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Solution of Volterra integral equations of the first kind with discontinuous kernels by using the Adomian decomposition method

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Abstract

In this paper, for solving linear and nonlinear Volterra integral equations of the first kind with discontinuous kernels, the Adomian decomposition method, and the modified Adomian decomposition method are presented. We convert the main equation into the Volterra integral equation of the second kind. The numerical examples are given to denote the accuracy of the suggested method.

Keywords. Volterra integral equation, Piecewise continuous kernel, Modified Adomian decomposition method.

2010 Mathematics Subject Classification. 45D05.

1. Introduction

Volterra integral equation (VIE) of the first kind with discontinuous kernels studied by a few writers [8]. Outcomes of utilized the quadrature scheme for integral equations with discontinuous kernels offered in [2]. Article [15] introduced the asymptotic behavior of solution of VIE of the first kind with discontinuous kernels. The direct quadrature method and an iterative numerical technique introduced for solving weakly regular VIEs of the first kind with piecewise continuous kernels in [8]. The existence of continuous solutions to parametric families of solutions for this class of equations is considered in [13, 16]. The equiconvergence of expansion of functions of the integral operators with discontinuous kernels is studied in [6]. In order to solve VIEs with piecewise continuous kernels, the Taylor-collocation method based on the stochastic arithmetic presented in [9]. Article [7] suggested a numerical scheme to solve the nonlinear Volterra integral equations (NVIE) of the first kind with discontinuous kernels by using the midpoint quadrature rule.

The VIEs are extensively used in several fields, such as populace biology, engineering, physics, demography, and economics [4]. In order to explain the storage systems, the Volterra integral models are used in article [17]. Article [18] by employment Volterra integral equations proposed a new model of storage control. Article [19] suggested a novel dynamic model based on VIEs of the first kind with piecewise continuous kernels as an adaptive solution to the load leveling problem. VIEs with piecewise continuous kernels can be solved by using shifted Legendre polynomials, which convert the main equation into a system of linear algebraic equations [1]. For solving VIEs of the first kind with special piecewise continuous kernels, the polynomial spline collocation technique is suggested in [20]. Recently, for the outcomes obtained from solving the integral equations with piecewise continuous kernels by the Adomian decomposition method (ADM), a new scheme and tool has been obtained for its validation [10].

We consider the VIE of the form

$$\int_0^x K(x,s)y(s)ds = r(x) \qquad x \in [0, X], \tag{1.1}$$

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where

$$K(x,s)=\begin{cases} K_1(x,s), & \alpha_0(x)<s<\alpha_1(x),\\ K_2(x,s), & \alpha_1(x)<s<\alpha_2(x),\\ \quad\vdots & \\ K_n(x,s), & \alpha_{n-1}(x)<s<\alpha_n(x),\end{cases} \tag{1.2}$$

is discontinuous kernel on curves $\alpha_i(x)$ $i=1,...,n$, and

$$0=\alpha_0(x)<\alpha_1(x)<\cdots<\alpha_n(x)=x, \qquad r(0)=0.$$

By rewriting the equation (1.1) we obtain

$$\sum_{i=1}^{n}\int_{\alpha_{i-1}(x)}^{\alpha_i(x)} K_i(x,s)\,y(s)\,ds=r(x). \tag{1.3}$$

Suppose that the functions $K_i(x,s)$ and $r(x)$ are sufficiently smooth functions and continuous. In addition, $\alpha_i(x)\in C^1[0,X]$ are not decreasing, and

$$0<\alpha_1^{'}(0)\leq\alpha_2^{'}(0)\leq\cdots\leq\alpha_{n-1}^{'}(0)<1.$$

The NVIE of the first kind with piecewise continuous kernel is introduced as follows

$$\sum_{i=1}^{n}\int_{\alpha_{i-1}(x)}^{\alpha_i(x)} K_i(x,s)\,G(y(s))\,ds=r(x), \tag{1.4}$$

where

$$\alpha_{i-1}(x)<s<\alpha_i(x), \quad i=1,...,n.$$

The functions $r(x)$, $\alpha_i(x)$ have continuous derivatives with respect to x. Assume $0=\alpha_0(x)<\alpha_1(x)<\cdots<\alpha_n(x)=x$, $K_n(x,x)\neq 0$, $\alpha_i(0)=0$, $r(0)=0$. $K_i(x,s)$, $i=1,\ldots,n$ have continuously differentiable w.r.t. x in $0\leq s\leq x\leq X$. The functions $\alpha_1(x),\ldots,\alpha_{n-1}(x)$ increase at least in a small neighborhood.

In this paper, we use the ADM for solving equations (1.3) and (1.4). This method is one of the essential methods for solving integral equations that can be employed directly. The paper is organized into five sections: In section 2, the existence of a unique solution is studied. Section 3 discusses the suggested method. Illustrative examples are reported in order to denote the accuracy of the proposed technique in section 4. The conclusion is stated in section 5.

2. Existence of a unique continuous solution

Introduce the function

$$I(x)=\sum_{i=1}^{n-1}|\alpha_i^{'}(x)\;K_n^{-1}(x,x)|\;|K_i(x,\alpha_i(x))-K_{i+1}(x,\alpha_i(x))|. \tag{2.1}$$

The following theorem shows sufficient conditions for a unique solution to the equation (1.3).

Theorem 2.1. *Suppose the following requirements satisfied when $x\in[0,X]$:*
continuous functions $K_i(x,s)$, $i=1,2,\ldots,n$, $\alpha_i(x)$, $r(x)$ have continuous derivatives w.r.t. x. $K_n(x,x)\neq 0$, $0=\alpha_0(x)<\alpha_1(x)<\cdots<\alpha_n(x)=x$, $x\in(0,X]$, $\alpha_i(0)=0$, $r(0)=0$, $I(0)<1$.
So, the equation (1.3) has a unique local solution in $C[0,\tau]$ where $\tau>0$.

Proof: See Ref. [11].

Theorem 2.2. *Assume the conditions of the Theorem* 2.1 *are fulfilled. Also, suppose*

$$\min_{\tau \leq x \leq X} \left(x - \alpha_{n-1}\left(x\right)\right) = d > 0.$$

Then equation (1.3) has a unique global solution in $C\left[0, X\right]$.

Proof: See Ref. [11].

3. Study of the Method

For solving the equation (3.1) we convert the equation (3.1) into an equation of the second kind and then apply the ADM.

$$\sum_{i=1}^{n} \int_{\alpha_{i-1}(x)}^{\alpha_i(x)} K_i\left(x, s\right) G\left(y(s)\right) ds = r(x). \tag{3.1}$$

By differentiating two sides of the equation (3.1) w.r.t. x, we obtain

$$\begin{aligned} r^{'}\left(x\right) = \sum_{i=1}^{n} \Big(\alpha_i^{'}(x) K_i(x, \alpha_i(x)) G(y(\alpha_i(x))) - \alpha_{i-1}^{'}(x) K_i(x, \alpha_{i-1}(x)) G(y(\alpha_{i-1}(x)))\Big) \\ + \sum_{i=1}^{n} \int_{\alpha_{i-1}(x)}^{\alpha_i(x)} \frac{\partial K_i\left(x, s\right)}{\partial x} G(y(s)) ds, \end{aligned} \tag{3.2}$$

let us consider

$$K_n\left(x, x\right) G\left(y(x)\right) = K_n\left(x, x\right) y(x), \tag{3.3}$$

we rewrite the equation (3.2) in the following way

$$\begin{aligned} r^{'}\left(x\right) = K_n\left(x, x\right) y(x) &+ \sum_{i=1}^{n-1} \Big(\alpha_i^{'}(x) K_i(x, \alpha_i(x)) G(y(\alpha_i(x)))\Big) \\ &- \sum_{i=1}^{n} \Big(\alpha_{i-1}^{'}(x) K_i(x, \alpha_{i-1}(x)) G(y(\alpha_{i-1}(x)))\Big) \\ &+ \sum_{i=1}^{n} \int_{\alpha_{i-1}(x)}^{\alpha_i(x)} \frac{\partial K_i\left(x, s\right)}{\partial x} G(y(s)) ds, \end{aligned} \tag{3.4}$$

by dividing both sides of the equation (3.4) by $K_n\left(x, x\right)$ with assumption $K_n\left(x, x\right) \neq 0$ for values of x we have

$$\begin{aligned} \frac{r^{'}\left(x\right)}{K_n\left(x, x\right)} = y(x) &+ \sum_{i=1}^{n-1} \left(\frac{\alpha_i^{'}(x)}{K_n\left(x, x\right)} \left[K_i(x, \alpha_i(x)) G(y(\alpha_i(x)))\right]\right) \\ &- \sum_{i=1}^{n} \left(\frac{\alpha_{i-1}^{'}(x)}{K_n\left(x, x\right)} \left[K_i(x, \alpha_{i-1}(x)) G_i(y(\alpha_{i-1}(x)))\right]\right) \\ &+ \sum_{i=1}^{n} \int_{\alpha_{i-1}(x)}^{\alpha_i(x)} \frac{1}{K_n\left(x, x\right)} \frac{\partial K_i\left(x, s\right)}{\partial x} G(y(s)) ds, \end{aligned} \tag{3.5}$$

also assume

$$\frac{r^{'}\left(x\right)}{K_n\left(x, x\right)} = R(x), \quad \frac{\alpha_i^{'}(x)}{K_n\left(x, x\right)} = \beta_i(x), \quad \frac{1}{K_n\left(x, x\right)} \frac{\partial K_i\left(x, s\right)}{\partial x} = Z_i\left(x, s\right), \tag{3.6}$$

substituting (3.6) in (3.5) gives

$$R(x) = y(x) + \sum_{i=1}^{n-1} \Big(\beta_i(x) \left[K_i(x, \alpha_i(x)) G_i(y(\alpha_i(x))) \right] \Big) - \sum_{i=1}^{n} \Big(\beta_{i-1}(x) [K_i(x, \alpha_{i-1}(x)) G(y(\alpha_{i-1}(x)))] \Big) + \sum_{i=1}^{n} \int_{\alpha_{i-1}(x)}^{\alpha_i(x)} Z_i\,(x, s)\, G(y(s)) ds, \tag{3.7}$$

we rewrite equation (3.7) in the following form

$$y(x) = R(x) - \sum_{i=1}^{n-1} \Big(\beta_i(x) \left[K_i(x, \alpha_i(x)) G(y(\alpha_i(x))) \right] \Big) + \sum_{i=1}^{n} \Big(\beta_{i-1}(x) [K_i(x, \alpha_{i-1}(x)) G(y(\alpha_{i-1}(x)))] \Big) - \sum_{i=1}^{n} \int_{\alpha_{i-1}(x)}^{\alpha_i(x)} Z_i\,(x, s)\, G(y(s)) ds. \tag{3.8}$$

In this paper, in order to solve the equation (3.8) by using the ADM, the kernels should be chosen in such a way that after derivation the values of $y(\alpha_i(x)), \quad i = 1, \ldots, n-1$ be omitted and only the value of $y(\alpha_n(x))$ remains for the calculation. The term $y(x)$ can be denoted by an infinite series as follows [10]

$$y(x) = \sum_{j=0}^{\infty} y_j(x), \tag{3.9}$$

by the following series, the nonlinear function $G\,(y)$ is constructed

$$G\,(y) = \sum_{k=0}^{\infty} A_k, \tag{3.10}$$

where

$$A_k = \left(\frac{1}{k!} \right) \left(\frac{d^k}{d\lambda^k} \right) \left[G \left(\sum_{j=0}^{\infty} \lambda^j y_j \right) \right]_{\lambda=0}, \tag{3.11}$$

introduced as Adomian polynomials. Nonlinear terms in equations can be calculated by using the Adomian polynomials. In equation (3.8), if we have only one term outside the integral for example $f(x)$ we use of the ADM and for more than one term outside the integral we use of the modified Adomian decomposition method (MADM), then we will have the following recursive formula.

$$y_0(x) = f(x) \tag{3.12}$$

$$y_j(x) = -\sum_{i=1}^{n} \int_{\alpha_{i-1}(x)}^{\alpha_i(x)} Z_i\,(x, s)\, A_{j-1} ds, \quad j \geq 1. \tag{3.13}$$

The uniqueness and convergence of the ADM denoted in the following lemma and theorem [3, 10]. Consider $|K_i(x, s)| < M_i \quad i = 1, \ldots, n, \quad 0 \leq s \leq x \leq X \quad x \in [0, X]$, the nonlinear term G(y) is Lipschitz continuous with $|G(y) - G(z)| \leq L|y - z|$.

Lemma 3.1. *The obtained solution of using the ADM for solving equations will be unique when* $0 < \eta < 1$, *where* $\eta = L \sum_{i=1}^{n} M_i(\alpha_i - \alpha_{i-1})$.
Proof. *See Ref. [3].*

Theorem 3.2. *The series solution for solving equations by utilizing the ADM converges if* $0 < \eta < 1$ *and* $|y_1| < \infty$.
Proof. *See Ref. [10].*

4. Numerical Examples

This section contains two examples of the VIEs with discontinuous kernels.

Example 4.1. Let us consider the following VIE of the first kind with discontinuous kernels

$$\int_0^{\frac{x}{4}}(xs)y(s)ds+\int_{\frac{x}{4}}^{\frac{x}{2}}(\frac{x^2}{4})y(s)ds+\int_{\frac{x}{2}}^{x}(\frac{xs}{2})y(s)ds=\frac{67x^4}{384}, \tag{4.1}$$

with the exact solution $y(x)=x, \quad x\in[0,1]$.
By differentiating both sides of the equation (4.1) w.r.t. x, we have

$$y(x)=\frac{67x}{48}-\int_0^{\frac{x}{4}}\frac{2s}{x^2}y(s)ds-\int_{\frac{x}{4}}^{\frac{x}{2}}\frac{1}{x}y(s)ds-\int_{\frac{x}{2}}^{x}\frac{s}{x^2}y(s)ds, \tag{4.2}$$

using the ADM, we get

$$y_0(x)=\frac{67x}{48}, \tag{4.3}$$

$$y_1(x)=-\int_0^{\frac{x}{4}}\frac{2s}{x^2}\frac{67s}{48}ds-\int_{\frac{x}{4}}^{\frac{x}{2}}\frac{1}{x}\frac{67s}{48}ds-\int_{\frac{x}{2}}^{x}\frac{s}{x^2}\frac{67s}{48}ds=-\frac{1273x}{2304}, \tag{4.4}$$

$$y_2(x)=-\int_0^{\frac{x}{4}}\frac{2s}{x^2}(-\frac{1273s}{2304})ds-\int_{\frac{x}{4}}^{\frac{x}{2}}\frac{1}{x}(-\frac{1273s}{2304})ds-\int_{\frac{x}{2}}^{x}\frac{s}{x^2}(-\frac{1273s}{2304})ds=\frac{24187x}{110592}, \tag{4.5}$$

$$\vdots$$

the series solution is in the following form

$$\begin{aligned} y_{n+1}(x)&=\frac{67x}{48}-\frac{1273x}{2304}+\frac{24187x}{110592}-\cdots \\ &=\frac{67x}{48}(1-\frac{19}{48}+\frac{361}{2304}-\cdots) \\ &=\frac{67x}{48}(\frac{1}{1+\frac{19}{48}})=x. \end{aligned} \tag{4.6}$$

Example 4.2. Consider the following NVIE of the first kind with piecewise continuous kernels

$$\int_0^{\frac{x}{2}}e^{x-s}e^{y(s)}ds+\int_{\frac{x}{2}}^{x}e^{s}e^{y(s)}ds=\frac{1}{2}e^{x}x-\frac{1}{2}e^{x}+\frac{1}{2}e^{2x}, \tag{4.7}$$

with the exact solution $y(x)=x, \quad x\in[0,1]$.
Similar to the one of the examples of [5], we assume

$$u(s)=e^{y(s)}, \tag{4.8}$$

so, we can write

$$\int_0^{\frac{x}{2}}e^{x-s}u(s)ds+\int_{\frac{x}{2}}^{x}e^{s}u(s)ds=\frac{1}{2}e^{x}x-\frac{1}{2}e^{x}+\frac{1}{2}e^{2x}, \tag{4.9}$$

by differentiating both sides of the equation (4.9) w.r.t. x, we have

$$u(x)=\frac{x}{2}+e^{x}-\frac{1}{e^{x}}\int_0^{\frac{x}{2}}e^{x-s}u(s)ds, \tag{4.10}$$

so, using the MADM, we obtain the recurrence relation

$$u_0(x) = e^x, \tag{4.11}$$

$$u_1(x) = \frac{x}{2} - \frac{1}{e^x}\int_0^{\frac{x}{2}} e^{x-s}u_0(s)ds, \tag{4.12}$$

$$u_{n+1}(x) = -\frac{1}{e^x}\int_0^{\frac{x}{2}} e^{x-s}u_n(s)ds. \tag{4.13}$$

According to the above recurrence relation, we have

$$u_0(x) = e^x, \tag{4.14}$$

$$u_1(x) = \frac{x}{2} - \frac{1}{e^x}\int_0^{\frac{x}{2}} e^{x-s}e^s ds = 0, \tag{4.15}$$

therefore,

$$u(x) = u_0(x) + u_1(x) = e^x, \tag{4.16}$$

$$y(x) = lne^x = x. \tag{4.17}$$

5. Conclusion

In this study, we utilized the ADM and MADM to solve the VIEs of the first kind with discontinuous kernels by converting them into equations of the second kind. The ADM is one of the best methods for solving VIEs of the second kind. The given numerical examples denoted the accuracy of the Adomian decomposition method.

Future research should examine the blow-up phenomena for nonlinear Volterra integral equations of the first kind with discontinuous kernels. Readers may refer to [12, 14].

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