x)$ to u^k at Θ_n, is rewritten as follows

$$u^k(x) \approx u_n^k(x) = \sum_{i=1}^{n} u^k(x_i)\overline{u}_i(x). \tag{2.16}$$

Theorem 2.10. *Suppose that* $u_n^k(x) \in \mathbf{W}_{K_2}^m[x_{\min}, x_{\max}]$ *and* $u^k(x)$ *are the approximate solution and the exact solution to (2.4), respectively. Then, we have*

$$|u^k(x) - u_n^k(x)| \le \|u^k\|_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]} |K_2(x,x) - \sum_{i=1}^{n} \overline{u}_i^2(x)|, \ \forall u^k(x) \in \mathbf{W}_{K_2}^m[x_{\min}, x_{\max}]. \tag{2.17}$$

Proof. The reproducing property allows us to write

$$\begin{aligned} u_n^k(x) &= \sum_{i=1}^{n} u^k(x_i)\overline{u}_i(x) \\ &= \sum_{i=1}^{n} \langle u^k, u_i \rangle_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]} \overline{u}_i(x) \\ &= \langle u^k, \sum_{i=1}^{n} \overline{u}_i(x) u_i \rangle_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]}. \end{aligned} \tag{2.18}$$

Thus,

$$\begin{aligned} |u^k(x) - u_n^k(x)|^2 &= |\langle u^k, K_2(x,.) - \sum_{i=1}^{n} \overline{u}_i(x)\overline{u}_i \rangle_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]}| \\ &\le \|u^k\|_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]} \|K_2(x,.) - \sum_{i=1}^{n} \overline{u}_i(x) u_i\|_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]}. \end{aligned} \tag{2.19}$$

We know that

$$\langle K_2(x,.) - \sum_{i=1}^{n} \overline{u}_i(x) u_i, \sum_{i=1}^{n} \overline{u}_i(x) u_i \rangle_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]} = 0. \tag{2.20}$$

Therefore, we obtain

$$\begin{aligned} \|K_2(x,.)\|_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]} &= \|K_2(x,.) - \sum_{i=1}^{n} \overline{u}_i(x) u_i + \sum_{i=1}^{n} \overline{u}_i(x) u_i\|_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]} \\ &= \|K_2(x,.) - \sum_{i=1}^{n} \overline{u}_i(x) u_i\|_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]} + \|\sum_{i=1}^{n} \overline{u}_i(x) u_i\|_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]}. \end{aligned} \tag{2.21}$$

By using (2.19) and (2.21), it is straightforward to show that

$$|u^k(x) - u_n^k(x)| \leq \|u^k\|_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]} |K_2(x,x) - \sum_{i=1}^{n} \overline{u}_i^2(x)|. \tag{2.22}$$

□

Lemma 2.11. *([13]) Suppose that $u \in C^m[x_{\min}, x_{\max}]$, and $\Theta_n = \{x_i\}_{i=1}^n \subset [x_{\min}, x_{\max}]$ is a distinct subset of $[x_{\min}, x_{\max}]$. Then*

$$\|u\|_{L^2[x_{\min},x_{\max}]} \leq d \max_{x_j \in \Theta_n} |u(x_j)| + ch^m \|u^{(m)}\|_{L^2[x_{\min},x_{\max}]}, \tag{2.23}$$

where c and d are real constants and $h = sup_{x \in [x_{\min},x_{\max}]} \min_{x_j \in \Theta_n} \|x - x_j\|$.

Theorem 2.12. *If u_n^k is the approximate solution of (2.4) in the space $\mathbf{W}_{K_2}^m[x_{\min}, x_{\max}]$, then*

$$\|u^k - u_n^k\|_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]} \leq ch^m \|u^k\|_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]}, \tag{2.24}$$

where c is a real constant.

Proof. By Lemma 2.11, we can obtain

$$\begin{aligned} \|u^k - u_n^k\|_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]} &\leq c_1 \max_{x_j \in \Theta_n} |u^k(x_j) - u_n^k(x_j)| + c_2 h^m \|(u^k - u_n^k)^{(m)}\|_{L^2[x_{\min},x_{\max}]} \\ &\leq ch^m \|u^k - u_n^k\|_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]}. \end{aligned} \tag{2.25}$$

We know that

$$\|u^k\|^2_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]} \leq \|u_n^k\|_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]} + \|u^k - u_n^k\|_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]} + 2\langle u^k - u_n^k, u_n^k \rangle_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]}.$$

Since $< u^k - u_n^k, u_n^k >_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]} = 0$, then we have

$$\|u^k\|^2_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]} \leq \|u_n^k\|_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]} + \|u^k - u_n^k\|_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]}. \tag{2.26}$$

Now, from (2.25) and (2.26) we obtain

$$\|u^k - u_n^k\|_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]} \leq ch^m \|u^k\|_{\mathbf{W}_{K_2}^m[x_{\min},x_{\max}]}, \tag{2.27}$$

where c is a real constant. Therefore, the proof is complete. □

3. Numerical Results

We used some experimental tests to demonstrate the accuracy and effectiveness of the proposed method. In the process of computation, all the symbolic and numerical computations were performed by using Maple 18 on a computer with Corei3 and 4.00 GB memory RAM. In this section, we compare the results obtained by using the proposed method with those exact solutions and the corresponding experimental results obtained by the methods presented in [21, 22].

Example 3.1. In this example, we consider the generalized B-S equation (1.1)-(1.3) with the parameters

$$Test\ I : (r, D, \sigma) = (0.06, 0.02, 0.4),$$
$$Test\ II : (r, D, \sigma) = (0.08, 0.02, 0.3).$$

The exact solution is given by

$$\mathbf{U}(s, \tau) = sN(d_1)e^{-D(T-\tau)} - EN(d_2)e^{-r(T-\tau)},$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy,$$
$$d_1(s,\tau) = \frac{\ln(\frac{s}{E}) + (r - D + \frac{1}{2}\sigma^2)(T-\tau)}{\sigma\sqrt{T-\tau}},$$
$$d_2(s,\tau) = \frac{\ln(\frac{s}{E}) + (r - D - \frac{1}{2}\sigma^2)(T-\tau)}{\sigma\sqrt{T-\tau}}.$$

Let $x_{\min}$ be sufficiently small, and $x_{\max}$ be a suitably chosen positive number. Then, by using $t = T - \tau$ and $x = \ln(s)$, the original model can be rewritten in the form

$$\begin{cases} \frac{\partial U}{\partial t}(x,t) = (r - D - \frac{1}{2}\sigma^2)\frac{\partial U}{\partial x}(x,t) + \frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial x^2}(x,t) - rU(x,t), & (x,t) \in Q_{T,x_{\min},x_{\max}}, \\ U(x,0) = \max(e^x - E, 0), & x_{\min} < x < x_{\max}, \\ U(x_{\min},t) = 0, & 0 \le t \le T, \\ U(x_{\max},t) = e^{x_{\max}}e^{-Dt} - Ee^{-rt}, & 0 \le t \le T, \end{cases} \tag{3.1}$$

where $Q_{T,x_{\min},x_{\max}} = \{(x,t) : x \in (x_{\min}, x_{\max}), t \in (0,T)\}$. Since the derivative of $\max(e^x - E, 0)$ is discontinuous at $x = \ln(E)$, we approximate $\max(e^x - E, 0)$ by a sufficiently smooth function $\Psi(e^x - E)$, where

$$\Psi(\zeta) = \begin{cases} y, & \zeta > \varepsilon, \\ \frac{35}{256}\varepsilon + \frac{1}{2}\zeta + \frac{35}{64\varepsilon}\zeta^2 - \frac{35}{128\varepsilon^3}\zeta^4 + \frac{7}{64\varepsilon^5}\zeta^6 - \frac{5}{256\varepsilon^7}\zeta^8, & -\varepsilon \le \zeta \le \varepsilon, \\ 0, & \zeta < -\varepsilon. \end{cases} \tag{3.2}$$

Define the transformation

$$u(x,t) = U(x,t) + \vartheta(x,t), \tag{3.3}$$

where

$$\vartheta(x,t) = -(e^{x_{\max}}e^{-Dt} - Ee^{-rt})\frac{x - x_{\min}}{x_{\max} - x_{\min}}.$$

Using this transformation, the model (1.10) can be written in the equivalent form

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = (r - D - \frac{1}{2}\sigma^2)\frac{\partial u}{\partial x}(x,t) + \frac{1}{2}\sigma_1^2 \frac{\partial^2 u}{\partial x^2}(x,t) - ru(x,t) + f(x,t), & (x,t) \in Q_{T,x_{\min},x_{\max}}, \\ u(x,0) = \overline{\Psi}(x), & x_{\min} < x < x_{\max}, \\ u(x_{\min},t) = 0, & 0 \le t \le T, \\ u(x_{\max},t) = 0, & 0 \le t \le T, \end{cases} \tag{3.4}$$

where

$$\begin{cases} \overline{\Psi}(x) = \Psi(e^x - E) - (e^{x_{\max}} - E)\frac{x - x_{\min}}{x_{\max} - x_{\min}}, \\ f(x,t) = \frac{\partial \vartheta}{\partial t}(x,t) - (r - D - \frac{1}{2}\sigma^2)\frac{\partial \vartheta}{\partial x}(x,t) - \frac{1}{2}\sigma^2 \frac{\partial^2 \vartheta}{\partial x^2}(x,t) + r\vartheta(x,t). \end{cases}$$

We define the RKHS $\mathbf{W}^3_{K_2}[x_{\min}, x_{\max}]$. Using the proposed method, taking $x_i = (x_{\max} - x_{\min})\frac{j}{n+1} + x_{\min}$, $j = 1, \dots, n$, on $[x_{\min}, x_{\max}]$, the approximate solution $U_n^N(x)$ is given by

$$U_n^N(x) = u_n^N(x) - \vartheta(x, t_N), \tag{3.5}$$

where

$$u_n^N(x) = \sum_{j=1}^{n} \gamma_i^N u_j(x). \tag{3.6}$$

Since the exact solution is known, the absolute error (AE) $e_{n,N}(x)$ and the maximum error (ME) $E_{n,N}$ can be obtained as

$$e_{n,N}(x) = |U_n^N(x) - U(x, t_N)|, \tag{3.7}$$
$$E_{n,N} = \max_{1 \le j \le n} |U_n^N(x_j) - U(x_j, t_N)|. \tag{3.8}$$

TABLE 1. Maximum AE obtained by the proposed method and the methods in [21, 22] at $t = 1$ for $r = 0.06$, $D = 0.02$ and $\sigma = 0.4$ (Example 3.1).

Exercise Price	$E = 1$	$E = 2$	$E = 3$
Our method			
$E_{30,15}$	1.143273e-003	2.232828e-003	5.657515e-003
$E_{30,20}$	5.177540e-004	3.825059e-003	3.351245e-003
Cubic B-spline collocation method [22]($\theta = 1$)			
$M_x = N_t = 10$	1.132668e-002	-	-
$M_x = N_t = 20$	3.242000e-003	-	-
$M_x = N_t = 40$	1.088152e-003	-	-
$M_x = N_t = 80$	4.148162e-004	-	-
Cubic B-spline collocation method [22]($\theta = \frac{1}{2}$)			
$M_x = N_t = 10$	9.542612e-003	-	-
$M_x = N_t = 20$	2.438174e-003	-	-
$M_x = N_t = 40$	6.209743e-004	-	-
$M_x = N_t = 80$	1.554989e-004	-	-
Crank-Nicolson finite-difference method [21]			
$M_x = N_t = 11$	2.319e-02	-	-
$M_x = N_t = 21$	4.914e-03	-	-
$M_x = N_t = 41$	1.175e-03	-	-
$M_x = N_t = 81$	2.934e-04	-	-
B-spline collocation method [21]			
$M_x = 10, N_t = 8$	8.144e-03	-	-
$M_x = 20, N_t = 17$	1.922e-03	-	-
$M_x = 40, N_t = 35$	4.433e-04	-	-
$M_x = 80, N_t = 71$	1.067e-04	-	-

We select $x_{\min} = -2$, $x_{\max} = +2$ and $\varepsilon = 10^{-3}$. The graphs of the obtained results of *Test I* and *Test II* are depicted in Figure 1 and Figure 2, respectively. The MEs in our computed solutions of *Test I* and *Test II* are given in Tables 1-5, respectively. Numerical results in these tables reveal that spatial convergence rates obtained are in accordance with the results given in Theorem 2.12. For instance, in Example 3.1, we chose $m = 3$, where m is the order of Sobolev space, i.e., $\mathbf{W}^3_{K_2}[x_{\min}, x_{\max}]$. As one expects from Theorem 2.12, the convergence rate in the space $\mathbf{W}^3_{K_2}[x_{\min}, x_{\max}]$, should be $O(h^3)$, which can be deduced from the results given in Tables 4 and 5.

4. Conclusion

In this paper, a kernel-based method was developed to solve the generalized Black-Scholes option pricing model. In addition, with discretization of the time variable we used the finite-difference scheme to analogize the time derivative term in the European option pricing model. Then, we solved the semi-discrete problem created in the reproducing kernel space. The successful application of this scheme demonstrated the finite-difference method was effective and needed less computational work for solving the option pricing models. The proposed kernel-based method provides a closed-form approximate solution on the entire domain and because of its simple implementation and sensible

TABLE 2. Maximum AE and temporal convergence rate (TCR) obtained by the proposed method at $t = 1$ and $E = 3$ for $n = 20$ (Example 3.1).

$(r, D, \sigma) = (0.06, 0.02, 0.4)$			
	$N = 30$	$N = 60$	$N = 120$
$E_{n,N}$	$3.037380e-003$	$1.391429e-003$	$6.116744e-004$
TCR	-	1.126260	1.185731
$(r, D, \sigma) = (0.08, 0.02, 0.3)$			
	$N = 30$	$N = 60$	$N = 120$
$E_{n,N}$	$4.536742e-003$	$2.025085e-003$	$9.130655e-004$
TCR	-	1.163498	1.149235

TABLE 3. Maximum AE obtained by the proposed method at $t = 1$ for $r = 0.08$, $D = 0.02$ and $\sigma = 0.3$ (Example 3.1).

Exercise Price	$E = 1$	$E = 2$	$E = 3$	$E = 4$
Our method				
$E_{20,15}$	1.228499e-003	2.872244e-003	2.938095e-003	3.773764e-003
$E_{20,20}$	1.242719e-003	4.999861e-003	1.870095e-003	3.035534e-003
$E_{20,25}$	1.477709e-003	3.200233e-003	1.598622e-003	2.494168e-003

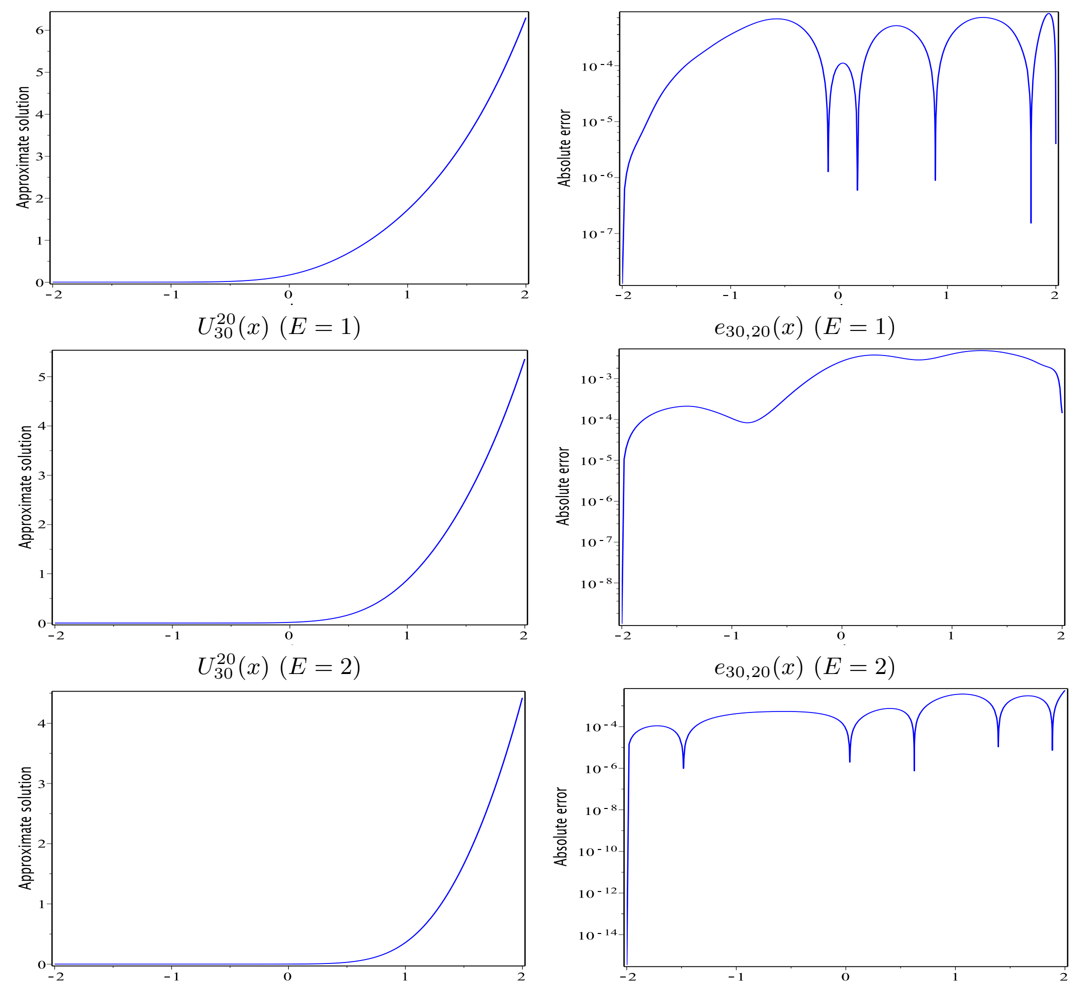
TABLE 4. Maximum AE and spatial convergence rate (SCR) obtained by the proposed method at $t = 1$ for $r = 0.06$, $D = 0.02$, $\sigma = 0.4$ with $N = 30$ (Example 3.1).

	$n = 5$	$n = 10$	$n = 20$	$n = 40$
$E = 1$				
$E_{n,N}$	2.796155e-002	4.315218e-003	5.577259e-004	1.200146e-004
SCR	-	2.695983	2.953611	2.265537
$E = 3$				
$E_{n,N}$	3.990749e-002	5.817834e-003	1.0373808e-003	2.326292e-004
SCR	-	2.778294	2.487859	2.160219

TABLE 5. Maximum AE and spatial convergence rate (SCR) obtained by the proposed method at $t = 1$ for $r = 0.08$, $D = 0.02$, $\sigma = 0.3$ with $N = 20$ (Example 3.1).

	$n = 5$	$n = 10$	$n = 20$	$n = 40$
$E = 1$				
$E_{n,N}$	5.079230e-002	7.024137e-003	1.211107e-003	2.135867e-004
SCR	-	2.854236	2.536094	2.507274
$E = 3$				
$E_{n,N}$	6.051967e-002	6.385631e-003	1.126318e-003	1.519592e-004
SCR	-	3.244629	2.503479	2.898586

accuracy is computationally effective. Numerical experiments were selected in order to indicate the proposed method convergence and stability. Also, the obtained results have high accuracy. We believe that the proposed kernel-based method has a remarkable potential and efficiency to solve other models of the option pricing. According to Theorem 2.12, in m-order Sobolev space, convergence order is $O(h^m)$, which can also be deduced from the numerical results. In fact, the results obtained from the tables reveal that better approximations are obtained in the higher-order Sobolev space. In the end, we mention some future research directions as follows:



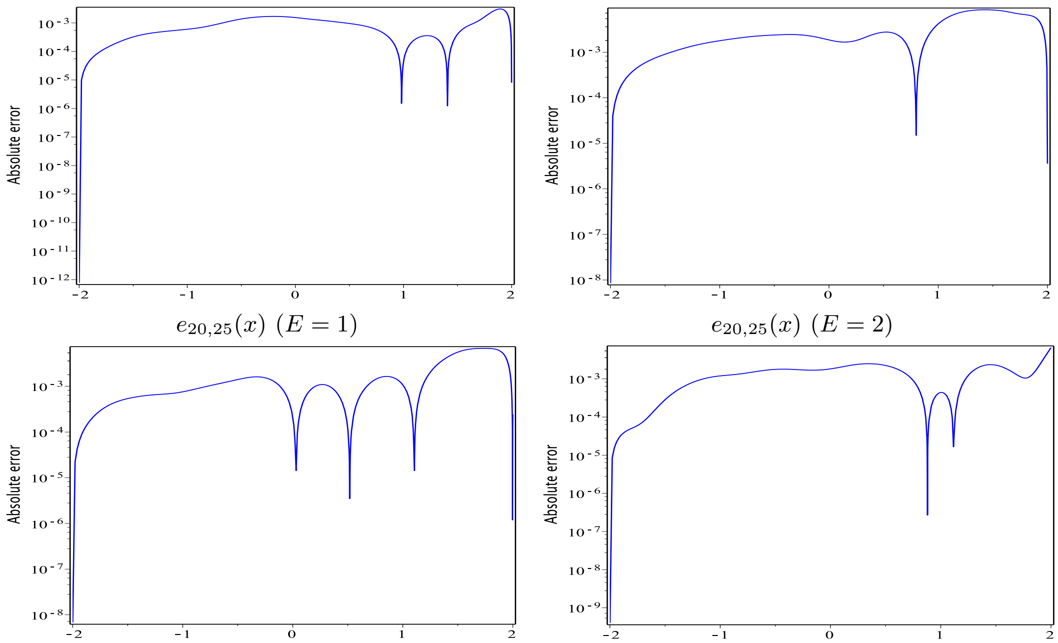
$U_{30}^{20}(x)$ $(E = 1)$ $e_{30,20}(x)$ $(E = 1)$

$U_{30}^{20}(x)$ $(E = 2)$ $e_{30,20}(x)$ $(E = 2)$

$U_{30}^{20}(x)$ $(E = 3)$ $e_{30,20}(x)$ $(E = 3)$

FIGURE 1. Graphs of the numerical solution $U_{30}^{20}(x)$ and AE $e_{30,20}(x)$ at $t = 1$ for $r = 0.06$, $D = 0.02$ and $\sigma = 0.4$ (Example 3.1).

i)**:** The proposed method can be used for other famous financial models.
ii)**:** We can use kernels based on polynomials to solve financial models.



$e_{20,25}(x)$ $(E = 1)$ $e_{20,25}(x)$ $(E = 2)$

$e_{20,25}(x)$ $(E = 3)$ $e_{20,25}(x)$ $(E = 4)$

FIGURE 2. Graphs of the AE $e_{20,25}(x)$ at $t = 1$ for $r = 0.08$, $D = 0.02$ and $\sigma = 0.3$ (Example 3.1).

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