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An operational vector method based on Chebyshev wavelet and hybrid functions for Riccati differential equations: Application in nonlinear physics equations

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Abstract

In this paper, we introduce an operational vector approach that uses Chebyshev wavelets and hybrid functions to approximate the solution of the Riccati differential equation arising in nonlinear physics equations celebrated as cosmology problems. The scheme's main features include a simple structure based on certain matrices and vectors, low computational complexity, and high accuracy. The method is direct, which means that projection methods are not used throughout the approximation procedure in order to reduce computational cost. Error analyses are provided, and several numerical examples and comparisons confirm the proposed scheme's superiority.

Keywords. Riccati equation, Direct method, Chebyshev wavelets, Hybrid functions, Operational vectors.
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1. Introduction

Some physical problems, including certain types of Newton's laws of motion under the effect of a power law central potential function$V(r) = kr^c$ with zero total energy, which is celebrated as Barotropic Friedmann-Robertson-Lemaitre cosmology, can be expressed in the Riccati form [24].

Since the nonlinear Riccati equations are broadly studied, converting these physical equations to the well-known Riccati equation has many undeniable advantages. This equation is a well-known mathematical model utilized in various fields of applied science for example in [13, 28]. For instance, super-symmetric quantum mechanics [7], variational calculus [36], nonlinear physics [22], renormalization group equations for running coupling constants in quantum field theories [23], and mathematical finance [4] are few applications of Riccati equations. As another application, the generalized Gross-Pitaevskii equations for the BEC wave functions and cosmology problems can be reduced to a Riccati equation [6].

The main purpose of this paper is to introduce an efficient direct scheme for solving the following Riccati differential equation:

$$u'(t) = A(t) + B(t)u(t) + C(t)u^2(t), \quad 0 \leq t \leq 1, \tag{1.1}$$

with initial value $u(0) = \alpha$.
It is important to note that only certain special cases of these equations have been solved analytically [30], highlighting the significance of selecting the most suitable numerical schemes in terms of convergence speed and accuracy.

In recent years, researchers have focused on operational matrices and their applications in solving problems involving continuous operators such as integration, differentiation, and delay, among others. These matrices have been employed for ordinary, fractional, and stochastic problems, utilizing various basis functions [2, 14, 18, 20]. Numerical methods utilizing operational matrices are known for their ease of implementation and high accuracy, thanks to their sparse nature. Additionally, these matrices can serve as preconditioners for inverse problems.

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Recent researches have explored numerical solutions of the Riccati equation using operational matrices, employing techniques such as B-spline scaling and Chebyshev cardinal functions with the collocation method [15], Chebyshev polynomials with the Tau method [27], Legendre wavelets [11], Bernstein polynomials [35], and others [31, 33].

Orthogonal functions play an important role in approximation theory and numerical analysis [37], and one of the most significant polynomials is Chebyshev polynomials [1] which have unique properties.

In our recent works [9, 17], we present an operational vector for Chebyshev polynomials (CP), hybrid Chebyshev polynomials and block-pulse functions (HCP) to approximate nonlinear Volterra integral equations. Here, we construct the operational vectors for Chebyshev wavelets (CW). These functions satisfy multi-resolution analysis and furnish some appropriate features for them. A key characteristic of this method is its low computational cost in setting up the equations, as it does not require the use of any projection method. The following is an outline of how the paper is organized: Section 2 provides a brief interpretation of some basic concepts about Chebyshev polynomials and their variants.

As the main idea, an operational vector is constructed for Chebyshev wavelets. In section 3, we present an outline of our direct scheme for solving Riccati equations. Some theorems for the convergence analysis are presented in section 4. In section 5, numerical results are reported to verify the applicability of the method in comparison with some projection methods.

2. Preliminaries

In order to make the paper self-contained, we recall some basic concepts which useful is in sequel.

• Chebyshev polynomials of the first kind are defined by

$$T_n(x) = cos(n\theta),\ \theta = Arc\ cos(x),$$

The orthogonality condition for CP with a weight function $w(x) = \dfrac{1}{\sqrt{1-x^2}}$ is as follows

$$\langle T_i(x), T_j(x)\rangle_w = \int_{-1}^{1} w(x)T_i(x)T_j(x)dx = \delta_{ij} \begin{cases} \dfrac{\pi}{2},\ i \neq 0, \\ \\ \pi,\ i = 0. \end{cases}$$

Since the CP form a complete orthogonal set with respect to the weight function $\tilde{w}(t) := w(2t-1)$, then for any $f\ L^2_{\tilde{w}}([0,1])$,

$$f(t) = \sum_{r=0}^{\infty} c_r T_r(t). \tag{2.1}$$

• Hybrid Chebyshev polynomials and block pulse functions (HCP) $T_{im}(t)$ have two parameters where $i = 1, ..., N$ and $m = 0, 1, ..., M-1$ are the order of block-pulse functions and Chebyshev polynomials, respectively. On the interval $[0, 1]$, they are defined by

$$T_{im}(t) = \begin{cases} T_m(2Nt - 2i + 1), & t \in \left[\frac{i-1}{N}, \frac{i}{N}\right], \\ 0, & otherwise. \end{cases}$$

$T_m(t)$ is Chebyshev polynomial of order m.

For brevity, let $H_r(t) := \{T_{im}(t)\}_{i,m}, r = 1, ..., NM$. Since $\mathbf{H}(t) := \{H_r(t)\}_{r=1}^{NM}$ form a complete orthogonal set with respect to the weight function $\tilde{w}_n(x) = w(2Nx - 2n + 1), n = 1, ..., N$, then the representation of $f \in L^2_{\tilde{w}}[0, 1]$ can be as $f(t) = \sum\limits_{r=1}^{\infty} c_r H_r(t)$.

• Chebyshev wavelets (CW) $\Psi(t) = \psi_{n,m} = \psi(k, n, m, t)$ form a complete orthogonal set for $L^2_{\tilde{w}}[0, 1]$ with respect to the weight function $\tilde{w}_n(x) = w(2^k x - 2n + 1)$ and have four arguments, $n = 1, 2, ..., N$, $N := 2^{k-1}$, k is an arbitrary positive integer, m is the degree of CP and t denotes the time.

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}} T_m(2^k t - 2n + 1), & t \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right], \\ 0, & otherwise, \end{cases}$$

where the orthogonality condition with respect to $\tilde{w}$ for CW is as follows

$$\langle \psi_{n,m}(t), \psi_{n',m'}(t)\rangle_{\tilde{w}} = \delta_{nn'}\delta_{mm'} \begin{cases} \dfrac{\pi}{2}, & m \neq 0, \\ \pi, & m = 0. \end{cases}$$

2.1. **Operational matrix of integration and product.** Let $\phi = [\phi_0, \phi_1, ..., \phi_n]$ be a desire basis functions. The operational matrix of integration for them fulfills in the following relation

$$\int_{t_0}^{t} \phi(s)ds \simeq \mathbf{P}\ \phi(t), \tag{2.2}$$

and the product operational matrix $\overline{\mathbf{C}}^T$ satisfies in the following formula

$$\phi(t)\phi^T(t)\mathbf{C} = \overline{\mathbf{C}}^T \phi(t),$$

where the vector $\mathbf{C} := [c_0, c_1, ..., c_{M-1}]$ and the matrix $\overline{\mathbf{C}}$ is a square matrix of order N+1.
Opertational matrix of integration for CP can be found in [8] as

$$\mathbf{P} = \frac{1}{A}\begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{4} & 0 & \frac{1}{4} & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{3} & \frac{-1}{2} & 0 & \frac{1}{6} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{(-1)^{m-1}}{(m-1)(m-3)} & 0 & 0 & 0 & \cdots & \frac{-1}{2(m-3)} & 0 & \frac{1}{2(m-1)} \\ \frac{(-1)^m}{m(m-2)} & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2(m-2)} & 0 \end{bmatrix}, \tag{2.3}$$

where $A = \frac{2}{t_f - t_0}$.
Regarding (2.3), one can conclude

$$\int_{t_0}^{t_f} T_m(s)ds = \frac{1}{A}(\frac{(-1)^m}{(m-1)^2-1}T_0 - \frac{1}{2(m-2)}T_{m-1} + \frac{1}{2m}T_{m+1}),\ m \geq 3. \tag{2.4}$$

The Operational matrix of product $\overline{\mathbf{C}}^T$ is a square matrix of order N+1 as

$$\overline{\mathbf{C}} = \frac{1}{2}\begin{bmatrix} 2c_0 & c_1 & \cdots & c_i & \cdots & c_{M-2} & c_{M-1} \\ 2c_1 & 2c_0 + c_2 & \cdots & c_{i-1} + c_{i+1} & \cdots & c_{M-3} + c_{M-1} & c_{M-2} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\ 2c_i & c_{i-1} + c_{i+1} & \cdots & 2c_0 + c_{2i} & \cdots & c_{M-2-i} & c_{M-1-i} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2c_{M-2} & c_{M-3} + c_{M-1} & \cdots & c_{M-2-i} & \cdots & 2c_0 & c_1 \\ 2c_{M-1} & c_{M-2} & \cdots & c_{M-1-i} & \cdots & c_1 & 2c_0 \end{bmatrix}, \tag{2.5}$$

where $i = [\frac{M}{2}]$ and the vector $\mathbf{C} := [c_0, c_1, ..., c_{M-1}]$.
Similarly, the operational matrices of integration and product for HCP is defined [29].
The operational matrix of integration for CW was obtained in [2] as follows

$$\int_0^t \psi(t)dt \approx \mathbf{P}\psi(t),$$

where the vector Ψ and matrix $\mathcal{P}$ are obtained as follows

$$\Psi(t) = [\psi_{1,0}, \psi_{1,1}, \cdots, \psi_{1,M-1}, \cdots, \psi_{2^{k-1},0}(t), \cdots, \psi_{2^{k-1},M-1}(t)], \tag{2.6}$$

and

$$\mathbf{P} = \begin{bmatrix} L & F & F & \cdots & F \\ O & L & F & \cdots & F \\ \vdots & O & L & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & F \\ O & O & \cdots & O & L \end{bmatrix}. \tag{2.7}$$

O is $M \times M$ matrix. L and F are $M \times M$ matrices which are defined as follows

$$L = 2^{-k} \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \cdots & 0 \\ \frac{-\sqrt{2}}{4} & 0 & \frac{1}{4} & 0 & 0 & \cdots & 0 \\ \frac{-\sqrt{2}}{3} & \frac{-1}{2} & 0 & \frac{1}{6} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\sqrt{2}}{2}\left(\frac{(-1)^{r-1}}{r-1} - \frac{(-1)^{r+1}}{r+1}\right) & 0 & 0 & \cdots & \frac{-1}{2(r-1)} & 0 & \frac{1}{2(r-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\sqrt{2}}{2}\left(\frac{(-1)^{M-2}}{M-2} - \frac{(-1)^{M}}{M}\right) & 0 & 0 & 0 & \cdots & \frac{-1}{2(M-2)} & 0 \end{bmatrix}, \tag{2.8}$$

and

$$F = 2^{-k} \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \frac{-2\sqrt{2}}{3} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^{r+1}}{r+1} - \frac{1-(-1)^{r-1}}{r-1}\right) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^{M}}{M} - \frac{1-(-1)^{M-2}}{M-2}\right) & 0 & \cdots & 0 \end{bmatrix}. \tag{2.9}$$

The operational matrix of product for Chebyshev wavelets is introduced in [2] as follows

$$\Psi\Psi^T C \approx \tilde{C}\Psi,$$

Where $\tilde{C} = \mathrm{diag}\left(\tilde{c}_1, \tilde{c}_2, \cdots, \tilde{c}_{2^{k-1}}\right)$ is diagonal blocked matrix which $\tilde{c}_i$ is defined in [16]

$$\tilde{c}_i = \frac{2^{\frac{k}{2}}}{\sqrt{\pi}} \begin{bmatrix} c_{i0} & c_{i1} & c_{i2} & c_{i3} & \cdots & c_{i,M-2} & c_{i,M-1} \\ c_{i1} & c_{i0} + \frac{1}{\sqrt{2}}c_{i2} & \frac{1}{\sqrt{2}}\left(c_{i1} + c_{i3}\right) & \frac{1}{\sqrt{2}}\left(c_{i2} + c_{i4}\right) & \cdots & \frac{1}{\sqrt{2}}\left(c_{i,M-3} + c_{i,M-1}\right) & \frac{1}{\sqrt{2}}c_{i,M-2} \\ c_{i2} & \frac{1}{\sqrt{2}}\left(c_{i1} + c_{i3}\right) & \frac{1}{\sqrt{2}}\left(c_{i2} + c_{i4}\right) & \frac{1}{\sqrt{2}}\left(c_{i1} + c_{i5}\right) & \cdots & \frac{1}{\sqrt{2}}c_{i,M-4} & \frac{1}{\sqrt{2}}c_{i,M-3} \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & c_{i0} + \frac{1}{\sqrt{2}}c_{i\mu} & c_{i1} + \frac{1}{\sqrt{2}}c_{i,\mu+1} & \cdots & \frac{1}{\frac{2}{c}\, i,\nu} \\ \cdots & \cdots & \cdots & c_{i1} + \frac{1}{\sqrt{2}}c_{i,\mu+1} & c_{i0} & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & c_{i0} & \frac{1}{\sqrt{2}} \\ c_{i,M-1} & \frac{1}{\sqrt{2}}c_{i,M-2} & \cdots & \cdots & \cdots & \frac{1}{\sqrt{2}}c_{i1} & c_{i0} \end{bmatrix}, \tag{2.10}$$

with

$$\mu = \begin{cases} M-2 & M \quad even, \\ M-1 & M \quad odd, \end{cases}$$

and

$$\nu = \begin{cases} \frac{M}{2} & M \quad even, \\ \frac{M-1}{2} & M \quad odd, \end{cases}$$

2.2. **Operational vector of product.** The operational vector $\mathbf{f}\hat{B}$ for a vector of basis functions, $\phi = [\phi_0, \phi_1, ..., \phi_n]$ can be described as

$$\phi^T(t)\ \mathbf{B}\ \phi(t) = \hat{\mathbf{B}}\ \phi(t), \tag{2.11}$$

where the matrix $\mathbf{B}$ is a square matrix of order n+1.

Operational vectors play a critical role in simplifying the approximation process in various problems [3]. Several significant operational vectors have been introduced for block-pulse functions, Chebyshev polynomials, and hybrid Chebyshev and block-pulse functions [9, 12, 17]. In this work, our objective is to derive an explicit formula for the operational vector of product specifically for Chebyshev wavelets (CW).

2.2.1. *CW operational vector.* Suppose that $\alpha := \dfrac{2^{\frac{k}{2}}}{\sqrt{\pi}}$ and $\beta := \dfrac{2^{\frac{k-1}{2}}}{\sqrt{\pi}}$, the CW operational vector is constructed by the following property [2]:

$$\begin{cases} \psi_{i0}\psi_{il} = \alpha\psi_{i0}, & l = 0, ..., M-1, \\ \psi_{il}\psi_{ir} = \alpha\psi_{i0} + \beta\psi_{i,l+r}, & r = l \neq 0, \\ \psi_{il}\psi_{ir} = \beta(\psi_{i,|l-r|} + \psi_{i,l+r}), & r \neq l,\ r, l \neq 0, \end{cases}$$

The above expressions hold for $l + r < M, i = 1, ..., 2^{k-1}$. For $l + r \geq M$, the second terms in the above formulas have been deleted. Let us consider the matrix $\mathbf{B}$ in relation (2.11) as $\mathbf{B} = [B_{ij}]_{i,j=1}^{N}$, where each B_{ij} is an M-square matrix and $N = 2^{k-1}$. To obtain the mentioned operational vector, suppose that two conditions $|i - j| = k - (l-1)M$ and $i + j - 2 = k + (l-1)M$ as A and B, respectively. Due to the definition (2.11), the corresponding operational vector $\hat{\mathbf{B}}_l$ is constructed as follows:

$$\hat{\mathbf{B}}_l(k) = \sum_{i,j} c_{i,j}(B_{ll})_{i,j},\ \ i, j, k = (l-1)M + 1, ..., lM,\ \ l = 1, ..., N,$$

where

$$c_{i,j} = \begin{cases} \beta,\ (A \wedge (i \neq j)) \vee B, \\ \alpha,\ (A \wedge (i = j)) \vee ((i = 1) \vee (j = 1)), \\ 0,\ otherwise. \end{cases}$$

3. Description of the approximation method

In this section, we outline the implementation of the operational vectors discussed earlier for solving the main problem (1.1). While our discussion and notation primarily focus on Chebyshev wavelets $\Psi(t)$, it can be readily extended to CP and HCP. To approximate these equations using CW, all functions involved need to be expanded in terms of CW and considering the operational vector (2.11). To facilitate this, let

$$\begin{aligned} u(t) &\simeq u_N(t) = \mathbf{U}^T\Psi(t), \\ B(t)u(t) &\simeq \Psi^T(t)\mathbf{B}\mathbf{U}^T\Psi(t) \simeq \hat{\mathbf{U}}_{1B}^T\Psi(t), \\ C(t)u^2(t) &\simeq \Psi^T(t)\mathbf{C}\mathbf{U}_2^{*T}\Psi(t) \simeq \hat{\mathbf{U}}_{2C}^T\Psi(t). \end{aligned} \tag{3.1}$$

Then, we consider the integral form of equation (1.1) as below:

$$u(t) = u(0) + \int_0^t (A(s) + B(s)u(s) + C(s)u^2(s))ds,$$

now, by taking into account the explanations from the relations (3.1), $u(0) = \alpha$ and $R(t) := \int_0^t A(s)ds$, the above relation can be written as

$$\mathbf{f}U^T\Psi(t) = \alpha E\Psi(t) + \mathbf{R}^T\Psi(t) + \int_0^t (\Psi^T(s)\mathbf{f}B\mathbf{U}^T\Psi(s) + \Psi^T(s)\mathbf{C}\mathbf{U}_2^{*T}\Psi(s))ds,$$

where the vector $E := (1, 0, 0, ..., 0)_{1\times N+1}$ and $\mathbf{P}$ is the operational matrix mentioned in (2.3). Consequently, by the relations (3.1), matrix representation of Riccati equation is obtained as

$$\mathbf{f}U^T = \alpha E + \mathbf{f}R^T + \hat{\mathbf{U}}_{1B}^T\mathbf{P} + \hat{\mathbf{U}}_{2C}^T\mathbf{P}. \tag{3.2}$$

By employing the Newton's method for seeking the unknown vector $\mathbf{U}$, then the desired approximation $u_N(t)$ can be obtained from $u_N(t) = \mathbf{U}^T\Psi(t)$.

4. Error analysis

Theorem 4.1. *([5]) If $u(t) \in H_w^k(D)$(sobolev space) and $u_M(t) = \sum_{r=0}^{J} c_rT_r(t) = \mathbf{C}^T\mathbf{T}(t)$ be the best approximation polynomials of $u(t)$ in L_w^2-norm, then*

$$\|u(t) - u_J(t)\|_{L_w^2(D)} \le C_0J^{-k}\|u(t)\|_{H_w^k(D)}.$$

This theorem demonstrates that by the greater smoothness of $f(t)$, the rate of convergence increase.

Theorem 4.2. *([19]) If $u(t) \in H_w^k(D)$, $I_i = [\frac{i-1}{n}, \frac{i}{n}]$ and $u_{NM}(t) = \sum_{j=1}^{N}\sum_{i=0}^{M-1} c_{ij}H_{ij}(t)$, then*

$$\|u(t) - u_{NM}(t)\|_{L_w^2(D)} \le C_0(NM)^{-k} \max_{1\le i\le N} \|u(t)\|_{H_w^k(I_i)}.$$

Theorem 4.3. *([32]) If $u(t) \in L_w^2(D)$, $|u''(t)| \le \gamma$ and $u_{NM}(t) = \sum_{j=1}^{N}\sum_{i=0}^{M-1} c_{ij}\psi_{ij}(t)$, then*

$$\|u(t) - u_{NM}(t)\|_{L_w^2(D)} \le \frac{\sqrt{\pi}\gamma}{2} \sum_{n=N+1}^{\infty}\sum_{m=M}^{\infty} \frac{1}{n^{\frac{5}{2}}(m^2-1)},$$

where N denotes 2^{k-1} defined in section 2.

The subsequent remark demonstrates the stability of these types of approaches.

Remark 4.4. ([25]) Let $u(t) \in H_w^k(D)$ and define $I_i = [\frac{i-1}{n}, \frac{i}{n}]$. Consider the approximation $u_{NM}(t) = \sum_{j=1}^{N}\sum_{i=0}^{M-1} c_{ij}H_{ij}(t)$. By applying Theorem 4.2 and using the triangular inequality, we can obtain the following inequalities:

$$\begin{aligned}
\|u_{NM}(t)\|_{L_w^2(D)} &\le \|u(t)\|_{L_w^2(D)} + \|u(t) - u_{NM}(t)\|_{L_w^2(D)} && (4.1)\\
&\le \|u(t)\|_{L_w^2(D)} + C_0(NM)^{-k} \max_{1\le i\le N} \|u(t)\|_{H_w^k(I_i)} && (4.2)\\
&\le C\|u(t)\|_{L_w^2(D)}. && (4.3)
\end{aligned}$$

Therefore, the stability of the method is derived from the above argument.

In order to indicate the error of using the operational matrix, the following theorems are presented.

Theorem 4.5. *([17]) Assume* $\boldsymbol{b}T_{M}(t) = [T_0(t), ..., T_M(t)]$ *be the shifted Chebyshev basis functions with weight function* $\tilde{w}$ *and the matrix* $\boldsymbol{P}$ *be the operational matrix of integration defined in (2.3), then the error of using operational matrix of integration is as follows:*

$$\| \int_{t_0}^{t} \boldsymbol{T}_M(s)ds - \boldsymbol{P}_{M\times M}\, \boldsymbol{T}_M(t)\|_{L^2_{\tilde{w}}(D)} \leq \frac{\sqrt{\pi}}{2\sqrt{2}AM}, \tag{4.4}$$

where $A = \frac{2}{t_f - t_0}$.

Theorem 4.6. *([9]) Assume* $\boldsymbol{H}_M(t) = [H_0(t), ..., H_M(t)]$ *be the hybrid Chebyshev polynomials and block pulse functions (HCP) and the matrix* $\boldsymbol{P}$ *be the HCP operational matrix of integration [29]. Therefore, the error estimate of utilizing HCP operational matrix of integration is obtained as*

$$\| \int_{0}^{t} \boldsymbol{H}(s)ds - \boldsymbol{P}\ \boldsymbol{H}(t)\|_{L^2_{\tilde{w}}(D)} \leq \frac{\sqrt{\pi}}{4\sqrt{2N}M}. \tag{4.5}$$

Theorem 4.7. *Assume* $\Psi(t)$ *denote the Chebyshev wavelets defined in (2.6), and* $\boldsymbol{P}$ *represent the operational matrix of integration for Chebyshev wavelets (CW). Hence, we can derive the error estimate associated with the utilization of the CW operational matrix of integration as follows:*

$$\| \int_{0}^{t} \Psi(s)ds - \boldsymbol{P}\ \Psi(t)\|_{L^2_{\tilde{w}}(D)} \leq \frac{\sqrt{\pi}}{4\sqrt{2N}M}. \tag{4.6}$$

Proof. If we consider the equality form of equation (2.3) for CW, then one can deduce that:

$$\int_{t_0}^{t} \Psi_{NM}(s)ds = \mathbf{P}_{NM\times N(M+1)}\Psi_{N(M+1)}, \tag{4.7}$$

where the indices verify the dimension of the above vectors and matrix $\mathbf{P}$.
Let

$$R := \int_{t_0}^{t} \Psi_{NM}(s)ds - \mathbf{P}_{NM\times NM}\Psi_{NM}. \tag{4.8}$$

Now, by comparing the equations (4.8) and (4.7) and with respect to the (2.7), R must be $\sum\limits_{i=1}^{N} \frac{\Psi_{i,M+1}}{2N(M-2)}$.
Therefore, the error estimate can be as

$$\begin{aligned}\| \int_{t_0}^{t} \Psi(s)ds - \mathbf{P}\ \Psi(t)\|_{L^2_{\tilde{w}}(D)} &= (\sum_{i=1}^{N} \int_{t_0}^{t_f} \tilde{w}_i(s) |\frac{\Psi_{i,M+1}}{N(M-2)}|^2 ds)^{\frac{1}{2}} \\ &\leq \frac{\sqrt{\pi}}{4\sqrt{2N}M}.\end{aligned}$$

The above theorem shows that if the parameters M and N increase, then the error will decrease.

5. Numerical examples

In order to solve the nonlinear system (3.2), we apply Newton iteration method with an initial guess obtained by steepest descent method. In order to study the convergence behavior of the proposed method, let employ the maximum absolute error which may be estimated approximately as

$$E_\infty = max\{|u(t) - u_N(t)|, t \in D\}.$$

For convenience, we denote the parameter L as the quantity of the used basis functions in the approximation methods.

Example 5.1. *Consider nonlinear Riccati differential equation [26]*

$$u'(x) = 2u(x) - u^2(x) + 1,$$

subject to the initial condition $u(0) = 0$, *The exact solution of this problem is*

$$u(x) = 1 + \sqrt{2}\tanh(\sqrt{2}x + \frac{1}{2}\ln(\frac{\sqrt{2}-1}{\sqrt{2}+1})).$$

This problem was first considered in 2008 [26]. In [26], the modified homotopy perturbation method was applied by using fourth-order term and the maximum absolute error was obtained 3.44×10^{-2}. In 2010, Lakestani and Dehghan [15] applied two collocation method for the approximate solution of the Eq. (1.1) using the operational matrices. The first method was based on B-spline basis functions and the second was based on Chebyshev cardinal functions. At last, the best results was achieved with $L = 20, 30$ and the maximum absolute errors were $7 \times 10^{-6}, 5 \times 10^{-8}$ by using method 1 and method 2, respectively. In the same year, Li [34] applied the Chebyshev wavelet operational matrix method and obtained E_∞ at least 1.8×10^{-5} for $L = 192$. In 2014, Bernulli wavelet method [14] was achieved in the best obtained results for the E_∞ with $L = 20$, 1×10^{-7}.

Here, we also compared our scheme with two recently methods in 2013 and 2015. Table 5 demonstrates the absolute error of the approximate solutions obtained for some $t \in [0, 1]$ by employing the present method with $L = 9, 12$, the Chebyshev polynomial operational matrix method [27] with $L = 9$ and the Bernstein polynomial collocation method [35] for $L = 12$, together. Regarding the above discussion, our method is more accurate with less basis functions L than the other aforementioned methods. Although Chebyshev operational matrix of integration and derivative method via collocation method [21, 27] have close results to ours, our method has better results.

The maximum absolute error of our scheme in terms of $\log(E_\infty)$ for various L is plotted in Figure 1. The graph justifies that the error function is monotone decreasing, which continuously depends on L and also represents that the convergence behavior is similar to the exponential convergence.

Table 1. Comparison of the absolute error of the present method, and references [14, 27, 35].

	$L = 9$		L=11	
t	present method	Ref. [27]	present method	Ref. [35]
0.0	1.63×10^{-6}	2×10^{-11}	3.00×10^{-8}	0
0.2	3.29×10^{-8}	7.62×10^{-6}	1.71×10^{-9}	2.35×10^{-7}
0.4	9.71×10^{-7}	2.02×10^{-6}	2.26×10^{-8}	3.05×10^{-7}
0.6	2.20×10^{-6}	2.62×10^{-6}	2.25×10^{-8}	3.38×10^{-7}
0.8	1.24×10^{-6}	2.25×10^{-6}	8.03×10^{-10}	3.42×10^{-7}
1.0	1.76×10^{-6}	1.90×10^{-10}	2.64×10^{-8}	1.17×10^{-5}
E_∞	2.33×10^{-6}	7.62×10^{-6}	3.02×10^{-8}	1.17×10^{-5}

Example 5.2. *Consider nonlinear Riccati differential equation [15]*

$$u'(x) = u(x) - u^2(x) - \frac{1}{1+t},$$

subject to the initial condition $u(0) = 1$, *whose exact solution is as follows:*

$$u(x) = \frac{1}{1+t}.$$

As we expected, the present method produced good accuracy with respect to other methods such as Chebyshev collocation method [21], B-spline and Chebyshev cardinal methods [15]. This claim is verified in Table 5. Although Chebyshev cardinal method and the present methods are closely contested in their results, Our approach has still better results.

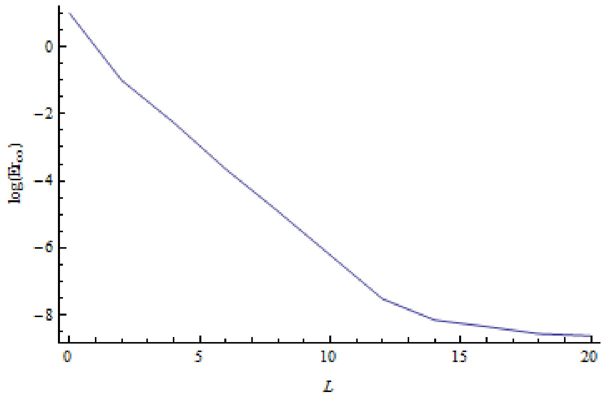


FIGURE 1. The maximum absolute error in terms of $\log E_\infty$ for various L.

TABLE 2. Comparison of the absolute error of the present method, Chebyshev cardinal and B-spline methods [15] and Chebyshev collocation method [21].

t	present method ($L = 10$)	Chebyshev cardinal ($L = 10$)	B-spline ($L = 12$)	collocation ($L = 10$)
0.0	3.29×10^{-8}	–	–	1.46×10^{-7}
0.2	3.06×10^{-8}	1.8×10^{-9}	5.9×10^{-6}	1.07×10^{-3}
0.4	3.76×10^{-9}	3.3×10^{-8}	7.7×10^{-6}	9.62×10^{-4}
0.6	2.36×10^{-8}	6.4×10^{-8}	6.3×10^{-6}	2.57×10^{-5}
0.8	2.90×10^{-8}	2.1×10^{-8}	5.9×10^{-6}	7.34×10^{-5}
1.0	3.33×10^{-8}	8.7×10^{-8}	5.7×10^{-6}	1.27×10^{-4}
E_∞	3.34×10^{-8}	8.7×10^{-8}	1.5×10^{-5}	1.41×10^{-3}

Example 5.3. *Consider nonlinear Riccati differential equation [11]*

$$u'(x) = 1 - u^2(x),$$

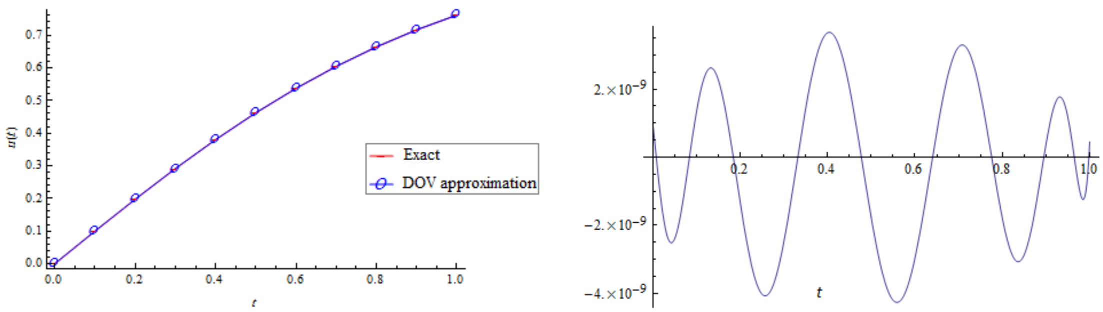
subject to the initial condition $u(0) = 1$. *The exact solution of this problem is*

$$u(x) = \frac{e^{2t} - 1}{e^{2t} + 1}.$$

The approximate solution using the present scheme is in high agreement with the exact solution. The comparison of the approximate solutions by using Bernulli wavelet [14], Bernstein polynomials [35] and the proposed approach are listed in Table 5 in the sense of the maximum absolute error. Also, the approximate and the exact solutions for this problem using Legendre wavelet method for k=1, m=25 is plotted in [11]. From our Figures (Figure 2) and those in [11], it is obvious that our obtained result has less error compared to Legendre wavelet method. The solution obtained through the proposed scheme demonstrates excellent agreement with the exact solution. A comparison of the approximate solutions using Bernoulli wavelets [14], Bernstein polynomials [35], and the proposed approach is provided in Table 5, considering the maximum absolute error. Additionally, the approximate and exact solutions for this problem, obtained using the Legendre wavelet method with k=1 and m=25, are plotted in [11]. By comparing our figures (Figure 2) with those in [11], it is evident that our results have lower error compared to the Legendre wavelet method. It should be pointed out that Chebyshev collocation method [21] has the same result as ours.

TABLE 3. Comparison of the maximum absolute error of the present method, Bernulli wavelet [14] and Bernstein polynomials [35].

	present method ($L=6$)	Bernstein poly. ($L=6$)	present method($L=10$)	Bernulli wavelet ($L=10$)
E_∞	2.37×10^{-5}	7.51×10^{-4}	4.26×10^{-9}	1×10^{-6}



(a) Comparison of the approximate and the exact solution.

(b) Plot of error for the present method.

FIGURE 2. Results of Ex. 5.3 for L=10.

6. Conclusions

In conclusion, the operational vector based on Chebyshev wavelets proves to be a valuable tool for simplifying the approximation procedure in solving integral and differential equations, particularly for Riccati differential equations. By introducing a direct operational vector method, we achieve highly accurate solutions with the conversion of Riccati differential equations into algebraic equations. The analysis of the operational matrix of integration for these basis functions ensures error control. Through numerical experiments, we have verified the effectiveness and applicability of the proposed method, thus establishing its validity as an advantageous approach in practical settings.

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