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Almost sure exponential numerical stability of balanced Maruyama with two step approximations of stochastic time delay Hopfield neural networks

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Abstract

This study examines the balanced Maruyama with two step approximations of stochastic Hopfield neural networks with delay. The main aim of this paper is to discover the conditions under which the exact solutions remain stable for the balanced Maruyama with two-step approximations of stochastic delay Hopfield neural networks (SDHNN). The semi-martingale theorem for convergence is used to demonstrate the almost sure exponential stability of balanced Maruyama with two-step approximations of stochastic delay Hopfield networks. Additionally, the numerical balanced Euler approximation's stability conditions are compared. Our theoretical findings are illustrated with numerical experiments.

Keywords. Almost sure exponential stability, Balanced two step Maruyama numerical approximations, Hopfield neural networks, Stochastic delay differential equations.

2010 Mathematics Subject Classification. 60G42, 60H10, 65L20.

1. Introduction

Stochastic Hopfield neural networks are a subject of great interest among young researchers due to their diverse range of applications, such as pattern recognition, classification, associative memory, optimization, signal and image processing, and parallel computation. However, finding analytical solutions for stochastic differential equations are often limited, and therefore, numerical methods are commonly used to handle them. In this context, the investigation of the stability properties of numerical methods is essential.

Several studies have explored the stability properties of numerical methods for stochastic differential equations, including mean square stability and almost sure exponential stability. The almost sure (a.s.) exponential stability of numerical methods was investigated in [4]. Moreover, the stability analysis of numerical methods for SDE with a delay term has been explored by many authors, including Cao et al. [3, 6], Higuchi [4], Liu et al. [7], Mao [8], Tan et al. [12]. Such as, the exponential stability of SDHNN was examined in [13, 14], and the stability analysis using the theorem of sem-imartingale convergence and Lyapunov function was explored in [5]. Blythe et al. in [1] initiated to study the stability of stochastic neural networks. In subsequent studies [7, 11], the exponential mean-square stability of numerical solutions for SDHNN was explored.

Furthermore, specific numerical methods have also been studied for their stability properties. Examples of these investigations include the exploration of the exponential stability in mean square of two-step Maruyama methods for SDEs with time delay in [3], and the study of asymptotic stability in mean square of two-step Maruyama methods for nonlinear neutral SDEs with constant time delay in [6]. Additionally, [2] delved into the method of solutions using linear multi-step Maruyama schemes.

The primary focus of this paper is to investigate the almost sure (a.s.) exponentially stable of balanced Maruyama with two-step approximations for non-linear SDHNN. The goal is to determine the stability criteria for these methods as applied to the mentioned networks. The paper consists of five sections, with the second section presenting the

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notations and stability conditions for analytical solutions. In section three, establishing the a.s exponential stability for the balanced Maruyama with two-step approximations for these networks. Finally, numerical examples to validate the theoretical findings.

2. Model Equations

Consider the stochastic Hopfield neural networks with delays of the form

$$\begin{cases} d\mathbf{y}(t) &= [-C\mathbf{y}(t) + A\mathbf{f}(\mathbf{y}(t)) + B\mathbf{g}(\mathbf{u}(t))]\,dt + \sigma(\mathbf{y}(t))\,dW(t), \quad t \geq 0, \\ \mathbf{y}(t) &= \zeta(t), \qquad -\tau_i \leq t \leq 0. \end{cases} \tag{2.1}$$

The number of neurons in the SDHNN, denoted by $n \geq 1$, is specified in the aforementioned system, under consideration involves a state vector $\mathbf{y}(t) = [y_1(t), ..., y_n(t)]^T \in \mathbb{R}^n$, which corresponds to n neurons. The vector $\mathbf{u}(t) = [y_1(t-\tau_1), ..., y_n(t-\tau_n)]^T$. Additionally, the vector functions $\mathbf{f}(\mathbf{y}(t)) = [f_1(y_1(t)), ..., f_n(y_n(t))]^T$ and $\mathbf{g}(\mathbf{u}(t)) = [g_1(y_1(t-\tau_1)), ..., g_n(y_n(t-\tau_n))]^T$ represents the neuron activation functions $y_j(t)$ is the state variable of the jth neuron at time t, f_j, and g_j denote the output of the jth unit at time t and $t-\tau_j$ respectively, and the diagonal matrix $C = (c_1, c_2, ..., c_n)$ has positive entries. In the given context, c_i is a positive constant that denotes the speed at which the ith unit resets its potential to the resting state when it is not connected to the network and is subjected to external stochastic perturbations. The values a_{ij} and b_{ij} determine the influence of the jth unit on the ith unit, while τ_j is a non-negative constant representing the transmission delay. The connection weight matrix $A = (a_{ij})_{n\times n}$ and the discretely delayed connection weight matrix $B = (b_{ij})_{n\times n}$ are also part of the system. The state vector $\mathbf{y}(t)$ satisfies $\mathbf{y}(t) = \zeta(t)$ on the initial segment $[-\tau, 0]$, where $\zeta(t) = [\zeta_1(t), ..., \zeta_n(t)]^T$ is a given function in $C([-\tau, 0], \mathbb{R}^n)$, and τ is the maximum value of the delays. Finally, the diagonal matrix $\sigma(\mathbf{y}(t)) = (\sigma_1(\mathbf{y}_1(t)), ..., \sigma_n(\mathbf{y}_n(t)))$ has $\sigma_i(0) = 0$, and the vector $W(t) = [W_1(t), ..., W_n(t)]^T \in \mathbb{R}^n$. σ_i are continuous functions. Let f_i and g_i be functions in $C^2(D;\mathbb{R}) \bigcap \mathscr{L}^2([0,T];\mathbb{R})$ and σ_i be in $C^1(D;\mathbb{R}) \bigcap \mathscr{L}^2([0,T];\mathbb{R})$. Here $C^l(D;\mathbb{R})$ denotes the family of continuously l-times differentiable real-valued function defined on D, while $\mathscr{L}^l([0,T];\mathbb{R})$ denotes the family of all real-valued measurable $\{\mathscr{F}_t\}$-adapted stochastic processes $\{f(t)\}_{t\in[0,T]}$ such that $\int_0^T |f(t)|^l\, dt < +\infty$. We put into practise the fundamental presumptions assumptions to get our results in [10, 11].

($\mathbb{H}$1) $f(0) \equiv 0$, $g(0) \equiv 0$, and $\sigma(0) \equiv 0$.

($\mathbb{H}$2) Both $f(x)$ and $g(x)$ satisfy the Lipschitz condition. That is for each $i = 1, 2, \cdots, n$, there exist constants $\kappa_i > 0$, $\rho_i > 0$, such that

$$\begin{aligned} |f_i(x) - f_i(y)| &\leq \kappa_i |x-y|, \\ |g_i(x) - g_i(y)| &\leq \rho_i |x-y|, \qquad \forall x, y \in \mathbb{R}^n. \end{aligned}$$

($\mathbb{H}$3) $\sigma_i(y)$ satisfies the Lipschitz condition, and there are non-negative constants μ_i such that

$$|\sigma_i(y_1) - \sigma_i(y_2)| \leq \mu_i |y_1 - y_2|.$$

Definition 2.1. [8, 10] The incidental solution for equations (2.1) is said to be exponential stability with mean square in case there is a combination constants $\lambda > 0, J > 0$ such that

$$\mathbb{E}\,|y(t,\zeta)|^2 \leq J\,|\zeta|^2\, e^{-\lambda t}, \quad t \geq 0,$$

Is true for every $\zeta \in C^b_{\mathscr{F}_0}([-\tau, 0], \mathbb{R}^n)$. In this case

$$limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{E}\,|y(t,\zeta)|^2 \leq -\lambda.$$

Theorem 2.2. *[11, 14] If (2.1) satisfies ($\mathbb{H}$1)-($\mathbb{H}$3), the following holds.*

($\mathbb{H}$4) For $i = 1, 2, \cdots, n$,

$$-2c_i + \sum_{j=1}^n |a_{ij}|\,\kappa_j + \sum_{j=1}^n |b_{ij}|\,\rho_j + \sum_{j=1}^n |a_{ji}|\,\kappa_i + \sum_{j=1}^n |b_{ji}|\,\rho_i + \sum_{j=1}^n \left(\mu_{ij}^2\right) < 0.$$

Then (2.1) is exponentially stable in mean square.

($\mathbb{H}5$)

$$\sum_{j=1}^{n} |a_{ij}|\,\kappa_j + \sum_{j=1}^{n} |b_{ij}|\,\rho_j \leq \sum_{j=1}^{n} |a_{ji}|\,\kappa_i + \sum_{j=1}^{n} |b_{ji}|\,\rho_i.$$

Theorem 2.3. *[7, 10] Under the assumptions of hypotheses ($\mathbb{H}1$)-($\mathbb{H}5$), it can be concluded that the solution to equation (2.1) exhibits exponential stability with almost sure.*

Definition 2.4. [8] The incidental solution for equation (2.1) is said to be a.s. exponentially stable in case there is a constant $\eta > 0$ such that

$$limsup_{t\to\infty} \frac{1}{t} \ln |y(t,\zeta)| \leq -\eta, \quad \text{almost surely}$$

for any initial data $\zeta \in C^b_{\mathscr{F}_0}([-\tau, 0], \mathbb{R}^n)$.

3. Almost sure exponential stability for balanced Maruyama with two-step Methods

Our focus in this section is to examine the almost sure exponential stability properties of the balanced Maruyama with two-step methods when applied to SDHNN as presented in equation (2.1). Using the balanced Maruyama with two-step approximations on equation (2.1), we obtain

$$\begin{aligned}\sum_{r=-1}^{1} \alpha_r \mathbf{z}^{k-r} &= \sum_{r=-1}^{1} \beta_r \left[-C\mathbf{z}^{k-r} + A\mathbf{f}\left(\mathbf{z}^{k-r}\right) + B\mathbf{g}\left(\mathbf{z}^{k-m-r}\right)\right]\Delta \\ &\quad + \sum_{r=0}^{1} \gamma_r \left(\sigma\left(\mathbf{z}^{k-r}\right)\Delta W^{k-r}\right) + B_k[\mathbf{z}^k - \mathbf{z}^{k+1}], \qquad k = 1, 2, \cdots, N. \end{aligned} \tag{3.1}$$

Here the $n \times n$ matrix $B(\mathbf{z}^k)$ is given by

$$B_k = B_0(\mathbf{z}^k)\Delta + B_1(\mathbf{z}^k)\ |\Delta W_n^k|,$$

The matrices B_0 and B_1 are referred to as control functions and are typically chosen as constants. These control functions are generally represented as matrices and are uniformly bounded. It is important to note that specific conditions, as defined in [9, 12], must be satisfied by these control functions. If the parameters of the balanced Maruyama with two-step approximations are selected such that they satisfy the consistency condition.

$$\sum_{j=-1}^{1} \alpha_j = 0, \quad 2\alpha_{-1} + \alpha_0 = \sum_{j=-1}^{1} \beta_j, \quad \alpha_{-1} = \gamma_0, \quad \alpha_{-1} + \alpha_0 = \gamma_1. \tag{3.2}$$

and

$$\alpha_{-1} = 1, \qquad -1 \leq \alpha_0 < 0, \qquad \beta_0 = \beta_1 = 0, \tag{3.3}$$

then we get

$$\alpha_1 = -1 - \alpha_0, \qquad \beta_{-1} = 2 + \alpha_0, \qquad \gamma_0 = 1, \qquad \gamma_1 = 1 + \alpha_0. \tag{3.4}$$

Upcoming theorem, we utilize the approach outlined in [7] to examine the nature of the balanced Maruyama with two-step approximations [2, 3] under the exponential stability with almost sure. However, when applying the proof method described in [7, 9], our calculation results in a value that differs for γ in the numerical section. Our numerical analysis shows that the balanced Maruyama with two-step numerical approximations outperforms the two-step Maruyama numerical approximations in terms of stability. With that being said, we will now provide the proof of the almost sure stability of the two-step Maruyama methods for SDHNN.

Theorem 3.1. *Assuming that hypotheses ($\mathbb{H}1$)-($\mathbb{H}5$) are satisfied, let γ be a positive number determined by the expression in question, $\gamma_0 > 0$ is the number defined by equation (3.12), let γ^* be the maximum of γ and γ_0, and let $\epsilon \in (0, \frac{\gamma^*}{2})$ be an arbitrary value. Provided that Δ is less than a certain Δ_0, then for all finite-valued random variables that are measurable with respect to $\mathscr{F}_0$, $\zeta(k\Delta)$, $k = -m, -m+1, -m+2, \cdots, 0$, the balanced Maruyama with two-step approximations (3.2) are a.s. exponentially stable. In other words, we obtain:*

$$\limsup_{k\to\infty} \frac{\log\left|\mathbf{z}^k\right|}{k\Delta} \le -\frac{\gamma^*}{2} + \epsilon, \quad a.s. \tag{3.5}$$

Proof. Expanding (3.1) and substituting in (3.3), we get

$$\begin{aligned}\mathbf{z}^{k+1} + \alpha_0\mathbf{z}^k - (1+\alpha_0)\mathbf{z}^{k-1} \;&=\; (2+\alpha_0)\left[-C\mathbf{z}^{k+1} + Af\left(\mathbf{z}^{k+1}\right) + Bg\left(\mathbf{z}^{k-m+1}\right)\right]\Delta \\ &\quad + \sigma\left(\mathbf{z}^k\right)\Delta W^k + (1+\alpha_0)\sigma\left(\mathbf{z}^{k-1}\right)\Delta W^{k-1} + B_k(\mathbf{z}^k - \mathbf{z}^{k+1}).\end{aligned}$$

Thus, we get

$$\begin{aligned}\mathbf{z}^{k+1}\left[1 + (2+\alpha_0)C\Delta + B_k\right] \;&=\; (B_k - \alpha_0)\mathbf{z}^k + (1+\alpha_0)\mathbf{z}^{k-1} + (2+\alpha_0)\left[Af\left(\mathbf{z}^{k+1}\right)\right]\Delta \\ &\quad + (2+\alpha_0)Bg\left(\mathbf{z}^{k-m+1}\right)\Delta + \sigma\left(\mathbf{z}^k\right)\Delta W^k + (1+\alpha_0)\sigma\left(\mathbf{z}^{k-1}\right)\Delta W^{k-1}.\end{aligned} \tag{3.6}$$

Utilizing the balanced Maruyama with two-step methods and under ($\mathbb{H}2$) and ($\mathbb{H}3$), we obtain:

$$\begin{aligned}&\left[1 + (2+\alpha_0)C\Delta + B_k\right]^2\left|\mathbf{z}^{k+1}\right|^2 = \Big\langle (B_k - \alpha_0)\mathbf{z}^k + (1+\alpha_0)\mathbf{z}^{k-1} + (2+\alpha_0)\left[Af\left(\mathbf{z}^{k+1}\right) + Bg\left(\mathbf{z}^{k-m+1}\right)\right]\Delta \\ &\quad + \sigma\left(\mathbf{z}^k\right)\Delta W^k + (1+\alpha_0)\sigma\left(\mathbf{z}^{k-1}\right)\Delta W^{k-1}, (B_k - \alpha_0)\mathbf{z}^k + (1+\alpha_0)\mathbf{z}^{k-1} \\ &\quad + (2+\alpha_0)\left[Af\left(\mathbf{z}^{k+1}\right) + Bg\left(\mathbf{z}^{k-m+1}\right)\right]\Delta + \sigma\left(\mathbf{z}^k\right)\Delta W^k + (1+\alpha_0)\sigma\left(\mathbf{z}^{k-1}\right)\Delta W^{k-1}\Big\rangle, \\ &\quad = (B_k - \alpha_0)^2\left|\mathbf{z}^k\right|^2 + (1+\alpha_0)^2\left|\mathbf{z}^{k-1}\right|^2 + 2(B_k - \alpha_0)\mathbf{z}^k\left[(1+\alpha_0)\mathbf{z}^{k-1}\right] \\ &\quad + 2(B_k - \alpha_0)\mathbf{z}^k\left[(2+\alpha_0)\left[Af\left(\mathbf{z}^{k+1}\right) + Bg\left(\mathbf{z}^{k-m+1}\right)\right]\Delta\right] \\ &\quad + 2(1+\alpha_0)\mathbf{z}^{k-1}\left[(2+\alpha_0)\left[Af\left(\mathbf{z}^{k+1}\right) + Bg\left(\mathbf{z}^{k-m+1}\right)\right]\Delta\right] + \Big[(2+\alpha_0)Af\left(\mathbf{z}^{k+1}\right)\Delta \\ &\quad + (2+\alpha_0)Af\left(\mathbf{z}^{k+1}\right)\Delta\Big]^2 + \left(\sigma\left(\mathbf{z}^k\right)\Delta W^k\right)^2 \\ &\quad + (1+\alpha_0)^2\left(\sigma\left(\mathbf{z}^{k-1}\right)\Delta W^{k-1}\right)^2 + 2\Big\langle (B_k - \alpha_0)\mathbf{z}^k + (1+\alpha_0)\mathbf{z}^{k-1} + (2+\alpha_0)\left[Af\left(\mathbf{z}^{k+1}\right)\right] \\ &\qquad + (2+\alpha_0)\left[Bg\left(\mathbf{z}^{k-m+1}\right)\right], \sigma\left(\mathbf{z}^k\right)\Delta W^k + (1+\alpha_0)\sigma\left(\mathbf{z}^{k-1}\right)\Delta W^{k-1}\Big\rangle \\ &\qquad \le (B_k - \alpha_0)^2\left|\mathbf{z}^k\right|^2 + (1+\alpha_0)^2\left|\mathbf{z}^{k-1}\right|^2 + 2(B_k - \alpha_0)\left|\mathbf{z}^k\right|(1+\alpha_0)\left|\mathbf{z}^{k-1}\right| \\ &\qquad + 2(B_k - \alpha_0)\left|\mathbf{z}^k\right|(2+\alpha_0)A\kappa\left|\mathbf{z}^{k+1}\right|\Delta + 2(B_k - \alpha_0)\left|\mathbf{z}^k\right|(2+\alpha_0)B\rho\left|\mathbf{z}^{k-m+1}\right|\Delta \\ &\qquad + 2(1+\alpha_0)|\mathbf{z}^{k-1}|(2+\alpha_0)A\kappa\left|\mathbf{z}^{k+1}\right|\Delta + 2(1+\alpha_0)|\mathbf{z}^{k-1}|(2+\alpha_0)B\rho\left|\mathbf{z}^{k-m+1}\right|\Delta \\ &\qquad + 2(2+\alpha_0)^2A^2\kappa^2\left|\mathbf{z}^{k+1}\right|^2\Delta^2 + 2(2+\alpha_0)^2B^2\rho^2\left|\mathbf{z}^{k-m+1}\right|^2\Delta^2 + \mu^2\left|\mathbf{z}^k\right|^2 \cdot \left|\Delta W^k\right|^2 \\ &\qquad + (1+\alpha_0)^2\mu^2\left|\mathbf{z}^{k-1}\right|^2 \cdot \left|\Delta W^{k-1}\right|^2 + 2\Big\langle (B_k - \alpha_0)\mathbf{z}^k + (1+\alpha_0)\mathbf{z}^{k-1} + (2+\alpha_0)\left[Af\left(\mathbf{z}^{k+1}\right)\right]\Delta \\ &\qquad + (2+\alpha_0)\left[Bg\left(\mathbf{z}^{k-m+1}\right)\right]\Delta, \sigma\left(\mathbf{z}^k\right)\Delta W^k + (1+\alpha_0)\sigma\left(\mathbf{z}^{k-1}\right)\Delta W^{k-1}\Big\rangle,\end{aligned}$$

We rewrite above equation then we get,

$$\begin{aligned}
&\left[(1+(2+\alpha_0)C\Delta+B_k)^2-(2+\alpha_0)^2\Delta^2\left(A^2\kappa^2+A\kappa B\rho\right)-(B_k+1)(2+\alpha_0)A\kappa\Delta\right]\left|\mathbf{z}^{k+1}\right|^2\\
&\le (B_k-\alpha_0)^2\left|\mathbf{z}^k\right|^2+(1+\alpha_0)^2\left|\mathbf{z}^{k-1}\right|^2+(B_k-\alpha_0)(1+\alpha_0)\left|\mathbf{z}^k\right|^2+(B_k-\alpha_0)(1+\alpha_0)\left|\mathbf{z}^{k-1}\right|^2\\
&\quad+(B_k-\alpha_0)\left|\mathbf{z}^k\right|^2(2+\alpha_0)A\kappa\Delta+(B_k-\alpha_0)\left|\mathbf{z}^k\right|^2(2+\alpha_0)B\rho\Delta+(B_k-\alpha_0)(2+\alpha_0)B\rho\left|\mathbf{z}^{k-m+1}\right|^2\Delta\\
&\quad+(1+\alpha_0)|\mathbf{z}^{k-1}|^2(2+\alpha_0)A\kappa\Delta+(1+\alpha_0)|\mathbf{z}^{k-1}|^2(2+\alpha_0)B\rho\Delta+(1+\alpha_0)(2+\alpha_0)B\rho\left|\mathbf{z}^{k-m+1}\right|^2\Delta\\
&\quad+(2+\alpha_0)^2AB\kappa\rho\left|\mathbf{z}^{k-m+1}\right|^2\Delta^2+(2+\alpha_0)^2B^2\rho^2\left|\mathbf{z}^{k-m+1}\right|^2\Delta^2+\mu^2\left|\mathbf{z}^k\right|^2\Delta\\
&\quad+(1+\alpha_0)^2\mu^2\left|\mathbf{z}^{k-1}\right|^2\Delta+2\Big\langle(B_k-\alpha_0)\mathbf{z}^k+(1+\alpha_0)\mathbf{z}^{k-1}+(2+\alpha_0)\left[Af\left(\mathbf{z}^{k+1}\right)\right]\Delta\\
&\quad+(2+\alpha_0)\left[Bg\left(\mathbf{z}^{k-m+1}\right)\right]\Delta,\sigma\left(\mathbf{z}^k\right)\Delta W^k+(1+\alpha_0)\sigma\left(\mathbf{z}^{k-1}\right)\Delta W^{k-1}\Big\rangle\\
&\le\left\{(B_k-\alpha_0)^2+(B_k-\alpha_0)(1+\alpha_0)+(B_k-\alpha_0)(2+\alpha_0)A\kappa\Delta+(B_k-\alpha_0)(2+\alpha_0)B\rho\Delta+\mu^2\Delta\right\}\left|\mathbf{z}^k\right|^2\\
&\quad+\left\{(1+\alpha_0)^2+(B_k-\alpha_0)(1+\alpha_0)+(1+\alpha_0)(2+\alpha_0)A\kappa\Delta+(1+\alpha_0)(2+\alpha_0)B\rho\Delta+(1+\alpha_0)^2\mu^2\Delta\right\}\left|\mathbf{z}^{k-1}\right|^2\\
&\quad+\left\{(B_k+1)(2+\alpha_0)B\rho\Delta+(2+\alpha_0)^2AB\kappa\rho\Delta^2+(2+\alpha_0)^2B^2\rho^2\Delta^2\right\}\left|\mathbf{z}^{k-m+1}\right|^2\Delta\\
&\quad+2\Big\langle(B_k-\alpha_0)\mathbf{z}^k+(1+\alpha_0)\mathbf{z}^{k-1}+(2+\alpha_0)\left[Af\left(\mathbf{z}^{k+1}\right)\right]\Delta+(2+\alpha_0)\left[Bg\left(\mathbf{z}^{k-m+1}\right)\right]\Delta,\\
&\quad\sigma\left(\mathbf{z}^k\right)\Delta W^k+(1+\alpha_0)\sigma\left(\mathbf{z}^{k-1}\right)\Delta W^{k-1}\Big\rangle, \qquad (3.7)
\end{aligned}$$

From equation (3.7) we get,

$$\begin{aligned}
Q|\mathbf{z}^{k+1}|^2 \le &\Big[\left\{(B_k-\alpha_0)^2+(B_k-\alpha_0)(1+\alpha_0)+(B_k-\alpha_0)(2+\alpha_0)\left(A\kappa+B\rho\right)\Delta\right.\\
&\left.+\mu^2\Delta\right\}|\mathbf{z}^k|^2+\Big\{(1+\alpha_0)^2+(B_k-\alpha_0)(1+\alpha_0)+(1+\alpha_0)(2+\alpha_0)A\kappa\Delta\\
&+(1+\alpha_0)(2+\alpha_0)B\rho\Delta+(1+\alpha_0)^2\mu^2\Delta\Big\}|\mathbf{z}^{k-1}|^2+\Big\{(B_k+1)(2+\alpha_0)B\rho\Delta\\
&+(2+\alpha_0)^2AB\kappa\rho\Delta^2+(2+\alpha_0)^2B^2\rho^2\Delta^2\Big\}|\mathbf{z}^{k-m+1}|^2+m_k\Big], \qquad (3.8)
\end{aligned}$$

where

$$\begin{aligned}
Q &= (1+(2+\alpha_0)C\Delta+B_k)^2-(2+\alpha_0)^2\Delta^2\left(A^2\kappa^2+A\kappa B\rho\right)-(B_k+1)(2+\alpha_0)A\kappa\Delta,\\
m_k &= 2\Big\langle(B_k-\alpha_0)\mathbf{z}^k+(1+\alpha_0)\mathbf{z}^{k-1}+(2+\alpha_0)\left[Af\left(\mathbf{z}^{k+1}\right)\right]\Delta+(2+\alpha_0)\left[Bg\left(\mathbf{z}^{k-m+1}\right)\right]\Delta,\\
&\qquad\sigma\left(\mathbf{z}^k\right)\Delta W^k+(1+\alpha_0)\sigma\left(\mathbf{z}^{k-1}\right)\Delta W^{k-1}\Big\rangle.
\end{aligned}$$

For any constant $C>1$, we have

$$D^{(k+1)\Delta}|\mathbf{z}^{k+1}|^2-D^{k\Delta}|\mathbf{z}^k|^2=D^{(k+1)\Delta}\left(|\mathbf{z}^{k+1}|^2-|\mathbf{z}^k|^2\right)+\left(D^{(k+1)\Delta}-D^{k\Delta}\right)|\mathbf{z}^k|^2$$

By this fact and equation (3.8) give

$$
\begin{aligned}
&Q\left[D^{(k+1)\Delta}|\mathbf{z}^{k+1}|^2 - D^{k\Delta}|\mathbf{z}^k|^2\right] \\
&\le \{(B_k-\alpha_0)^2 + (B_k-\alpha_0)(1+\alpha_0) + (B_k-\alpha_0)(2+\alpha_0)A\kappa\Delta + (B_k-\alpha_0)(2+\alpha_0)B\rho\Delta \\
&\quad +\mu^2\Delta^2 - QD^{-\Delta}\}\, D^{(k+1)\Delta}|\mathbf{z}^k|^2 + \Big\{(1+\alpha_0)^2 + (B_k-\alpha_0)(1+\alpha_0) + (1+\alpha_0)(2+\alpha_0)A\kappa\Delta \\
&\quad +(1+\alpha_0)(2+\alpha_0)B\rho\Delta + (1+\alpha_0)^2\mu^2\Delta\Big\} D^{(k+1)\Delta}|\mathbf{z}^{k-1}|^2 + \Big\{(B_k+1)(2+\alpha_0)B\rho\Delta \\
&\quad +(2+\alpha_0)^2 AB\kappa\rho\Delta^2 + (2+\alpha_0)^2B^2\rho^2\Delta^2\Big\} D^{(k+1)\Delta}|\mathbf{z}^{k-m+1}|^2 + m_k D^{(k+1)\Delta}.
\end{aligned}
$$

Hence, we acquire

$$
\begin{aligned}
Q\left[D^{k\Delta}|\mathbf{z}^k|^2\right] \quad &\le Q\left[|\mathbf{z}^0|^2\right] + \{(B_k-\alpha_0)^2 + (B_k-\alpha_0)(1+\alpha_0) + (B_k-\alpha_0)(2+\alpha_0)A\kappa\Delta \\
&\quad +(B_k-\alpha_0)(2+\alpha_0)B\rho\Delta + \mu^2\Delta - QD^{-\Delta}\} \sum_{s=0}^{k-1} D^{(s+1)\Delta}|\mathbf{z}^s|^2 + \Big\{(1+\alpha_0)^2 \\
&\quad +(B_k-\alpha_0)(1+\alpha_0) + (1+\alpha_0)(2+\alpha_0)A\kappa\Delta + (1+\alpha_0)(2+\alpha_0)B\rho\Delta \\
&\quad +(1+\alpha_0)^2\mu^2\Delta\Big\} \sum_{s=0}^{k-1} D^{(s+1)\Delta}|\mathbf{z}^{s-1}|^2 + \Big\{(B_k+1)(2+\alpha_0)B\rho\Delta \\
&\quad +(2+\alpha_0)^2AB\kappa\rho\Delta^2 + (2+\alpha_0)^2B^2\rho^2\Delta^2\Big\} \sum_{s=0}^{k-1} D^{(s+1)\Delta}|\mathbf{z}^{s-m+1}|^2 + m_k \sum_{s=0}^{k-1} D^{(s+1)\Delta},
\end{aligned}
$$

where $\sum_{s=0}^{k-1} D^{(s+1)\Delta} m_k = M_k$ is a martingale with $M_0 = 0$. Note that

$$
\begin{aligned}
\sum_{s=0}^{k-1} D^{(s+1)\Delta}|\mathbf{z}^{s-m+1}|^2 \quad &= D^{(m-1)\Delta} \sum_{s=-m+1}^{-1} D^{(s+1)\Delta}|\mathbf{z}^s|^2 + D^{(m-1)\Delta} \sum_{s=0}^{k-1} D^{(s+1)\Delta}|\mathbf{z}^s|^2 \\
&\quad - D^{(m-1)\Delta} \sum_{s=k-m+1}^{k-1} D^{(s+1)\Delta}|\mathbf{z}^s|^2,
\end{aligned}
$$

and

$$
\sum_{s=0}^{k-1} D^{(s+1)\Delta}|\mathbf{z}^{s-1}|^2 = D^{\Delta} \sum_{s=-1}^{-1} D^{(s+1)\Delta}|\mathbf{z}^s|^2 + D^{\Delta} \sum_{s=0}^{k-1} D^{(s+1)}|\mathbf{z}^s|^2 - D^{\Delta} \sum_{s=k-1}^{k-1} D^{(s+1)\Delta}|\mathbf{z}^s|^2.
$$

Therefore, we have

$$\begin{aligned} Q\left[D^{k\Delta}|\mathbf{z}^k|^2\right] \quad &\leq Q\left[|\mathbf{z}^0|^2\right] + \big\{(B_k-\alpha_0)^2 + (B_k-\alpha_0)(1+\alpha_0) + (B_k-\alpha_0)(2+\alpha_0)A\kappa\Delta \\ &+(B_k-\alpha_0)(2+\alpha_0)B\rho\Delta + \mu^2\Delta - QD^{-\Delta}\big\}\sum_{s=0}^{k-1} D^{(s+1)\Delta}|\mathbf{z}^s|^2 + \Big\{(1+\alpha_0)^2 \\ &+(B_k-\alpha_0)(1+\alpha_0) + (1+\alpha_0)(2+\alpha_0)A\kappa\Delta + (1+\alpha_0)(2+\alpha_0)B\rho\Delta \\ &+(1+\alpha_0)^2\mu^2\Delta\Big\} \times \left(D^{\Delta}\sum_{s=-1}^{-1} D^{(s+1)\Delta}|\mathbf{z}^s|^2 + D^{\Delta}\sum_{s=0}^{k-1} D^{(s+1)}|\mathbf{z}^s|^2\right. \\ &\left.-D^{\Delta}\sum_{s=k-1}^{k-1} D^{(s+1)\Delta}|\mathbf{z}^s|^2\right) + \Big\{(B_k-\alpha_0)(2+\alpha_0)B\rho + (1+\alpha_0)(2+\alpha_0)B\rho\Delta \\ &+(2+\alpha_0)^2AB\kappa\rho\Delta^2 + (2+\alpha_0)^2B^2\rho^2\Delta^2\Big\} \times \left(D^{(m-1)\Delta}\sum_{s=-m+1}^{-1} D^{(s+1)\Delta}|\mathbf{z}^s|^2\right. \\ &\left.+D^{(m-1)\Delta}\sum_{s=0}^{k-1} D^{(s+1)\Delta}|\mathbf{z}^s|^2 - D^{(m-1)\Delta}\sum_{s=k-m+1}^{k-1} D^{(s+1)\Delta}|\mathbf{z}^s|^2\right) + M_k. \end{aligned}$$

Thus, we get

$$\begin{aligned} &Q\left[D^{k\Delta}|\mathbf{z}^k|^2\right] + \Big\{(1+\alpha_0)^2 + (B_k-\alpha_0)(1+\alpha_0) + (1+\alpha_0)(2+\alpha_0)A\kappa\Delta + (1+\alpha_0)(2+\alpha_0)B\rho\Delta \\ &\quad+(1+\alpha_0)^2\mu^2\Delta\Big\} \times \left(D^{\Delta}\sum_{s=k-1}^{k-1} D^{(s+1)\Delta}|\mathbf{z}^s|^2\right) + \Big\{(B_k-\alpha_0)(2+\alpha_0)B\rho + (1+\alpha_0) \\ &\quad\cdot(2+\alpha_0)B\rho\Delta + (2+\alpha_0)^2AB\kappa\rho\Delta^2 + (2+\alpha_0)^2B^2\rho^2\Delta^2\Big\} \times \left(D^{(m-1)\Delta}\sum_{s=k-m+1}^{k-1} D^{(s+1)\Delta}|\mathbf{z}^s|^2\right) \\ &\leq Q\left[|\mathbf{z}^0|^2\right] + \big\{(B_k-\alpha_0)^2 + (B_k-\alpha_0)(1+\alpha_0) + (B_k-\alpha_0)(2+\alpha_0)A\kappa\Delta + (B_k-\alpha_0)(2+\alpha_0)B\rho\Delta \\ &\quad+\mu^2\Delta - QD^{-\Delta}\big\}\sum_{s=0}^{k-1} D^{(s+1)\Delta}|\mathbf{z}^s|^2 + \Big\{(1+\alpha_0)^2 + (B_k-\alpha_0)(1+\alpha_0) + (1+\alpha_0)(2+\alpha_0)A\kappa\Delta \\ &\quad+(1+\alpha_0)(2+\alpha_0)B\rho\Delta + (1+\alpha_0)^2\mu^2\Delta\Big\} \times \left(D^{\Delta}\sum_{s=-1}^{-1} D^{(s+1)\Delta}|\mathbf{z}^s|^2 + D^{\Delta}\sum_{s=0}^{k-1} D^{(s+1)}|\mathbf{z}^s|^2\right) \\ &\quad+\Big\{(B_k-\alpha_0)(2+\alpha_0)B\rho + (1+\alpha_0)(2+\alpha_0)B\rho\Delta + (2+\alpha_0)^2AB\kappa\rho\Delta^2 + (2+\alpha_0)^2B^2\rho^2\Delta^2\Big\} \\ &\quad\times\left(D^{(m-1)\Delta}\sum_{s=-m+1}^{-1} D^{(s+1)\Delta}|\mathbf{z}^s|^2 + D^{(m-1)\Delta}\sum_{s=0}^{k-1} D^{(s+1)\Delta}|\mathbf{z}^s|^2\right) + M_k. \end{aligned}$$

Therefore, we have

$$Q\left[D^{k\Delta}|\mathbf{z}^k|^2\right]+\Big\{(1+\alpha_0)^2+(B_k-\alpha_0)(1+\alpha_0)+(1+\alpha_0)(2+\alpha_0)A\kappa\Delta+(1+\alpha_0)(2+\alpha_0)B\rho\Delta$$
$$+(1+\alpha_0)^2\mu^2\Delta\Big\}\times\left(D^{m\Delta}\sum_{s=k-1}^{k-1}D^{(s+1)\Delta}|\mathbf{z}^s|^2\right)+\Big\{(B_k-\alpha_0)(2+\alpha_0)B\rho+(1+\alpha_0)(2+\alpha_0)B\rho\Delta$$
$$+(2+\alpha_0)^2AB\kappa\rho\Delta^2+(2+\alpha_0)^2B^2\rho^2\Delta^2\Big\}\times\left(D^{(m-1)\Delta}\sum_{s=k-m+1}^{k-1}D^{(s+1)\Delta}|\mathbf{z}^s|^2\right)\le Y_k,$$

where

$$Y_k=\quad Q\left[|\mathbf{z}^0|^2\right]+\left\{(B_k-\alpha_0)^2+(B_k-\alpha_0)(1+\alpha_0)+(B_k-\alpha_0)(2+\alpha_0)A\kappa\Delta+(B_k-\alpha_0)(2+\alpha_0)B\rho\Delta\right.$$
$$\left.+\mu^2\Delta-QD^{-\Delta}\right\}\sum_{s=0}^{k-1}D^{(s+1)\Delta}|\mathbf{z}^s|^2+\Big\{(1+\alpha_0)^2+(B_k-\alpha_0)(1+\alpha_0)+(1+\alpha_0)(2+\alpha_0)A\kappa\Delta$$
$$+(1+\alpha_0)(2+\alpha_0)B\rho\Delta+(1+\alpha_0)^2\mu^2\Delta\Big\}\times\left(D^{\Delta}\sum_{s=-1}^{-1}D^{(s+1)\Delta}|\mathbf{z}^s|^2+D^{\Delta}\sum_{s=0}^{k-1}D^{(s+1)}|\mathbf{z}^s|^2\right)$$
$$+\Big\{(B_k-\alpha_0)(2+\alpha_0)B\rho\Delta+(1+\alpha_0)(2+\alpha_0)B\rho\Delta+(2+\alpha_0)^2AB\kappa\rho\Delta^2+(2+\alpha_0)^2B^2\rho^2\Delta^2\Big\}$$
$$\times\left(D^{(m-1)\Delta}\sum_{s=-m+1}^{-1}D^{(s+1)\Delta}|\mathbf{z}^s|^2+D^{(m-1)\Delta}\sum_{s=0}^{k-1}D^{(s+1)\Delta}|\mathbf{z}^s|^2\right)+M_k.$$

Let us consider the following function:

$$\Psi(D)\quad=\quad\left\{(B_k-\alpha_0)^2+(B_k-\alpha_0)(1+\alpha_0)+(B_k-\alpha_0)(2+\alpha_0)A\kappa\Delta+(B_k-\alpha_0)(2+\alpha_0)B\rho\Delta\right.$$
$$\left.+\mu^2\Delta-QD^{-\Delta}\right\}\sum_{s=0}^{k-1}D^{(s+1)\Delta}|\mathbf{z}^s|^2+\Big\{(1+\alpha_0)^2+(B_k-\alpha_0)(1+\alpha_0)+(1+\alpha_0)(2+\alpha_0)A\kappa\Delta$$
$$+(1+\alpha_0)(2+\alpha_0)B\rho\Delta+(1+\alpha_0)^2\mu^2\Delta\Big\}\times\left(D^{\Delta}\sum_{s=0}^{k-1}D^{(s+1)}|\mathbf{z}^s|^2\right)$$
$$+\Big\{(B_k-\alpha_0)(2+\alpha_0)B\rho\Delta+(1+\alpha_0)(2+\alpha_0)B\rho\Delta+(2+\alpha_0)^2AB\kappa\rho\Delta^2+(2+\alpha_0)^2B^2\rho^2\Delta^2\Big\}$$
$$\times\left(D^{(m-1)\Delta}\sum_{s=0}^{k-1}D^{(s+1)\Delta}|\mathbf{z}^s|^2\right).$$

It is obvious that

$$\Psi(D)\quad=\quad\left\{(B_k-\alpha_0)^2+(B_k-\alpha_0)(1+\alpha_0)+(B_k-\alpha_0)(2+\alpha_0)A\kappa\Delta+(B_k-\alpha_0)(2+\alpha_0)B\rho\Delta\right.$$
$$\left.+\mu^2\Delta-QD^{-\Delta}\right\}D^{\Delta}+\Big\{(1+\alpha_0)^2+(B_k-\alpha_0)(1+\alpha_0)+(1+\alpha_0)(2+\alpha_0)A\kappa\Delta$$
$$+(1+\alpha_0)(2+\alpha_0)B\rho\Delta+(1+\alpha_0)^2\mu^2\Delta\Big\}D^{2\Delta}+\Big\{(B_k+1)(2+\alpha_0)B\rho\Delta$$
$$+(2+\alpha_0)^2AB\kappa\rho\Delta^2+(2+\alpha_0)^2B^2\rho^2\Delta^2\Big\}D^{m\Delta}.$$

For any $D > 1$, it is evident that $\Psi^{'}(D)$ is positive. Additionally,

$$\begin{aligned}\Psi(1) \quad = \quad & \{(B_k-\alpha_0)^2+(B_k-\alpha_0)(1+\alpha_0)+(B_k-\alpha_0)(2+\alpha_0)A\kappa\Delta+(B_k-\alpha_0)(2+\alpha_0)B\rho\Delta \\ & +\mu^2\Delta-Q\}+\Big\{(1+\alpha_0)^2+(B_k-\alpha_0)(1+\alpha_0)+(1+\alpha_0)(2+\alpha_0)A\kappa\Delta \\ & +(1+\alpha_0)(2+\alpha_0)B\rho\Delta+(1+\alpha_0)^2\mu^2\Delta\Big\}+\Big\{(B_k+1)(2+\alpha_0)B\rho\Delta \\ & +(2+\alpha_0)^2AB\kappa\rho\Delta^2+(2+\alpha_0)^2B^2\rho^2\Delta^2\Big\}.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\Psi(1) \quad = \quad & \Big[(2+\alpha_0)^2\left(A^2\kappa^2+AB\kappa\rho+B^2\rho^2\right)-(2+\alpha_0)^2D^2\Big]\Delta^2 \\ & +\Big[2(B_k+1)(2+\alpha_0)\left[-C+A\kappa+B\rho\right]+\left(1+(1+\alpha_0)^2\right)\mu^2\Big]\Delta.\end{aligned}$$

Hence, we get

$$\Delta_1=\left\{\frac{-\Big[2(B_k+1)(2+\alpha_0)\left[-C+A\kappa+B\rho\right]+\left(1+(1+\alpha_0)^2\right)\mu^2\Big]}{(2+\alpha_0)^2\Big[\left(A\kappa+B\rho\right)^2-D^2\Big]}\right\}.$$

From our calculations, it follows that $\Psi(1)$ is negative. Thus, for any $\Delta < \Delta_1$, there exists a unique value of D_0 such that $\Psi(D_0) = 0$. By selecting $D = D_0$, we get the following result:

$$\begin{aligned}Y_k = \quad & Q\left[|\mathbf{z}^0|^2\right]+\Big\{(1+\alpha_0)^2+(B_k-\alpha_0)(1+\alpha_0)+(1+\alpha_0)(2+\alpha_0)A\kappa\Delta \\ & +(1+\alpha_0)(2+\alpha_0)B\rho\Delta+(1+\alpha_0)^2\mu^2\Delta\Big\}\times\left(D^{\Delta}\sum_{s=-1}^{-1}D^{(s+1)\Delta}|\mathbf{z}^s|^2\right) \\ & +\Big\{(B_k+1)(2+\alpha_0)B\rho\Delta+(2+\alpha_0)^2AB\kappa\rho\Delta^2+(2+\alpha_0)^2B^2\rho^2\Delta^2\Big\} \\ & \times\left(D^{(m-1)\Delta}\sum_{s=-m+1}^{-1}D^{(s+1)\Delta}|\mathbf{z}^s|^2\right)+M_k.\end{aligned}$$

Taking into account the initial value $\mathbf{z}^k < \infty$ it can be inferred from Lemma 2 in [10] that for $D = D_0$, the following holds:

$$\lim_{k\to\infty} Y_k \leq \infty \quad \text{a.s.}$$

As a result, we obtain.

$$\begin{aligned}\limsup_{k\to\infty} QD_0|\mathbf{z}^k|^2 \;&\leq\; \limsup_{k\to\infty}\Bigg[QD_0^{k\Delta}|\mathbf{z}^k|^2+\Big\{(1+\alpha_0)^2+(B_k-\alpha_0)(1+\alpha_0)+(1+\alpha_0)(2+\alpha_0)A\kappa\Delta\\&\quad+(1+\alpha_0)(2+\alpha_0)B\rho\Delta+(1+\alpha_0)^2\mu^2\Delta\Big\}\times\left(D^{\Delta}\sum_{s=-1}^{-1}D^{(s+1)\Delta}|\mathbf{z}^s|^2\right)\\&\quad+\Big\{(B_k+1)(2+\alpha_0)B\rho\Delta+(2+\alpha_0)^2AB\kappa\rho\Delta^2+(2+\alpha_0)^2B^2\rho^2\Delta^2\Big\}\\&\quad\times\left(D^{(m-1)\Delta}\sum_{s=-m+1}^{-1}D^{(s+1)\Delta}|\mathbf{z}^s|^2\right)\Bigg]\\&\leq\; \lim_{k\to\infty}Y_k<\infty\quad \text{a.s.}\end{aligned}\tag{3.9}$$

Since τ_j can be expressed as $\tau_j=m_j\Delta$, we can conclude that

$$\begin{aligned}&(B_k-\alpha_0)^2+(B_k-\alpha_0)(1+\alpha_0)+(B_k-\alpha_0)(2+\alpha_0)A\kappa\Delta+(B_k-\alpha_0)(2+\alpha_0)B\rho\Delta\\&+\mu^2\Delta-QD^{-\Delta}+\Big\{(1+\alpha_0)^2+(B_k-\alpha_0)(1+\alpha_0)+(1+\alpha_0)(2+\alpha_0)A\kappa\Delta\\&+(1+\alpha_0)(2+\alpha_0)B\rho\Delta+(1+\alpha_0)^2\mu^2\Delta\Big\}D_0^{\Delta}+\Big\{(B_k+1)(2+\alpha_0)B\rho\Delta\\&+(2+\alpha_0)^2AB\kappa\rho\Delta^2+(2+\alpha_0)^2B^2\rho^2\Delta^2\Big\}\left(D_0^{(m-1)\Delta}\right)=0.\end{aligned}\tag{3.10}$$

We define the constant D as $D=e^{\varphi}$.

$$\begin{aligned}\Psi(\varphi)\;&=\;(B_k-\alpha_0)^2+(B_k-\alpha_0)(1+\alpha_0)+(B_k-\alpha_0)(2+\alpha_0)A\kappa\Delta+(B_k-\alpha_0)(2+\alpha_0)B\rho\Delta\\&\quad+\mu^2\Delta-Qe^{-\varphi\Delta}+\Big\{(1+\alpha_0)^2+(B_k-\alpha_0)(1+\alpha_0)+(1+\alpha_0)(2+\alpha_0)A\kappa\Delta\\&\quad+(1+\alpha_0)(2+\alpha_0)B\rho\Delta+(1+\alpha_0)^2\mu^2\Delta\Big\}e^{\varphi\Delta}+\Big\{(B_k+1)(2+\alpha_0)B\rho\Delta\\&\quad+(2+\alpha_0)^2AB\kappa\rho\Delta^2+(2+\alpha_0)^2B^2\rho^2\Delta^2\Big\}\left(e^{(m-1)\varphi\Delta}\right).\end{aligned}$$

Assuming $\varphi_0=\log(D_0)$, for any $\Delta<\Delta_1$, from (3.10), we obtain that

$$\begin{aligned}\bar{\Psi}(\varphi_0)\;&=\;\big\{(B_k-\alpha_0)^2+(B_k-\alpha_0)(1+\alpha_0)+(B_k-\alpha_0)(2+\alpha_0)A\kappa\Delta+(B_k-\alpha_0)(2+\alpha_0)B\rho\Delta\\&\quad+\mu^2\Delta-Qe^{-\varphi_0\Delta}\big\}\,e^{\varphi_0\Delta}+\Big\{(1+\alpha_0)^2+(B_k-\alpha_0)(1+\alpha_0)+(1+\alpha_0)(2+\alpha_0)A\kappa\Delta\\&\quad+(1+\alpha_0)(2+\alpha_0)B\rho\Delta+(1+\alpha_0)^2\mu^2\Delta\Big\}e^{2\varphi_0\Delta}+\Big\{(B_k-\alpha_0)(2+\alpha_0)B\rho\Delta+\\&\quad(1+\alpha_0)(2+\alpha_0)B\rho\Delta+(2+\alpha_0)^2AB\kappa\rho\Delta^2+(2+\alpha_0)^2B^2\rho^2\Delta^2\Big\}\left(e^{\varphi_0\tau_j}\right)=0.\end{aligned}$$

Noting that

$$\lim_{\Delta\to 0}\frac{(1+B_k)\,(1+\alpha_0)\,e^{2\varphi_0\Delta}+(1+B_k)\,(B_k-\alpha_0)\,e^{\varphi_0\Delta}-Q}{\Delta}=2\varphi_0(1+B_k)(2+\alpha_0+B_k)+(2+\alpha_0)\,(A\kappa-2C).$$

Then, we have

$$\begin{aligned}\lim_{\Delta\to 0}\bar{\Psi}\,(\varphi_0)\;&=\;\left((1+B_k)\,(2+\alpha_0)\,B\rho\right)e^{\varphi_0\tau_j}\\&\quad+(1+B_k)\,(2+\alpha_0)\,(\varphi_0-2C+2A\kappa+B\rho)+\left(1+(1+\alpha_0)^2\right)\mu^2.\end{aligned}\tag{3.11}$$

Let γ_0 be the unique positive root of equation (3.11). It follows from (3.11) that:

$$\lim_{\Delta \to 0} \varphi_0 = \gamma_0. \tag{3.12}$$

For any positive $\epsilon \in \left(0, \frac{\gamma^*}{2}\right)$, let Δ_2 be a value such that $\Delta < \Delta_2$. Then, we have $\gamma^* > \epsilon > 0$, which implies that

$$\varphi_0 > \gamma_0 - 2\epsilon > \gamma^* - 2\epsilon.$$

Therefore, by utilizing equation (3.9) and the definition of φ_0, it becomes apparent that

$$\limsup_{k \to \infty} e^{\varphi_0 \Delta} \left|\mathbf{z}^k\right|^2 < \infty.$$

Therefore, for any $\Delta < \Delta_0 = \min\{\Delta_1, \Delta_2\}$, we get

$$\limsup_{k \to \infty} \frac{\log \left|\mathbf{z}^k\right|}{k\Delta} \leq -\frac{\gamma^*}{2} + \epsilon, \quad \text{almost surely}$$

Hence the proof is complete. □

4. Numerical Examples

Example 4.1. Let us examine the subsequent SDHNN in the presence of a standard Brownian motion $W(t)$:

$$d\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = -C\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} dt + A\begin{pmatrix} f(y_1(t)) \\ f(y_2(t)) \end{pmatrix} dt + B\begin{pmatrix} g(y_1(t-1)) \\ g(y_2(t-2)) \end{pmatrix} dt + \begin{pmatrix} \sigma_1 y_1(t) \\ \sigma_2 y_2(t) \end{pmatrix} dW(t). \tag{4.1}$$

Let $f(x) = x$, $g(x) = sin(x)$. The case we are interested in is as follows:

$$C = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}, A = \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}, B = \begin{pmatrix} -4 & 3 \\ 3 & 2 \end{pmatrix} \text{ and } \sigma = \begin{pmatrix} 1 \\ -\sqrt{5} \end{pmatrix}.$$

It is obvious that $\kappa_j = \rho_j = 1$, for $j = 1, 2$. $\mu_{11} = 1$ and $\mu_{21} = \sqrt{5}$. So the conditions ($\mathbb{H}\mathbf{1}$)-($\mathbb{H}\mathbf{3}$) are satisfied. Let

$$\sum_{j=1}^{n} |a_{ij}|\kappa_j + \sum_{j=1}^{n} |b_{ij}|\rho_j = \sum_{j=1}^{n} |a_{ji}|\kappa_i + \sum_{j=1}^{n} |b_{ji}|\rho_i = \begin{cases} 17 & \text{if } i = 1, \\ 7 & \text{if } i = 2. \end{cases}$$

In this example, the two-step Maruyama numerical approximations are a.s. exponential stable for all $-1 \leq \alpha_0 < 0$ and $\Delta \in (0, 0.1)$. Let $\alpha_0 = -0.25$ with $\gamma_{10} = 1.3751$ and $\gamma_{20} = 1.0694$, the Figure 1 is unstable. By Theorem 3.1, the balanced Maruyama with two-step approximations are a.s. exponential stable for $\alpha_0 = -0.5$ with $\gamma_{10} = 1.3775$ and $\gamma_{20} = 1.0751$ (see Figure 2) is stable. Thus, we can verify the validity of Theorem 3.1 in this particular example.

5. Conclusion

The focus of this research is on the stability analysis of balanced Maruyama with two-step methods used in stochastic delay Hopfield neural networks (SDHNN). Our findings show that these numerical schemes provide better stability compared to earlier studies [10, 11]. We identify the step sizes that lead to almost sure exponential stability of SDHNN solutions when using the balanced Maruyama with two-step methods. Our future research aims to extend this investigation to more universal balanced Maruyama with two-step methods and also to balanced Maruyama with multi-step methods.

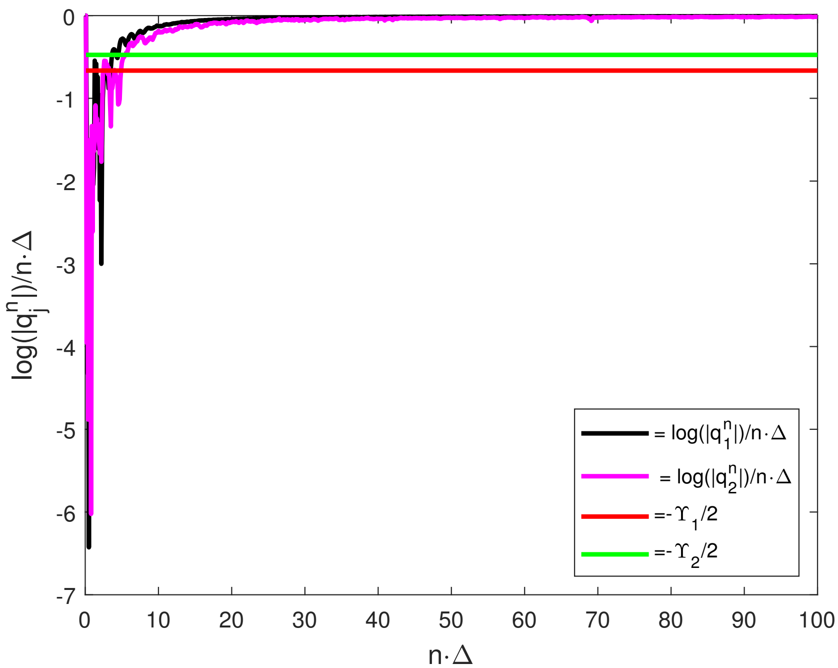

FIGURE 1. Two-step Maruyama numerical approximation is a.s. exponential unstable $\Delta = 0.2$.

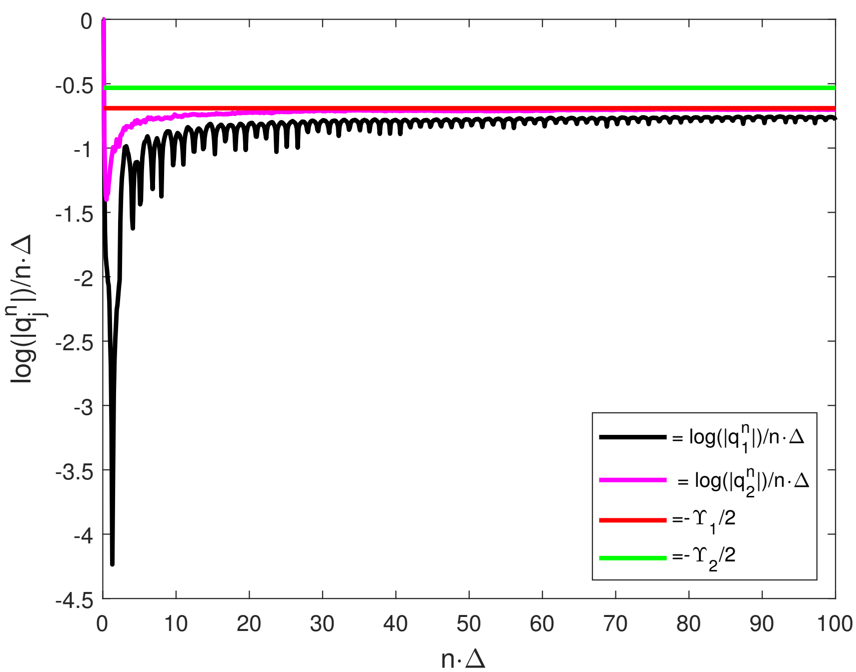

FIGURE 2. balanced Maruyama with two-step approximation is a.s. exponential stable with control function $B_n = 10$ and $\Delta = 0.2$.

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