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Application of fuzzy ABC fractional differential equations in infectious diseases

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Abstract

In this paper, for solving the HIV fuzzy mathematical model, it is first transformed into a system of three nonlinear fuzzy Atangana-Baleanu-Caputo (ABC) fractional differential equations with three unknowns and fuzzy initial values. Then, using the generalized Hukuhara difference and ABC fractional derivative and applying the fuzzy numerical ABC-PI method, its fuzzy solution is calculated. Moreover, some theorems are defined to prove the existence and uniqueness of the solution. Then, it is explained that the proposed method can be used for the system of any equations with unknowns.
Therefore, in order to determine the solution of the fuzzy mathematical model of the transmission of COVID-19, it is transformed into a system of six nonlinear fuzzy Atangana-Baleanu-Caputo (ABC) fractional differential equations with six unknowns and fuzzy initial values and is solved similarly. At the end, a numerical example is presented to verify the effectiveness of the proposed method.

Keywords. Fuzzy Atangana-Baleanu-Caputo(ABC) fractional derivative, Fuzzy ABC fractional differential equations, HIV fuzzy mathematical model.

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1. Introduction

Nowadays, viruses such as HIV and COVID-19 are the most dangerous viruses that have greatly affected human life. Various mathematical models have been proposed to describe such diseases. Among them, differential equation system-based models provide desirable results. Many studies have been performed in the field of HIV [3, 7, 13, 17, 20, 22, 25, 29]. In 2004, a system of differential equations with two equations was considered to model HIV infection, and the behavior of the solution was studied [15]. A more general model in the form of a system of differential equations with three equations was considered in [24], and after many investigations, it was stated that these findings not only propose a kinetic concept of pathogenicity for HIV-1 but also provide theoretical principles for developing treatment strategies.

Then, in [26], the previous model is extended to the following form by considering the anti-virus effects and studying the evolution of drug resistance:

$$\begin{cases} \frac{dx}{dt} = \beta - kxz - dx + by, \\ \frac{dy}{dt} = kzx - (b+\delta)x, \\ \frac{dz}{dt} = N\delta x - cz. \end{cases} \tag{1.1}$$

In the above equations, parameters are defined as follows: x, uninfected (susceptible) cells. y, infected cells in the eclipse phase. z, HIV virus particles in the blood cells. δ, the death rate of infected cells x; b, the rate at which infected cells return to the uninfected class. c, the death rate of viruses. N, the average number of viral particles produced by an infected cell, and k, the rate of infection.

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In [4], the model in (1.1) is considered under the Caputo fractional derivative, and numerical results are determined using the extended Euler method. Recently, Nazir et al. studied system (1.1) under the Caputo-Fabrizio derivative (CFD) with an exponential kernel. The existence and uniqueness of the solution were proved for this model using the fixed point theorem. Furthermore, the half-analytical solution of the model was achieved by the integral transform of Laplace coupled with the Adomian decomposition method [23]. On the other hand, according to [27], many parameters of the model that are based on medical information can be formulated as fuzzy if-then rule statements. Therefore, fuzzy modeling was considered in a symptomatic population infected by the HIV virus. Moreover, a comparison study between the fuzzy and Anderson classic models was performed.

Thereafter, a fuzzy delay system is considered to determine the fuzzy expected curve of HIV-positive individuals who receive antiretroviral therapy [28]. Zarei et al. introduced a system of linear differential equations with fuzzy parameters describing the ambiguous behavior of $CD4^{+}$ T cells and CTL level and the load of HIV virus in patients with weak, moderate, and strong immune systems. Moreover, by determining the implicit solution of the model, they claimed that despite the simplicity of the proposed model, it is able to determine the effect of antiretroviral therapy in preventing the spread of the HIV virus [32].

Another destructive virus is the coronavirus. Corona viruses are a large family of viruses that have a specified crown of sugary proteins. They were called Corona for the first time in 1960 due to their appearance. These kinds of viruses cause common diseases, such as the common cold, and fatal diseases. Middle East respiratory syndrome (MERS-CoV) and severe respiratory syndrome (SARS-CoV) are members of the Corona virus family. One of the new Corona viruses is COVID-19, which has recently affected various aspects of human life. Different approaches have been proposed to discuss and investigate this virus [6, 11, 12, 14, 18, 19, 21, 31, 33].

In [16], dynamic modeling of the Corona Virus (2019-nCov) was proposed using the Atangana-Baleanu derivative, and a numerical method was presented to solve it. Baleanu et al. proposed a fractional order model for transmission of COVID-19 under the Caputo-Fabrizio derivative. Using the homotopy analysis transformation method, which is a hybrid combination of homotopy analysis and Laplace transformation, the model was solved, and the existence and uniqueness of the solution were investigated [8]. Recently, qualitative modeling has been considered for COVID-19, and using Laplace transformation and Adomian decomposition, numerical results of the model have been achieved [1]. Since

- Modeling based on a system of differential equations yields good results in the mathematical modeling of HIV and COVID-19 viruses.
- It may be a good choice if one chooses a fuzzy mathematical model for HIV and COVID-19 that considers uncertainty and ambiguity, because in such diseases both humans and viruses are simultaneously involved, and the nature of these viruses is unknown and involves ambiguity. Additionally, in physical and environmental aspects, humans are different from each other.

Therefore, in this paper, a fuzzy mathematical model of HIV and COVID-19 viruses in the form of a fuzzy ABC fractional differential equations system is presented, and a numerical method is proposed to solve it.

The structure of the paper is as follows: in section 2, some necessary basic theories are represented. In section 3, a fuzzy mathematical model is considered for HIV and COVID-19 viruses in the form of the fuzzy ABC fractional differential equations system using the fuzzy numerical ABC-PI method. Then, a fuzzy numerical ABC-PI method is proposed to determine its solution. Moreover, the existence and uniqueness of the solutions and solving the system of fuzzy ABC fractional differential equations are investigated, and Ref [2] and its disadvantages are also reviewed. A numerical example is presented in section 4 to verify the proposed method, and finally, the conclusion is given in section 5.

2. Preliminaries

In this section, the basic definitions required for the next sections are presented.

Definition 2.1. A fuzzy number u in parametric form is a pair $\tilde{u} = [\underline{u}(r), \overline{u}(r)]$ of functions $\overline{u}(r)$ and $\overline{u}(r)$, $0 \le r \le 1$, which satisfy the following requirements:

1. $\underline{u}(r)$ is a bounded non-decreasing left continuous function in $(0, 1]$ and right-continuous at 0,

2. $\overline{u}(r)$ is a bounded non-increasing left continuous function in $(0,1]$ and right-continuous at 1,
3. $\underline{u}(r) \leq \overline{u}(r),\ 0 \leq r \leq 1$.

Definition 2.2. For arbitrary fuzzy numbers $\tilde{u} = [\underline{u}(r), \overline{u}(r)]$ and $\tilde{v} = [\underline{v}(r), \overline{v}(r)]$, we define addition and scalar multiplication for $0 \leq r \leq 1$ as follows:

$$\tilde{u} \oplus \tilde{v} = [\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)],$$

$$\lambda \odot \tilde{u} = \begin{cases} [\lambda \underline{u}(r), \lambda \overline{u}(r)], & \lambda \geq 0, \\ [\lambda \overline{u}(r), \lambda \underline{u}(r)] & \lambda < 0. \end{cases}$$

Definition 2.3. [5, 30] The generalized Hukhara difference of two fuzzy numbers is defined as follows:

$$\tilde{u} \ominus_{gh} \tilde{v} = w \Leftrightarrow \begin{cases} (i)\tilde{u} = \tilde{v} \oplus \tilde{w}, \\ or\ (ii)\tilde{v} = \tilde{u} \oplus (-1)\tilde{w}. \end{cases}$$

The first case is equivalent to the definition of the Hukuhara difference and is denoted by $\tilde{u} \ominus \tilde{v}$. If $\tilde{u} = [\underline{u}(r), \overline{u}(r)]$ and $\tilde{v} = [\underline{v}(r), \overline{v}(r)]$, there is:

$$\tilde{u} \ominus_{gh} \tilde{v} = [\min\{\underline{u}(r) - \underline{v}(r), \overline{u}(r) - \overline{v}(r)\}, \max\{\underline{u}(r) - \underline{v}(r), \overline{u}(r) - \overline{v}(r)\}],$$

and the conditions for the existence of $\tilde{u} \ominus_{gh} \tilde{v} = \tilde{w} \in R_F$ are:

$$case(i) : \begin{cases} \underline{w}(r) = \underline{u}(r) - \underline{v}(r) and\ \overline{w}(r) = \overline{u}(r) - \overline{v}(r), \\ with\ \underline{w}(r)\ increasing, \overline{w}(r)\ decreasing,\ \underline{w}(r) \leq \overline{w}(r). \end{cases}$$

$$case(ii) : \begin{cases} \underline{w}(r) = \overline{u}(r) - \overline{v}(r) and\ \overline{w}(r) = \underline{u}(r) - \underline{v}(r), \\ with\ \underline{w}(r)\ increasing,\ \overline{w}(r)\ decreasing,\ \underline{w}(r) \leq \overline{w}(r). \end{cases}$$

Definition 2.4. [30] If $x_0 \in (a,b)$ and h are such that $x_0 + h \in (a,b)$, then the gH-derivative of a function $f : (a,b) \to I$ can be defined as:

$$\tilde{f}'(x_0) = \lim_{h \to 0} \frac{\tilde{f}(x_0 + h) \ominus_{gh} \tilde{f}(x_0)}{h}.$$

If $\tilde{f}'(x_0) \in I$ satisfying the above exists, it can be said that $\tilde{f}$ is generalized Hukuhara differentiable (gH-differentiable, for short).

Definition 2.5.

- Case 1. (i-Differentiability)

$$\tilde{y}' = [\underline{y}'(t;r), \overline{y}'(t;r)],\ 0 \leq r \leq 1. \tag{2.1}$$

- Case 2. (ii-Differentiability)

$$\tilde{y}' = [\overline{y}'(t;r), \underline{y}'(t;r)],\ 0 \leq r \leq 1. \tag{2.2}$$

Definition 2.6. [2] The ABC fractional derivative in the sense of Caputo is defined in two cases as follows:

$${}_0^{ABC}D_t^{i,\alpha} \tilde{y}(t) = \left[{}_0^{ABC}D_t^{\alpha} \underline{y}(t;r), {}_0^{ABC}D_t^{\alpha} \overline{y}(t;r) \right],\ in\ case(1), \tag{2.3}$$

$${}_0^{ABC}D_t^{ii,\alpha} \tilde{y}(t) = \left[{}_0^{ABC}D_t^{\alpha} \overline{y}(t;r), {}_0^{ABC}D_t^{\alpha} \underline{y}(t;r) \right],\ in\ case(2), \tag{2.4}$$

where

$${}_0^{ABC}D_t^{*,\alpha} \tilde{y}(t) = \frac{B(\alpha)}{1-\alpha} \int_0^t E_\alpha(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha) \odot \tilde{y}'(\tau) d\tau\ * \in \{i, ii\}. \tag{2.5}$$

On other hand

$$
\begin{aligned}
&{}^{AB}_{0}I^{\alpha}_{0}({}^{AB}_{0}D^{i,\alpha}_{t}\tilde{y}(t)) = \tilde{y}(t) \ominus \tilde{y}(0),\\
&{}^{AB}_{0}I^{\alpha}_{t}({}^{AB}_{0}D^{ii,\alpha}_{t}\tilde{y}(t)) = \ominus(-1)(\tilde{y}(t) \ominus \tilde{y}(0)),
\end{aligned} \tag{2.6}
$$

where

$$
{}^{AB}_{0}I^{\alpha}_{t}(\tilde{y}(t)) = \frac{1-\alpha}{B(\alpha)} \odot \tilde{y}(t) \oplus \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \odot \int_0^t (t-\tau)^{\alpha-1} \odot \tilde{y}(\tau)d\tau. \tag{2.7}
$$

3. Fuzzy ABC Fractional Differential Equations System Application in HIV

Since the parameters involved in the mathematical model of HIV are inaccurate and the estimation of parameters is based on linguistic variables, qualitative modeling is investigated here to define a model that is meaningful and based on medical principles. The classic form is a special case of such a model.

Remark 3.1. In all models, variables and coefficients are positive, and they are solved by assuming the existence of a generalized Hukuhara derivative.

In this section, a HIV fuzzy mathematical model in the following form is solved:

$$
\begin{cases}
{}^{ABC}_{0}D^{*_1,\alpha}_{t}\tilde{x}(t) = \tilde{a} \ominus_{gh} \tilde{b} \odot \tilde{x}(t) \odot \tilde{z}(t) \ominus_{gh} \tilde{c} \odot \tilde{x}(t) \oplus \tilde{d} \odot \tilde{y}(t),\\
{}^{ABC}_{0}D^{*_2,\beta}_{t}\tilde{y}(t) = \tilde{b} \odot \tilde{z}(t) \odot \tilde{x}(t) \ominus_{gh} (\tilde{d} \oplus \tilde{e}) \odot \tilde{x}(t),\\
{}^{ABC}_{0}D^{*_3,\delta}_{t}\tilde{z}(t) = N.\tilde{e} \odot \tilde{x}(t) \ominus_{gh} \tilde{g} \odot \tilde{z}(t),\\
\tilde{x}(0) = \tilde{x}_0,\\
\tilde{y}(0) = \tilde{y}_0,\\
\tilde{z}(0) = \tilde{z}_0.
\end{cases} \tag{3.1}
$$

where $*_1, *_2, *_3 \in \{i, ii\},\ 0 < \alpha, \beta, \delta < 1, 0 < t < T < \infty, T \in R,$

$\tilde{x}(t)$	Uninfected (susceptible) cells
$\tilde{y}(t)$	Infected cells in the eclipse phase
$\tilde{z}(t)$	HIV virus particles in the blood cells
$\tilde{a}$	Supply rate of new T cells
$\tilde{c}$	Rate of natural death
$\tilde{b}$	Rate of infected T cells
$\tilde{e}$	Death rate of infected T cells
$\tilde{d}$	Rate of return of infected cells to the uninfected class
$\tilde{g}$	Death rate of viruses
N	Average number of particles infected by an uninfected cell

where $\tilde{x}(t)$, $\tilde{y}(t)$, and $\tilde{z}(t)$ are unknown positive fuzzy parameters, and $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$, $\tilde{e}$, and $\tilde{g}$ are known positive fuzzy parameters, and N is a known natural number.
To solve the HIV fuzzy mathematical model (3.1), it is considered that:

$$
\begin{aligned}
&\tilde{f}(t, \tilde{x}(t), \tilde{y}(t)), \tilde{z}(t)) = \tilde{a} \ominus_{gh} \tilde{b} \odot \tilde{x}(t) \odot \tilde{z}(t) \ominus_{gh} \tilde{c} \odot \tilde{x}(t) \oplus \tilde{d} \odot \tilde{y}(t),\\
&\tilde{g}(t, \tilde{x}(t), \tilde{y}(t)), \tilde{z}(t)) = \tilde{b} \odot \tilde{z}(t) \odot \tilde{x}(t) \ominus_{gh} (\tilde{d} \oplus \tilde{e} \odot \tilde{x}(t),\\
&\tilde{h}(t, \tilde{x}(t), \tilde{y}(t)), \tilde{z}(t)) = N.\tilde{e} \odot \tilde{x}(t) \ominus_{gh} \tilde{g} \odot \tilde{z}(t),
\end{aligned}
$$

As a result, the HIV mathematical model (3.1) changes to the following fuzzy ABC fractional differential equations system:

$$\begin{cases} {}_0^{ABC}D_t^{*_1,\alpha}\tilde{x}(t) = \tilde{f}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)), \\ {}_0^{ABC}D_t^{*_2,\beta}\tilde{y}(t) = \tilde{g}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)), \\ {}_0^{ABC}D_t^{*_3,\delta}\tilde{z}(t) = \tilde{h}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)), \\ \tilde{x}(0) = \tilde{x}_0, \\ \tilde{y}(0) = \tilde{y}_0, \\ \tilde{z}(0) = \tilde{z}_0. \end{cases} \tag{3.2}$$

Therefore, a practical method is presented in the following to solve the system.

Proposed Method

Existence. The fuzzy ABC fractional differential equations system (3.2) with continuous fuzzy functions $\tilde{f}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))$, $\tilde{g}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))$, and $\tilde{h}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))$, non-negative kernels for every t, α, β, and δ, and fuzzy number initial values has a fuzzy solution. Consider the fuzzy ABC fractional differential equations system (3.2). Applying the AB fractional integral on both sides of the equations, we have:

$$\begin{cases} {}_0^{AB}I_t^{\alpha}\left(D_t^{*_1,\alpha}\tilde{x}(t)\right) = {}_0^{AB}I_t^{\alpha}\left(\tilde{f}\left(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)\right)\right) \\ {}_0^{AB}I_t^{\alpha}\left(D_t^{*_2,\beta}\tilde{x}(t)\right) = {}_0^{AB}I_t^{\beta}\left(\tilde{g}\left(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)\right)\right) \\ {}_0^{AB}I_t^{\alpha}\left(D_t^{*_3,\delta}\tilde{x}(t)\right) = {}_0^{AB}I_t^{\beta}\left(\tilde{h}\left(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)\right)\right) \end{cases} \tag{3.3}$$

Now, different cases occur when solving system (3.2).

case	1	2	3	4	5	6	7	8
$*_1$	i	i	i	i	ii	ii	ii	ii
$*_2$	i	i	ii	ii	i	i	ii	ii
$*_3$	i	ii	i	ii	i	ii	i	ii

- **case 1:** $*_1 = *_2 = *_3 = i$.
  In this case, using (2.6) and (2.7) results in:

$$\begin{cases} \tilde{x}(t)\ominus\tilde{x}(0) = \frac{1-\alpha}{B(\alpha)}\odot\tilde{f}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))\oplus\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\odot\int_0^t(t-\tau)^{\alpha-1}\odot\tilde{f}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))d\tau, \\ \tilde{y}(t)\ominus\tilde{y}(0) = \frac{1-\beta}{B(\beta)}\odot\tilde{g}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))\oplus\frac{\beta}{B(\beta)\Gamma(\beta)}\odot\int_0^t(t-\tau)^{\beta-1}\odot\tilde{g}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))d\tau, \\ \tilde{z}(t)\ominus\tilde{z}(0) = \frac{1-\delta}{B(\delta)}\odot\tilde{h}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))\oplus\frac{\delta}{B(\delta)\Gamma(\delta)}\odot\int_0^t(t-\tau)^{\delta-1}\odot\tilde{h}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))d\tau \end{cases} \tag{3.4}$$

  Putting $t = t_n = 0 + nh$, where h is a constant step-size, it is obtained:

$$\begin{cases} \tilde{x}(t_n)\ominus\tilde{x}(0) = \frac{1-\alpha}{B(\alpha)}\odot\tilde{f}(t_n,\tilde{x}(t_n),\tilde{y}(t_n),\tilde{z}(t_n))\oplus\frac{\alpha}{B(\alpha)\Gamma(\alpha)} \\ \qquad\odot\sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}(t_n-\tau)^{\alpha-1}\odot\tilde{f}(\tau,\tilde{x}(\tau),\tilde{y}(\tau),\tilde{z}(\tau))d\tau, \\ \tilde{y}(t_n)\ominus\tilde{y}(0) = \frac{1-\beta}{B(\beta)}\odot\tilde{g}(t_n,\tilde{x}(t_n),\tilde{y}(t_n),\tilde{z}(t_n))\oplus\frac{\beta}{B(\beta)\Gamma(\beta)} \\ \qquad\odot\sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}(t_n-\tau)^{\beta-1}\odot\tilde{g}(\tau,\tilde{x}(\tau),\tilde{y}(\tau),\tilde{z}(\tau))d\tau, \\ \tilde{z}(t_n)\ominus\tilde{z}(0) = \frac{1-\delta}{B(\delta)}\odot\tilde{h}(t_n,\tilde{x}(t_n),\tilde{y}(t_n),\tilde{z}(t_n))\oplus\frac{\delta}{B(\delta)\Gamma(\delta)} \\ \qquad\odot\sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}(t_n-\tau)^{\delta-1}\odot\tilde{h}(\tau,\tilde{x}(\tau),\tilde{y}(\tau),\tilde{z}(\tau))d\tau. \end{cases} \tag{3.5}$$

Now, one can approximate the function $\tilde{f}(\tau,\tilde{x}(\tau),\tilde{y}(\tau),\tilde{z}(\tau))$ on $[t_i,t_{i+1}]$ by the first-order Lagrange interpolation as follows:

$$\begin{aligned}
&\tilde{f}(\tau,\tilde{x}(\tau),\tilde{y}(\tau),\tilde{z}(\tau)) \approx \tilde{f}(t_{i+1},\tilde{x}(t_{i+1}),\tilde{y}(t_{i+1}),\tilde{z}(t_{i+1}))\\
&\oplus \left(\frac{\tau-t_{i+1}}{h}\left(\ominus(-1)\left(\tilde{f}(t_{i+1},\tilde{x}(t_{i+1}),\tilde{y}(t_{i+1}),\tilde{z}(t_{i+1}))\ominus \tilde{f}(t_i,\tilde{x}(t_i),\tilde{y}(t_i),\tilde{z}(t_i))\right)\right)\right),\\
&\tilde{g}(\tau,\tilde{x}(\tau),\tilde{y}(\tau),\tilde{z}(\tau)) \approx \tilde{g}(t_{i+1},\tilde{x}(t_{i+1}),\tilde{y}(t_{i+1}),\tilde{z}(t_{i+1}))\\
&\oplus \left(\frac{\tau-t_{i+1}}{h}\left(\ominus(-1)\left(\tilde{g}(t_{i+1},\tilde{x}(t_{i+1}),\tilde{y}(t_{i+1}),\tilde{z}(t_{i+1}))\ominus \tilde{g}(t_i,\tilde{x}(t_i),\tilde{y}(t_i),\tilde{z}(t_i))\right)\right)\right),\\
&\tilde{h}(\tau,\tilde{x}(\tau),\tilde{y}(\tau),\tilde{z}(\tau)) \approx \tilde{h}(t_{i+1},\tilde{x}(t_{i+1}),\tilde{y}(t_{i+1}),\tilde{z}(t_{i+1}))\\
&\oplus \left(\frac{\tau-t_{i+1}}{h}\left(\ominus(-1)\left(\tilde{h}(t_{i+1},\tilde{x}(t_{i+1}),\tilde{y}(t_{i+1}),\tilde{z}(t_{i+1}))\ominus \tilde{h}(t_i,\tilde{x}(t_i),\tilde{y}(t_i),\tilde{z}(t_i))\right)\right)\right) \ \tau\in[t_i,t_{i+1}].
\end{aligned} \tag{3.6}$$

By substituting (3.6) in equations of (3.5) along with some algebraic operations, the following are achieved

$$\begin{cases}
\tilde{x}(t_n)\ominus\tilde{x}(0)=\frac{\alpha h^\alpha}{B(\alpha)}\odot\left(\varepsilon_n\odot\tilde{f}(0,\tilde{x}(0),\tilde{y}(0),\tilde{z}(0))\oplus\sum_{i=1}^n\mu_{n-i}\odot\tilde{f}(t_i,\tilde{x}(t_i),\tilde{y}(t_i),\tilde{z}(t_i))\right),\\
\tilde{y}(t_n)\ominus\tilde{y}(0)=\frac{\beta h^\beta}{B(\beta)}\odot\left(\varepsilon'_n\odot\tilde{g}(0,\tilde{x}(0),\tilde{y}(0),\tilde{z}(0))\oplus\sum_{i=1}^n\nu_{n-i}\odot\tilde{g}(t_i,\tilde{x}(t_i),\tilde{y}(t_i),\tilde{z}(t_i))\right),\\
\tilde{z}(t_n)\ominus\tilde{z}(0)=\frac{\delta h^\delta}{B(\delta)}\odot\left(\varepsilon''_n\odot\tilde{h}(0,\tilde{x}(0),\tilde{y}(0),\tilde{z}(0))\oplus\sum_{i=1}^n\gamma_{n-i}\odot\tilde{h}(t_i,\tilde{x}(t_i),\tilde{y}(t_i),\tilde{z}(t_i))\right).
\end{cases} \tag{3.7}$$

Finally, by solving the above system, where $B(x),\varepsilon_n,\varepsilon'_n,\varepsilon''_n,\mu_j,\nu_j,\gamma_j>0$ are defined as follows, the solution of the system of fuzzy ABC fractional differential equations is determined.

$$\begin{aligned}
&s(x)=1-x+\frac{x}{\Gamma(x)}, x=\alpha \ or \ \beta \ or \ \delta,\\
&\varepsilon_n=\frac{(n-1)^{\alpha+1}-n^\alpha(n-\alpha-1)}{\Gamma(\alpha+2)},\\
&\varepsilon'_n=\frac{(n-1)^{\beta+1}-n^\beta(n-\beta-1)}{\Gamma(\beta+2)},\\
&\varepsilon''_n=\frac{(n-1)^{\delta+1}-n^\delta(n-\delta-1)}{\Gamma(\delta+2)},\\
&\mu_j=\begin{cases}\frac{1}{\Gamma(\alpha+2)}+\frac{1-\alpha}{\alpha h^\alpha} & j=0,\\ \frac{(j-1)^{\alpha-1}-2j^{\alpha+1}+(j+1)^{\alpha+1}}{\Gamma(\alpha+2)} & j=1,2,\ldots,n-1,\end{cases}\\
&\nu_j=\begin{cases}\frac{1}{\Gamma(\beta+2)}+\frac{1-\beta}{\beta h^\beta} & j=0,\\ \frac{(j-1)^{\beta-1}-2j^{\beta+1}+(j+1)^{\beta+1}}{\Gamma(\beta+2)} & j=1,2,\ldots,n-1,\end{cases}\\
&\gamma_j=\begin{cases}\frac{1}{\Gamma(\delta+2)}+\frac{1-\delta}{\delta h^\delta} & j=0,\\ \frac{(j-1)^{\delta-1}-2j^{\delta+1}+(j+1)^{\delta+1}}{\Gamma(\delta+2)} & j=1,2,\ldots,n-1,\end{cases}
\end{aligned} \tag{3.8}$$

- **case 2:** $*_1=i, *_2=i, *_3=ii$.
  In this case, substituting (2.6) and (2.7) in (3.3) results in:

$$\begin{cases}
\tilde{x}(t)\ominus\tilde{x}(0)=\frac{1-\alpha}{B(\alpha)}\odot\tilde{f}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))\oplus\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\odot\int_0^t(t-\tau)^{\alpha-1}\odot\tilde{f}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))d\tau,\\
\tilde{y}(t)\ominus\tilde{y}(0)=\frac{1-\beta}{B(\beta)}\odot\tilde{g}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))\oplus\frac{\beta}{B(\beta)\Gamma(\beta)}\odot\int_0^t(t-\tau)^{\beta-1}\odot\tilde{g}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))d\tau,\\
\ominus(-1)(\tilde{z}(t)\ominus\tilde{z}(0))=\frac{1-\delta}{B(\delta)}\odot\tilde{h}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))\oplus\frac{\delta}{B(\delta)\Gamma(\delta)}\odot\int_0^t(t-\tau)^{\delta-1}\odot\tilde{h}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))d\tau.
\end{cases} \tag{3.9}$$

Therefore, the solution of system (3.2) can be achieved similar to the previous case by solving the following system:

$$\begin{cases} \tilde{x}(t_n) \ominus \tilde{x}(0) = \frac{\alpha h^{\alpha}}{B(\alpha)} \odot \Big(\varepsilon_n \odot \tilde{f}(0, \tilde{x}(0), \tilde{y}(0), \tilde{z}(0)\Big) \oplus \sum_{i=1}^{n} \mu_{n-i} \odot \tilde{f}(t_i, \tilde{x}(t_i), \tilde{y}(t_i), \tilde{z}(t_i)), \\ \tilde{y}(t_n) \ominus \tilde{y}(0) = \frac{\beta h^{\beta}}{B(\beta)} \odot (\varepsilon'_n \odot \tilde{g}(0, \tilde{x}(0), \tilde{y}(0), \tilde{z}(0)) \oplus \sum_{i=1}^{n} \nu_{n-i} \odot \tilde{g}(t_i, \tilde{x}(t_i), \tilde{y}(t_i), \tilde{z}(t_i)), \\ \ominus(-1)(\tilde{z}(t_n) \ominus \tilde{z}(0)) = \frac{\delta h^{\delta}}{B(\delta)} \odot \Big(\varepsilon''_n \odot \tilde{h}(0, \tilde{x}(0), \tilde{y}(0), \tilde{z}(0)\Big) \oplus \sum_{i=1}^{n} \gamma_{n-i} \odot \tilde{h}(t_i, \tilde{x}(t_i), \tilde{y}(t_i), \tilde{z}(t_i)), \end{cases} \tag{3.10}$$

in which coefficients are obtained from (3.8).

- **case 3:** $*_1 = i, *_2 = ii, *_3 = i$.
  In this case, substitution of (2.6) and (2.7) in (3.3) yields:

$$\begin{cases} \tilde{x}(t) \ominus \tilde{x}(0) = \frac{1-\alpha}{B(\alpha)} \odot \tilde{f}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) \oplus \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \odot \int_0^t (t-\tau)^{\alpha-1} \odot \tilde{f}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) d\tau, \\ \ominus(-1)(\tilde{y}(t) \ominus \tilde{y}(0)) = \frac{1-\beta}{B(\beta)} \odot \tilde{g}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) \oplus \frac{\beta}{B(\beta)\Gamma(\beta)} \odot \int_0^t (t-\tau)^{\beta-1} \odot \tilde{g}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) d\tau, \\ \tilde{z}(t) \ominus \tilde{z}(0) = \frac{1-\delta}{B(\delta)} \odot \tilde{h}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) \oplus \frac{\delta}{B(\delta)\Gamma(\delta)} \odot \int_0^t (t-\tau)^{\delta-1} \odot \tilde{h}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) d\tau. \end{cases} \tag{3.11}$$

Therefore, the solution of system (3.2) can be achieved similar to the previous case by solving the following system:

$$\begin{cases} \tilde{x}(t_n) \ominus \tilde{x}(0) = \frac{\alpha h^{\alpha}}{B(\alpha)} \odot \Big(\varepsilon_n \odot \tilde{f}(0, \tilde{x}(0), \tilde{y}(0), \tilde{z}(0)\Big) \oplus \sum_{i=1}^{n} \mu_{n-i} \odot \tilde{f}(t_i, \tilde{x}(t_i), \tilde{y}(t_i), \tilde{z}(t_i)), \\ \ominus(-1)(\tilde{y}(t_n) \ominus \tilde{y}(0)) = \frac{\beta h^{\beta}}{B(\beta)} \odot (\varepsilon'_n \odot \tilde{g}(0, \tilde{x}(0), \tilde{y}(0), \tilde{z}(0)) \oplus \sum_{i=1}^{n} \nu_{n-i} \odot \tilde{g}(t_i, \tilde{x}(t_i), \tilde{y}(t_i), \tilde{z}(t_i)), \\ \tilde{z}(t_n) \ominus \tilde{z}(0) = \frac{\delta h^{\delta}}{B(\delta)} \odot \Big(\varepsilon''_n \odot \tilde{h}(0, \tilde{x}(0), \tilde{y}(0), \tilde{z}(0)\Big) \oplus \sum_{i=1}^{n} \gamma_{n-i} \odot \tilde{h}(t_i, \tilde{x}(t_i), \tilde{y}(t_i), \tilde{z}(t_i)), \end{cases} \tag{3.12}$$

in which coefficients are obtained from (3.8).

- **case 4:** $*_1 = i, *_2 = ii, *_3 = ii$.
  In this case, substituting (2.6) and (2.7) in (3.3) leads to:

$$\begin{cases} \tilde{x}(t) \ominus \tilde{x}(0) = \frac{1-\alpha}{B(\alpha)} \odot \tilde{f}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) \oplus \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \odot \int_0^t (t-\tau)^{\alpha-1} \odot \tilde{f}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) d\tau, \\ \ominus(-1)(\tilde{y}(t) \ominus \tilde{y}(0)) = \frac{1-\beta}{B(\beta)} \odot \tilde{g}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) \oplus \frac{\beta}{B(\beta)\Gamma(\beta)} \odot \int_0^t (t-\tau)^{\beta-1} \odot \tilde{g}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) d\tau, \\ \ominus(-1)(\tilde{z}(t) \ominus \tilde{z}(0)) = \frac{1-\delta}{B(\delta)} \odot \tilde{h}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) \oplus \frac{\delta}{B(\delta)\Gamma(\delta)} \odot \int_0^t (t-\tau)^{\delta-1} \odot \tilde{h}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) d\tau \end{cases} \tag{3.13}$$

Therefore, the solution of system (3.2) can be achieved similar to the previous case by solving the following system:

$$\begin{cases} \tilde{x}(t_n) \ominus \tilde{x}(0) = \frac{\alpha h^{\alpha}}{B(\alpha)} \odot \Big(\varepsilon_n \odot \tilde{f}(0, \tilde{x}(0), \tilde{y}(0), \tilde{z}(0)\Big) \oplus \sum_{i=1}^{n} \mu_{n-i} \odot \tilde{f}(t_i, \tilde{x}(t_i), \tilde{y}(t_i), \tilde{z}(t_i)), \\ \ominus(-1)(\tilde{y}(t_n) \ominus \tilde{y}(0)) = \frac{\beta h^{\beta}}{B(\beta)} \odot (\varepsilon'_n \odot \tilde{g}(0, \tilde{x}(0), \tilde{y}(0), \tilde{z}(0)) \oplus \sum_{i=1}^{n} \nu_{n-i} \odot \tilde{g}(t_i, \tilde{x}(t_i), \tilde{y}(t_i), \tilde{z}(t_i)), \\ \ominus(-1)(\tilde{z}(t_n) \ominus \tilde{z}(0)) = \frac{\delta h^{\delta}}{B(\delta)} \odot \Big(\varepsilon''_n \odot \tilde{h}(0, \tilde{x}(0), \tilde{y}(0), \tilde{z}(0)\Big) \oplus \sum_{i=1}^{n} \gamma_{n-i} \odot \tilde{h}(t_i, \tilde{x}(t_i), \tilde{y}(t_i), \tilde{z}(t_i)), \end{cases} \tag{3.14}$$

in which coefficients are obtained from (3.8).

- **case 5:** $*_1 = ii, *_2 = i, *_3 = i$.
  In this case, substituting (2.6) and (2.7) in (3.3) results in:

$$\begin{cases} \ominus(-1)(\tilde{x}(t) \ominus \tilde{x}(0)) = \frac{1-\alpha}{B(\alpha)} \odot \tilde{f}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) \oplus \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \odot \int_0^t (t-\tau)^{\alpha-1} \odot \tilde{f}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) d\tau, \\ \tilde{y}(t) \ominus \tilde{y}(0) = \frac{1-\beta}{B(\beta)} \odot \tilde{g}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) \oplus \frac{\beta}{B(\beta)\Gamma(\beta)} \odot \int_0^t (t-\tau)^{\beta-1} \odot \tilde{g}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) d\tau, \\ \tilde{z}(t) \ominus \tilde{z}(0) = \frac{1-\delta}{B(\delta)} \odot \tilde{h}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) \oplus \frac{\delta}{B(\delta)\Gamma(\delta)} \odot \int_0^t (t-\tau)^{\delta-1} \odot \tilde{h}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) d\tau. \end{cases} \tag{3.15}$$

Therefore, the solution of system (3.2) can be achieved similar to the previous case by solving the following system:

$$\begin{cases} \ominus(-1)(\tilde{x}(t_n) \ominus \tilde{x}(0)) = \frac{\alpha h^\alpha}{B(\alpha)} \odot \Big(\varepsilon_n \odot \tilde{f}(0, \tilde{x}(0), \tilde{y}(0), \tilde{z}(0)\Big) \oplus \sum_{i=1}^{n} \mu_{n-i} \odot \tilde{f}(t_i, \tilde{x}(t_i), \tilde{y}(t_i), \tilde{z}(t_i)), \\ \tilde{y}(t_n) \ominus \tilde{y}(0) = \frac{\beta h^\beta}{B(\beta)} \odot (\varepsilon'_n \odot \tilde{g}(0, \tilde{x}(0), \tilde{y}(0), \tilde{z}(0)) \oplus \sum_{i=1}^{n} \nu_{n-i} \odot \tilde{g}(t_i, \tilde{x}(t_i), \tilde{y}(t_i), \tilde{z}(t_i)), \\ \tilde{z}(t_n) \ominus \tilde{z}(0) = \frac{\delta h^\delta}{B(\delta)} \odot \Big(\varepsilon''_n \odot \tilde{h}(0, \tilde{x}(0), \tilde{y}(0), \tilde{z}(0)\Big) \oplus \sum_{i=1}^{n} \gamma_{n-i} \odot \tilde{h}(t_i, \tilde{x}(t_i), \tilde{y}(t_i), \tilde{z}(t_i)), \end{cases} \tag{3.16}$$

in which coefficients are obtained from (3.8).

- **case 6:** $*_1 = ii, *_2 = i, *_3 = ii$.
  In this case, substitution of (2.6) and (2.7) in (3.3) yields:

$$\begin{cases} \ominus(-1)(\tilde{x}(t) \ominus \tilde{x}(0)) = \frac{1-\alpha}{B(\alpha)} \odot \tilde{f}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) \oplus \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \odot \int_0^t (t-\tau)^{\alpha-1} \odot \tilde{f}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) d\tau, \\ \tilde{y}(t) \ominus \tilde{y}(0) = \frac{1-\beta}{B(\beta)} \odot \tilde{g}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) \oplus \frac{\beta}{B(\beta)\Gamma(\beta)} \odot \int_0^t (t-\tau)^{\beta-1} \odot \tilde{g}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) d\tau, \\ \ominus(-1)(\tilde{z}(t) \ominus \tilde{z}(0)) = \frac{1-\delta}{B(\delta)} \odot \tilde{h}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) \oplus \frac{\delta}{B(\delta)\Gamma(\delta)} \odot \int_0^t (t-\tau)^{\delta-1} \odot \tilde{h}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) d\tau. \end{cases} \tag{3.17}$$

Therefore, the solution of system (3.2) can be achieved similar to the previous case by solving the following system:

$$\begin{cases} \ominus(-1)(\tilde{x}(t_n) \ominus \tilde{x}(0)) = \frac{\alpha h^\alpha}{B(\alpha)} \odot \Big(\varepsilon_n \odot \tilde{f}(0, \tilde{x}(0), \tilde{y}(0), \tilde{z}(0)\Big) \oplus \sum_{i=1}^{n} \mu_{n-i} \odot \tilde{f}(t_i, \tilde{x}(t_i), \tilde{y}(t_i), \tilde{z}(t_i)), \\ \tilde{y}(t_n) \ominus \tilde{y}(0) = \frac{\beta h^\beta}{B(\beta)} \odot (\varepsilon'_n \odot \tilde{g}(0, \tilde{x}(0), \tilde{y}(0), \tilde{z}(0)) \oplus \sum_{i=1}^{n} \nu_{n-i} \odot \tilde{g}(t_i, \tilde{x}(t_i), \tilde{y}(t_i), \tilde{z}(t_i)), \\ \ominus(-1)(\tilde{z}(t_n) \ominus \tilde{z}(0)) = \frac{\delta h^\delta}{B(\delta)} \odot \Big(\varepsilon''_n \odot \tilde{h}(0, \tilde{x}(0), \tilde{y}(0), \tilde{z}(0)\Big) \oplus \sum_{i=1}^{n} \gamma_{n-i} \odot \tilde{h}(t_i, \tilde{x}(t_i), \tilde{y}(t_i), \tilde{z}(t_i)), \end{cases} \tag{3.18}$$

in which coefficients are obtained from (3.8).

- **case 7:** $*_1 = ii, *_2 = ii, *_3 = i$.
  In this case, substituting (2.6) and (2.7) in (3.3) results in:

$$\begin{cases} \ominus(-1)(\tilde{x}(t) \ominus \tilde{x}(0)) = \frac{1-\alpha}{B(\alpha)} \odot \tilde{f}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) \oplus \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \odot \int_0^t (t-\tau)^{\alpha-1} \odot \tilde{f}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) d\tau, \\ \ominus(-1)(\tilde{y}(t) \ominus \tilde{y}(0)) = \frac{1-\beta}{B(\beta)} \odot \tilde{g}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) \oplus \frac{\beta}{B(\beta)\Gamma(\beta)} \odot \int_0^t (t-\tau)^{\beta-1} \odot \tilde{g}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) d\tau, \\ \tilde{z}(t) \ominus \tilde{z}(0) = \frac{1-\delta}{B(\delta)} \odot \tilde{h}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) \oplus \frac{\delta}{B(\delta)\Gamma(\delta)} \odot \int_0^t (t-\tau)^{\delta-1} \odot \tilde{h}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) d\tau. \end{cases} \tag{3.19}$$

Therefore, the solution of system (3.2) can be achieved similar to the previous case by solving the following system:

$$\begin{cases} \ominus(-1)(\tilde{x}(t_n) \ominus \tilde{x}(0)) = \frac{\alpha h^\alpha}{B(\alpha)} \odot \Big(\varepsilon_n \odot \tilde{f}(0, \tilde{x}(0), \tilde{y}(0), \tilde{z}(0)\Big) \oplus \sum_{i=1}^{n} \mu_{n-i} \odot \tilde{f}(t_i, \tilde{x}(t_i), \tilde{y}(t_i), \tilde{z}(t_i)), \\ \ominus(-1)(\tilde{y}(t_n) \ominus \tilde{y}(0)) = \frac{\beta h^\beta}{B(\beta)} \odot (\varepsilon'_n \odot \tilde{g}(0, \tilde{x}(0), \tilde{y}(0), \tilde{z}(0)) \oplus \sum_{i=1}^{n} \nu_{n-i} \odot \tilde{g}(t_i, \tilde{x}(t_i), \tilde{y}(t_i), \tilde{z}(t_i)), \\ \tilde{z}(t_n) \ominus \tilde{z}(0) = \frac{\delta h^\delta}{B(\delta)} \odot \Big(\varepsilon''_n \odot \tilde{h}(0, \tilde{x}(0), \tilde{y}(0), \tilde{z}(0)\Big) \oplus \sum_{i=1}^{n} \gamma_{n-i} \odot \tilde{h}(t_i, \tilde{x}(t_i), \tilde{y}(t_i), \tilde{z}(t_i)), \end{cases} \tag{3.20}$$

in which coefficients are obtained from (3.8).

- **case 8:** $*_1 = ii, *_2 = ii, *_3 = ii$.
  In this case, substituting (2.6) and (2.7) in (3.3) leads to:

$$\begin{cases} \ominus(-1)(\tilde{x}(t) \ominus \tilde{x}(0)) = \frac{1-\alpha}{B(\alpha)} \odot \tilde{f}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) \oplus \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \odot \int_0^t (t-\tau)^{\alpha-1} \odot \tilde{f}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) d\tau, \\ \ominus(-1)(\tilde{y}(t) \ominus \tilde{y}(0)) = \frac{1-\beta}{B(\beta)} \odot \tilde{g}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) \oplus \frac{\beta}{B(\beta)\Gamma(\beta)} \odot \int_0^t (t-\tau)^{\beta-1} \odot \tilde{g}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) d\tau, \\ \ominus(-1)(\tilde{z}(t) \ominus \tilde{z}(0)) = \frac{1-\delta}{B(\delta)} \odot \tilde{h}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) \oplus \frac{\delta}{B(\delta)\Gamma(\delta)} \odot \int_0^t (t-\tau)^{\delta-1} \odot \tilde{h}(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) d\tau. \end{cases} \tag{3.21}$$

Therefore, the solution of system (3.2) can be achieved similar to the previous case by solving the following system:

$$\begin{cases} \ominus(-1)(\tilde{x}(t_n)\ominus\tilde{x}(0)) = \frac{\alpha h^\alpha}{B(\alpha)}\odot\Big(\varepsilon_n\odot\tilde{f}(0,\tilde{x}(0),\tilde{y}(0),\tilde{z}(0)\Big)\oplus\sum_{i=1}^n\mu_{n-i}\odot\tilde{f}(t_i,\tilde{x}(t_i),\tilde{y}(t_i),\tilde{z}(t_i)),\\ \ominus(-1)(\tilde{y}(t_n)\ominus\tilde{y}(0)) = \frac{\beta h^\beta}{B(\beta)}\odot(\varepsilon'_n\odot\tilde{g}(0,\tilde{x}(0),\tilde{y}(0),\tilde{z}(0))\oplus\sum_{i=1}^n\nu_{n-i}\odot\tilde{g}(t_i,\tilde{x}(t_i),\tilde{y}(t_i),\tilde{z}(t_i)),\\ \ominus(-1)(\tilde{z}(t_n)\ominus\tilde{z}(0)) = \frac{\delta h^\delta}{B(\delta)}\odot\Big(\varepsilon''_n\odot\tilde{h}(0,\tilde{x}(0),\tilde{y}(0),\tilde{z}(0)\Big)\oplus\sum_{i=1}^n\gamma_{n-i}\odot\tilde{h}(t_i,\tilde{x}(t_i),\tilde{y}(t_i),\tilde{z}(t_i)), \end{cases} \tag{3.22}$$

in which coefficients are obtained from (3.8). In solving the fuzzy mathematical model (3.1), if the types of difference and derivative are not determined, all cases must be taken into consideration, and at the end, choosing the unique solution is up to the decision-maker.

Theorem 3.2. *Assume $\tilde{f}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))$, $\tilde{g}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))$, and $\tilde{h}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))$ are continuous functions on $[0,T)$. Then a necessary condition for the existence of a solution to the problem (3.2) is that $\tilde{f}(0,\tilde{x}(0),\tilde{y}(0),\tilde{z}(0)) = \tilde{g}(0,\tilde{x}(0),\tilde{y}(0),\tilde{z}(0)) = \tilde{h}(0,\tilde{x}(0),\tilde{y}(0),\tilde{z}(0)) = 0$.*

Proof. Since $\tilde{f}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))$, $\tilde{g}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))$, and $\tilde{h}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))$ are continuous, the following hold:

$$\lim_{t\to0^+}\int_0^t {}_{*_1-gh}\tilde{x}'(\tau)E_\alpha(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha)\ d\tau = 0,$$

$$\lim_{t\to0^+}\int_0^t {}_{*_2-gh}\tilde{y}'(\tau)E_\beta(-\frac{\beta}{1-\beta}(t-\tau)^\beta)\ d\tau = 0,$$

$$\lim_{t\to0^+}\int_0^t {}_{*_3-gh}\tilde{z}'(\tau)E_\delta(-\frac{\delta}{1-\delta}(t-\tau)^\delta)\ d\tau = 0.$$

Therefore,

$${}_0^{ABC}D_t^{*_1,\alpha}\tilde{x}(0^+) = {}_0^{ABC}D_t^{*_2,\beta}\tilde{y}(0^+) = {}_0^{ABC}D_t^{*_3,\delta}\tilde{z}(0^+) = 0 \ *_1,*_2,*3\in\{i,ii\}.$$

Thus, substituting the above equality in (3.2) yields
$\tilde{f}(0,\tilde{x}(0),\tilde{y}(0),\tilde{z}(0)) = \tilde{g}(0,\tilde{x}(0),\tilde{y}(0),\tilde{z}(0)) = \tilde{h}(0,\tilde{x}(0),\tilde{y}(0),\tilde{z}(0))$, which completes the proof. ☐

Uniqueness. Let

$$\tilde{Y}(t) = \begin{bmatrix}\tilde{x}(t)\\ \tilde{y}(t)\\ \tilde{z}(t)\end{bmatrix},\ \tilde{Y}(0) = \begin{bmatrix}\tilde{x}(0)\\ \tilde{y}(0)\\ \tilde{z}(0)\end{bmatrix},\ \tilde{Y}_0 = \begin{bmatrix}\tilde{x}_0\\ \tilde{y}_0\\ \tilde{z}_0\end{bmatrix},$$

$$F(t,\tilde{Y}(t)) = \begin{bmatrix}f(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))\\ g(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))\\ h(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))\end{bmatrix},$$

$${}_0^{ABC}D_t^{*,\sigma}F(t,\tilde{Y}(t)) = \begin{bmatrix}{}_0^{ABC}D_t^{*_1,\alpha}\tilde{f}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))\\ {}_0^{ABC}D_t^{*_2,\beta}\tilde{g}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))\\ {}_0^{ABC}D_t^{*_3,\delta}\tilde{h}(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))\end{bmatrix},$$

$$I = [0,T].$$

Then the system (3.2) can be written as:

$${}_0^{ABC}D_t^{*,\sigma}\tilde{F}(t,\tilde{Y}(t)) = \tilde{F}(t,\tilde{Y}(t)),\ t\in I,\ \tilde{Y}(0) = \tilde{Y}_0, \tag{3.23}$$

where the problem (3.23) has at least one solution. Consequently, the considered model (3.2) has at least one solution.

Proof. Refer to [2]. ☐

Definition 3.3. [10] It is said that a point $t_0 \in (a,b)$ is a switching point for the differentiability of $\tilde{f}$ if gH-differentiability changes from type (i) to type (ii) or from type (ii) to type (i).

Theorem 3.4. *If $\tilde{x}(0) = 0$ (or $\tilde{y}(0) = 0$, or $\tilde{z}(0) = 0$), where then $\tilde{x}$ (or $\tilde{y}$, or $\tilde{z}$) are not ii-differentiable.*

Proof. The proof by contradiction method is used. Therefore, it is assumed that $\tilde{x}$ is ii-differentiable. Hence, the following cases can occur:

- If the length of the support increases by increasing t, $\tilde{x}$ is i-differentiable.
- If the length of the support increases at first and then decreases by increasing t, $\tilde{x}(t)$ has a switch point.
- Assume $\tilde{x} = [\underline{x}(t,r), \overline{x}(t,r)]$. If $\underline{x}(t,r) = \overline{x}(t,r)$ holds for each r and increasing t, then $\tilde{x}$ is a real function.

In all three cases above, the contradiction proposition is false. Therefore, the claim is true. □

Remark 3.5. In this paper, a method is proposed to solve the system of fractional differential integral equations in the form of (3.2). The proposed method can also be applied to the general form of the system of fractional differential integral equations as follows:

$$\begin{cases} {}_0^{ABC}D_t^{*_1,\alpha_1}\tilde{u}_1(t) = f_{11}(t)\tilde{u}_1(t) + f_{12}(t)\tilde{u}_2(t) + \cdots + f_{1n}(t)\tilde{u}_n(t), \\ {}_0^{ABC}D_t^{*_2,\alpha_2}\tilde{u}_2(t) = f_{21}(t)\tilde{u}_1(t) + f_{22}(t)\tilde{u}_2(t) + \cdots + f_{2n}(t)\tilde{u}_n(t), \\ \vdots \\ {}_0^{ABC}D_t^{*_n,\alpha_n}\tilde{u}_n(t) = f_{n1}(t)\tilde{u}_1(t) + f_{n2}(t)\tilde{u}_2(t) + \cdots + f_{nn}(t)\tilde{u}_n(t), \end{cases}$$

where $*_1, *_2, \ldots, *_n \in \{i, ii\}$, $0 < \alpha_1, \alpha_2, \ldots, \alpha_n < 1$, $0 < t < T < \infty$, $T \in R, \tilde{u}_1(t), \tilde{u}_2(t), \ldots, \tilde{u}_n(t) \in C^F(I) \cap L^F(I), I = [0,T] \subseteq R$ and $f_{kj}(t)$ are real-valued functions for $1 \leq k, j \leq n$. The value of the parameters in the mathematical model of the transmission of COVID-19 is uncertain and varies over intervals, such as the transmission rate from $\tilde{I}(t)$ to $\tilde{S}(t)$ or from $\tilde{W}(t)$ to $\tilde{S}(t)$. Hence, a fuzzy mathematical model is considered for COVID-19, as the parameters in a fuzzy model are fuzzy numbers. This model can effectively reveal the ambiguity in real-world problems. Consequently, in the following, the goal is to provide a fuzzy mathematical model for the transmission of COVID-19 as an application of a system of fuzzy ABC fractional differential equations in the following form:

$$\begin{cases} {}_0^{ABC}D_t^{*_1,\alpha_1}\tilde{S}(t) = \tilde{\Lambda} \ominus_{gh} \tilde{m} \odot \tilde{S}(t) \ominus_{gh} \tilde{\beta}_p \odot \tilde{S}(t) \odot (\tilde{I}(t) \oplus \tilde{k} \odot \tilde{A}(t)) \ominus_{gh} \tilde{\beta}_w \odot \tilde{S}(t) \odot \tilde{W}(t), \\ {}_0^{ABC}D_t^{*_2,\alpha_2}\tilde{E}(t) = \tilde{\beta}_p \odot \tilde{S}(t) \odot (\tilde{I}(t) \oplus \tilde{k} \odot \tilde{A}(t)) \oplus \tilde{\beta}_w \odot \tilde{S}(t) \odot \tilde{W}(t) \\ \ominus_{gh}([1,1] \ominus_{gh} \tilde{\delta}) \odot \tilde{\omega} \odot \tilde{E}(t) \ominus_{gh} \tilde{\delta} \odot \tilde{\omega} \odot \tilde{E}(t) \ominus_{gh} \tilde{m} \odot \tilde{E}(t), \\ {}_0^{ABC}D_t^{*_3,\alpha_3}\tilde{I}(t) = ([1,1] \ominus_{gh} \tilde{\delta}) \odot \tilde{\omega} \odot \tilde{E}(t) \ominus_{gh} (\tilde{\gamma} \oplus \tilde{m}) \odot \tilde{I}(t), \\ {}_0^{ABC}D_t^{*_4,\alpha_4}\tilde{A}(t) = \tilde{\delta} \odot \tilde{\omega}_p' \odot \tilde{E}(t) \ominus_{gh} (\tilde{\gamma}' \oplus \tilde{m}) \odot \tilde{A}(t), \\ {}_0^{ABC}D_t^{*_5,\alpha_5}\tilde{R}(t) = \tilde{\gamma} \odot \tilde{I}(t) \oplus \tilde{\gamma}' \odot \tilde{A}(t) \ominus_{gh} \tilde{m} \odot \tilde{R}(t), \\ {}_0^{ABC}D_t^{*_6,\alpha_6}\tilde{W}(t) = \tilde{\mu} \odot \tilde{I}(t) \oplus \tilde{\mu}' \odot \tilde{A}(t) \ominus_{gh} \tilde{\varepsilon} \odot \tilde{W}(t), \\ \tilde{S}(0) = \tilde{S}_0, \quad \tilde{E}(0) = \tilde{E}_0, \quad \tilde{I}(0) = \tilde{I}_0, \\ \tilde{A}(0) = \tilde{A}_0, \quad \tilde{R}(0) = \tilde{R}_0, \quad \tilde{W}(0) = \tilde{W}_0, \end{cases} \tag{3.24}$$

where $*_1, *_2, *_3, *_4, *_5, *_6 \in \{i, ii\}$, $0 < \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 < 1$, $0 < t < T < \infty$, $T \in R$,

$\tilde{S}(t)$	Susceptible people
$\tilde{E}(t)$	Exposed people
$\tilde{I}(t)$	Symptomatic infected people
$\tilde{A}(t)$	Asymptomatic infected people
$\tilde{R}(t)$	Removed people, including recovered and dead people
$\tilde{W}(t)$	COVID-19 in the reservoir
$\tilde{\Lambda}$	$\tilde{\Lambda} = \tilde{n} \odot N$ N refers to the total number of people, and $\tilde{n}$ is the birth rate.
$\tilde{m}$	Death rate of people
$\tilde{\beta}_p$	Transmission rate from $\tilde{I}(t)$ to $\tilde{S}(t)$
$\tilde{k}$	Multiple of the transmissible of $\tilde{A}(t)$ to that of $\tilde{I}(t)$
$\tilde{\beta}_w$	Transmission rate from $\tilde{W}(t)$ to $\tilde{S}(t)$
$\tilde{\delta}$	Proportion of asymptomatic infection rates among people
$\frac{1}{\tilde{\omega}}$	Incubation period of people
$\frac{1}{\tilde{\omega}'}$	Latent period of people
$\frac{1}{\tilde{\gamma}}$	Infectious period of symptomatic infection in people
$\frac{1}{\tilde{\gamma}'}$	Infectious period of asymptomatic infection in people
$\tilde{\mu}$	Shedding coefficients from $\tilde{I}(t)$ to $\tilde{W}(t)$
$\tilde{\mu}'$	Shedding coefficients from $\tilde{A}(t)$ to $\tilde{W}(t)$
$\frac{1}{\tilde{\varepsilon}}$	Lifetime of the virus in $\tilde{W}(t)$

By considering:

$$\begin{aligned}
&\tilde{f}_1(\tilde{S}(t),\tilde{E}(t),\tilde{I}(t),\tilde{A}(t),\tilde{R}(t),\tilde{W}(t)) = \tilde{\Lambda} \ominus_{gh} \tilde{m} \odot \tilde{S}(t) \ominus_{gh} \tilde{\beta}_p \odot \tilde{S}(t) \odot (\tilde{I}(t) \oplus \tilde{k} \odot \tilde{A}(t)) \\
&\qquad\qquad \ominus_{gh} \tilde{\beta}_w \odot \tilde{S}(t) \odot \tilde{W}(t), \\
&\tilde{f}_2(\tilde{S}(t),\tilde{E}(t),\tilde{I}(t),\tilde{A}(t),\tilde{R}(t),\tilde{W}(t)) = \tilde{\beta}_p \odot \tilde{S}(t) \odot (\tilde{I}(t) \oplus \tilde{k} \odot \tilde{A}(t)) \oplus \tilde{\beta}_w \odot \tilde{S}(t) \odot \tilde{W}(t) \\
&\qquad\qquad \ominus_{gh} ([1,1] \ominus_{gh} \tilde{\delta}) \odot \tilde{\omega} \odot \tilde{E}(t) \ominus_{gh} \tilde{\delta} \odot \tilde{\omega} \odot \tilde{E}(t) \ominus_{gh} \tilde{m} \odot \tilde{E}(t), \\
&\tilde{f}_3(\tilde{S}(t),\tilde{E}(t),\tilde{I}(t),\tilde{A}(t),\tilde{R}(t),\tilde{W}(t)) = ([1,1] \ominus_{gh} \tilde{\delta}) \odot \tilde{\omega} \odot \tilde{E}(t) \ominus_{gh} (\tilde{\gamma} \oplus \tilde{m}) \odot \tilde{I}(t), \\
&\tilde{f}_4(\tilde{S}(t),\tilde{E}(t),\tilde{I}(t),\tilde{A}(t),\tilde{R}(t),\tilde{W}(t)) = \tilde{\delta} \odot \tilde{\omega}'_p \odot \tilde{E}(t) \ominus_{gh} (\tilde{\gamma}' \oplus \tilde{m}) \odot \tilde{A}(t), \\
&\tilde{f}_5(\tilde{S}(t),\tilde{E}(t),\tilde{I}(t),\tilde{A}(t),\tilde{R}(t),\tilde{W}(t)) = \tilde{\gamma} \odot \tilde{I}(t) \oplus \tilde{\gamma}' \odot \tilde{A}(t) \ominus_{gh} \tilde{m} \odot \tilde{R}(t), \\
&\tilde{f}_6(\tilde{S}(t),\tilde{E}(t),\tilde{I}(t),\tilde{A}(t),\tilde{R}(t),\tilde{W}(t)) = \tilde{\mu} \odot \tilde{I}(t) \oplus \tilde{\mu}' \odot \tilde{A}(t) \ominus_{gh} \tilde{\varepsilon} \odot \tilde{W}(t),
\end{aligned}$$

The model in (3.24) is transformed to a system of fuzzy ABC fractional differential equations as follows:

$$\begin{cases}
{}_0^{ABC}D_t^{*_1,\alpha_1}\tilde{S}(t) = \tilde{f}_1(\tilde{S}(t),\tilde{E}(t),\tilde{I}(t),\tilde{A}(t),\tilde{R}(t),\tilde{W}(t)), \\
{}_0^{ABC}D_t^{*_2,\alpha_2}\tilde{E}(t) = \tilde{f}_2(\tilde{S}(t),\tilde{E}(t),\tilde{I}(t),\tilde{A}(t),\tilde{R}(t),\tilde{W}(t)), \\
{}_0^{ABC}D_t^{*_3,\alpha_3}\tilde{I}(t) = \tilde{f}_3(\tilde{S}(t),\tilde{E}(t),\tilde{I}(t),\tilde{A}(t),\tilde{R}(t),\tilde{W}(t)), \\
{}_0^{ABC}D_t^{*_4,\alpha_4}\tilde{A}(t) = \tilde{f}_4(\tilde{S}(t),\tilde{E}(t),\tilde{I}(t),\tilde{A}(t),\tilde{R}(t),\tilde{W}(t)), \\
{}_0^{ABC}D_t^{*_5,\alpha_5}\tilde{R}(t) = \tilde{f}_5(\tilde{S}(t),\tilde{E}(t),\tilde{I}(t),\tilde{A}(t),\tilde{R}(t),\tilde{W}(t)), \\
{}_0^{ABC}D_t^{*_6,\alpha_6}\tilde{W}(t) = \tilde{f}_6(\tilde{S}(t),\tilde{E}(t),\tilde{I}(t),\tilde{A}(t),\tilde{R}(t),\tilde{W}(t)), \\
\tilde{S}(0) = \tilde{S}_0, \quad \tilde{E}(0) = \tilde{E}_0, \quad \tilde{I}(0) = \tilde{I}_0, \\
\tilde{A}(0) = \tilde{A}_0, \quad \tilde{R}(0) = \tilde{R}_0, \quad \tilde{W}(0) = \tilde{W}_0,
\end{cases}$$

that can be solved using the proposed method. The following example demonstrates the efficiency of the proposed method.

Review of [2]

1. In this paper, the solution of the following fuzzy differential equation is investigated:

$$\left[{}_0^{ABC}D_t^{*,\alpha}y(t)\right]_r = [f(t,y(t))]_r \; 0 < t < T < \infty, \; T \in R, \; * \in \{i, ii\}.$$

Substituting $t = 0$ in both sides of the above equation, (3.23) leads to:

$$\left[{}_0^{ABC}D_t^{*,\alpha}y(0)\right]_r = [f(0,y(0))]_r.$$

As presented in Theorem 3.2, $\left[{}_0^{ABC}D_t^{*,\alpha}y(0)\right]_r = 0$ always holds. Therefore, by substituting it in the above equation, the following result can be achieved:

$$[f(0,y(0))]_r = 0.$$

This main condition for the existence of the solution was not indicated in [2]. Moreover, in Example 1 [2], the sign of λ was not mentioned.

2. The general form of the nonlinear fractional differential equation was considered as [2]:

$${}_0^{ABC}D_t^{\alpha}y(t) = f(t,y(t)), \; 0 < t < T < \infty, \; T \in R, \; y(0) = y_0.$$

It means that the initial value was assumed to be zero and t was assumed to be positive, while in Examples 2 and 3 [2], the initial value was -1, and $t < 0$.

3. In Theorem 1 [2], the condition of non-negativity of E_t was considered for t and α, while it was neglected in the solving and proving procedures.
4. The type of derivative must be determined according to the problem, and if it is not determined, all cases must be considered, and the final solution is the one that is closer to the opinion of the decision-maker. For instance, in Example 1 [2], it is said that for $\lambda < 0$, the solution is of the second type derivative, but the figures indicate an incremental support.

In the present paper, all the aforementioned cases have been modified.

4. Numerical Example

In this section, a numerical example is utilized to investigate the validity and effectiveness of the proposed method. Since the data is not constant in practice and varies in intervals and is also used to determine output based on reality for future works, the data proposed in [9] is considered to be uncertain. As a result, the following data is selected for the model of the transmission of COVID-19:

$\tilde{\Lambda}$	$1.\tilde{1}46 = [1.1 + 0.046r, 1.2 - 0.054r]$
$\tilde{m}$	0.034
$\tilde{\beta}_p$	0.0025
$\tilde{k}$	0.05
$\tilde{\beta}_w$	0.001
$\tilde{\delta}$	$0.\tilde{2}5 = [0.25r, 0.4 - 0.15r$
$\tilde{\omega}$	0.071
$\tilde{\omega}'$	0.1
$\tilde{\gamma}$	0.047
$\tilde{\gamma}'$	0.1
$\tilde{\mu}$	0.002
$\tilde{\mu}'$	0.001
$\tilde{\varepsilon}$	0.033
$\tilde{S}(0)$	$\tilde{35} = [34.7 + 0.3r, 35.2 - 0.2r]$
$\tilde{E}(0)$	$\tilde{25} = [24.5 + 0.5r, 25.1 - 0.1r]$
$\tilde{I}(0)$	$\tilde{25} = [24.8 + 0.2r, 25.3 - 0.3r]$
$\tilde{A}(0)$	$\tilde{10} = [9.9 + 0.1r, 10.2 - 0.2r]$
$\tilde{R}(0)$	$\tilde{0} = [0.0]$
$\tilde{W}(0)$	$\tilde{5} = [4.75 + 0.25r, 5.15 - 0.15r]$

Therefore, model (3.24) is transformed to:

$$\left\{\begin{array}{l}
{}_{0}^{ABC}D_t^{i,\alpha_1}\tilde{S}(t) = [1.1 + 0.046r, 1.2 - 0.054r] \ominus 0.034\tilde{S}(t) \ominus 0.0025\tilde{S}(t) \odot (\tilde{I}(t) \oplus 0.05\tilde{A}(t)) \\
\ominus 0.001\tilde{S}(t) \odot \tilde{W}(t), \\
{}_{0}^{ABC}D_t^{i,\alpha_2}\tilde{E}(t) = 0.0025\tilde{S}(t) \odot (\tilde{I}(t) \oplus 0.05\tilde{A}(t)) \oplus 0.001\tilde{S}(t) \odot W \ominus ([1,1] \\
\ominus [0.25r, 0.4 - 0.15r]) \odot 0.071\tilde{E}(t) \\
\ominus [0.25r, 0.4 - 0.15r] \odot 0.1\tilde{E}(t) \ominus 0.034\tilde{E}(t), \\
{}_{0}^{ABC}D_t^{i,\alpha_3}\tilde{I}(t) = ([1,1] \ominus [0.25r, 0.4 - 0.15r]) \odot 0.071\tilde{E}(t) \ominus (0.047 + 0.034)\tilde{I}(t), \\
{}_{0}^{ABC}D_t^{i_4,\alpha_4}\tilde{A}(t) = [0.25r, 0.4 - 0.15r] \odot (0.1) \odot \tilde{E}(t) \ominus (0.1 + 0.034)\tilde{A}(t), \\
{}_{0}^{ABC}D_t^{i,\alpha_5}\tilde{R}(t) = 0.047\tilde{I}(t) \oplus (0.1) \odot \tilde{A}(t) \ominus 0.034\tilde{R}(t), \\
{}_{0}^{ABC}D_t^{i,\alpha_6}\tilde{W}(t) = 0.003\tilde{I}(t) \oplus 0.001\tilde{A}(t) \ominus 0.033\tilde{W}(t), \\
\tilde{S}(0) = [34.7_{0}.3r, 35.2 - 0.2r], \quad \tilde{E}(0) = [24.5 + 0.5r, 25.1 - 0.1r], \\
\tilde{I}(0) = [24.8 + 0.2r, 25.3 - 0.3r], \quad \tilde{A}(0) = [9.9 + 0.1r, 10.2 - 0.2r], \\
\tilde{R}(0) = [0, 0], \quad \tilde{W}(0) = [4.75 + 0.25r, 5.15 - 0.15r].
\end{array}\right. \tag{4.1}$$

Using the proposed method, the above system is solved. The results presented in Figure 1 are achieved for $\alpha = 0.96$. It can be seen that the achieved results have good accuracy, and the proposed method is more compatible with certain environments (for $r = 1$) and is valid for the fuzzy system of fractional ODEs. Furthermore, the dynamics of infected, susceptible, exposed, and asymptotically infected people can be investigated for different integer and fractional orders in a fuzzy environment.

5. Conclusion

Mathematics is a tool for modeling many real-world problems. Whenever a mathematical model is closer to a real system, it can provide a more accurate solution and statement of the problem. Since variables involved with HIV and Corona viruses are linguistic and ambiguous, in this paper, qualitative mathematical modeling is considered based on the fuzzy Atangana-Baleanu-Caputo (ABC) fractional differential equations system. A useful method is presented for the fuzzy Atangana-Baleanu-Caputo (ABC) fractional differential equations system using the ABC derivative, and it is

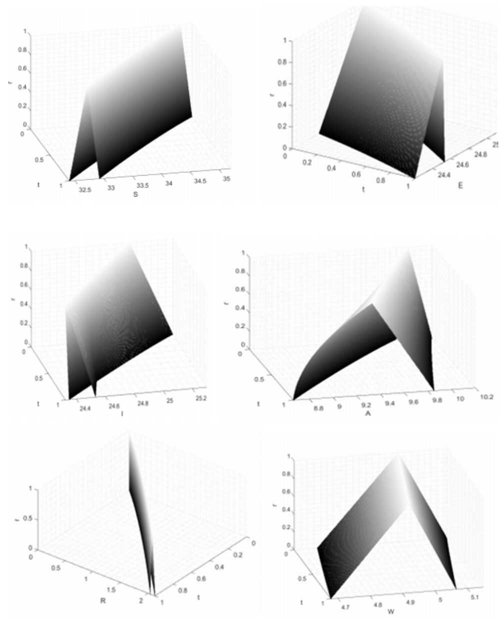

Figure 1. Simulation results corresponding to the numerical example for $\alpha = 0.96$.

shown that the achieved solution is a fuzzy number. Moreover, in a numerical example for a special case, a qualitative case study in China is considered. The results verify the performance of the proposed method. In the next study, various numerical methods with various fractional derivatives that are used to solve the fuzzy mathematical model of COVID-19 will be compared.

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