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Extrapolated triangular splitting method to interval system of linear algebraic equations

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Abstract

This paper presents an extrapolated triangular splitting method (`ETrnSplit`) to find the formal solution for the interval system of linear algebraic equations in which this method uses Kaucher interval arithmetic. Some numerical experiments are given to show the efficiency of this method.

Keywords. Interval systems of linear algebraic equations, Extrapolation methods, Kaucher interval arithmetic.
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1. Introduction

In this paper, the main subject is the presentation of an extrapolated triangular splitting method for finding, as quickly as possible, the formal solution of the interval system of linear algebraic equations

$$\begin{cases} \boldsymbol{a}_{11}x_1 + \boldsymbol{a}_{12}x_2 + \ldots + \boldsymbol{a}_{1n}x_n = \boldsymbol{b}_1, \\ \boldsymbol{a}_{21}x_1 + \boldsymbol{a}_{22}x_2 + \ldots + \boldsymbol{a}_{2n}x_n = \boldsymbol{b}_2, \\ \quad\vdots \qquad\quad \vdots \qquad\quad \ddots \qquad\quad \vdots \qquad\quad \vdots \\ \boldsymbol{a}_{n1}x_1 + \boldsymbol{a}_{n2}x_2 + \ldots + \boldsymbol{a}_{nn}x_n = \boldsymbol{b}_n, \end{cases} \tag{1.1}$$

and the matrix form

$$\boldsymbol{A}x = \boldsymbol{b}, \tag{1.2}$$

with interval $n \times n$-matrix $\boldsymbol{A} = (\boldsymbol{a}_{ij})$ and interval right-hand side n-vector $\boldsymbol{b} = (\boldsymbol{b}_i)$. Mathematical modeling in most engineering sciences, chemistry, physics, and economics involves inaccurate and uncertain quantities, and interval analysis is becoming increasingly popular when processing such models. For example, in the periodic table of chemical elements, the atomic weights of 13 elements are currently expressed as intervals [37]. Many engineering problems such as the estimation of physical parameters, robust control, and robotics lead to the interval system of linear equations [7, 16]. Because of the importance, many studies have been done for the an solving interval system of the form (1.1)-(1.2) by researchers such as Oettli and Prager [22, 23], Hansen [5], Moore [16], Rohn [24], Neumaier [20], Alefeld and Mayer [2], Shary [27–29] and so forth. In these studies, the main idea is to estimate the boundary enclosure of the solution set to an interval system of linear equations using the classical interval arithmetic. In some other methods for solving these problems, the Kaucher interval arithmetic [9–11] is used in many pioneering studies [13, 17, 19, 25, 27, 28, 39] in which their main idea is to find the "formal (algebraic) solutions" for the interval system of linear equations.

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Shary in [30, 31] has presented stationary single-step iterative methods for finding formal (algebraic) solutions to the interval system of linear equations. One of these methods so-called `TrnSplit` which this method is based on the triangular splitting of the matrix of the system. The convergence theorem of the `TrnSplit` method is described in [8].

The purpose in this paper is to present an extrapolated triangular splitting method for computing formal (algebraic) solutions of the interval linear systems of equations that considerably accelerates the method `TrnSplit`. The paper is organized as follow. Section 2 presents the necessary notations, review some classical interval arithmetic and Kaucher interval arithmetic. In section 3, first reviewed the extrapolation method to the system of linear equations. It is then shown by a counter-example that the extrapolation of methods using classical interval computation does not allow us to correctly compute formal solutions to the interval system of linear equations, and also, presents the extrapolated triangular splitting method (`ETrnSplit`). In section 4, the convergence speed of the `ETrnSplit` method in the numerical examples is shown. Finally, section 5 presents an overall summary and conclusions of the entire work.

2. Preliminaries

In this introductory section, we remind some necessary basic concepts of the classical interval arithmetic and Kaucher interval arithmetic.

A real interval is known to be a bounded closed and connected subset of the real axis, that is, the set of the form

$$\boldsymbol{a} := [\underline{a}, \overline{a}] = \{\, x \in \mathbb{R} \mid \underline{a} \le x \le \overline{a},\ \underline{a}, \overline{a} \in \mathbb{R} \,\}.$$

In the rest of the paper, all interval objects are denoted by the boldface letters. The notations $\underline{\boldsymbol{x}} and \overline{\boldsymbol{x}}$ mean the lower (left) and upper (right) endpoints of an interval $\boldsymbol{x}$ respectively. The set of all real intervals is denoted by $\mathbb{IR}$, and

$$\mathbb{IR}^n = \{\, (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n)^\top \mid \boldsymbol{x}_i \in \mathbb{IR}, 1 \le i \le n \,\},$$

stands for the set of all n-dimensional interval vectors. The latter are often called *boxes*. The interval matrix is a rectangular table of intervals, which is designated by $\boldsymbol{A} = (\boldsymbol{a}_{ij})_{m\times n}$, in which $\boldsymbol{a}_{ij} = [\underline{\boldsymbol{a}}_{ij}, \overline{\boldsymbol{a}}_{ij}]$.

An interval square matrix $\boldsymbol{A}$ is called a nonsingular (regular) matrix if all point matrices $A \in \boldsymbol{A}$ are *nonsingular* (regular) [6, 14]. An interval square matrix $\boldsymbol{A}$ is called a *singular* matrix if it is not nonsingular, which is equivalent to the fact that the matrix $\boldsymbol{A}$ contains at least one singular point matrix. The set of all interval $m \times n$-matrices is denoted as $\mathbb{IR}^{m\times n}$. If S is a nonempty bounded subset of $\mathbb{R}$ we denote by

$$\square S := [\inf(S), \sup(S)],$$

the *hull* of S, i.e. the tightest interval enclosing S [21]. Other symbols also follow the informal international standard [12].

In the classical interval arithmetic, the main arithmetic operations are defined "by representatives", that is, according to the following fundamental principle

$$\boldsymbol{a} * \boldsymbol{b} = \{a * b \mid a \in \boldsymbol{a}, b \in \boldsymbol{b}\}, \qquad * \in \{+, -, \cdot, /\}. \tag{2.1}$$

The above relation can be reformulated in the following constructive manner for separate arithmetic operations:

$$\begin{aligned}
&\boldsymbol{a} + \boldsymbol{b} = \left[\ \underline{\boldsymbol{a}} + \underline{\boldsymbol{b}},\ \overline{\boldsymbol{a}} + \overline{\boldsymbol{b}}\ \right],\\
&\boldsymbol{a} - \boldsymbol{b} = \left[\ \underline{\boldsymbol{a}} - \overline{\boldsymbol{b}}, \overline{\boldsymbol{a}} - \underline{\boldsymbol{b}}\ \right],\\
&\boldsymbol{a} \cdot \boldsymbol{b} = \left[\ \min\{\underline{\boldsymbol{a}}\ \underline{\boldsymbol{b}}, \underline{\boldsymbol{a}}\overline{\boldsymbol{b}}, \overline{\boldsymbol{a}}\underline{\boldsymbol{b}}, \overline{\boldsymbol{a}}\overline{\boldsymbol{b}}\}\ ,\ \max\{\underline{\boldsymbol{a}}\ \underline{\boldsymbol{b}}, \underline{\boldsymbol{a}}\overline{\boldsymbol{b}}, \overline{\boldsymbol{a}}\underline{\boldsymbol{b}}, \overline{\boldsymbol{a}}\overline{\boldsymbol{b}}\}\ \right],\\
&\boldsymbol{a}/\boldsymbol{b} = \boldsymbol{a} \cdot \left[\ 1/\overline{\boldsymbol{b}}, 1/\underline{\boldsymbol{b}}\ \right]; 0 \notin \boldsymbol{b}.
\end{aligned} \tag{2.2}$$

The algebraic properties of the classical interval arithmetic $\mathbb{IR}$ are much poorer in comparison with those of the real numbers $\mathbb{R}$. For example, each element $\boldsymbol{a}$ with nonzero width of $\mathbb{IR}$ does not have the inverse with respect to the first operation in (2.2), i. e.

$$\boldsymbol{a} - \boldsymbol{a} \neq 0.$$

The above inequality and other shortcomings of classical interval arithmetic [18, 21] brought to life the need for its algebraic and order completion. Its result was Kaucher interval arithmetic.

In fact, the importance of Kaucher interval arithmetic $\mathbb{KR}$ becomes clear when we want to find the solution of an interval equation $\boldsymbol{ax} = \boldsymbol{b}$ or in general case an interval system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ so that the solution to the equation or the interval system of linear equations satisfies. These kinds of solutions call *"formal solutions"* defined as follows:

Definition 2.1. [28, 32] An interval vector is called a formal solution to an interval equation or a system of equations, if substituting it into the equation or a system of equations and execution of all operations in interval arithmetic result in a valid equality.

In the following, we continue to introduce the Kaucher interval arithmetic and its properties. It is noted that most of the literature is mentioned in [31]. In examining the properties of classical interval arithmetic, we find that some concepts cannot be defined in $\mathbb{IR}$. Another of these weaknesses is related to the inclusion ordering "$\subseteq$" [31, 33]. In partially ordered sets, the possibility of taking, for any two elements, their lower bound "$\wedge$" and the upper bound "$\vee$" with respect to the order in question plays a huge role. In $\mathbb{IR}$, the corresponding operations are

$$\boldsymbol{a} \wedge \boldsymbol{b} := \inf_{\subseteq} \{\boldsymbol{a}, \boldsymbol{b}\} = \left[\max\{\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}\}, \min\{\overline{\boldsymbol{a}}, \overline{\boldsymbol{b}}\} \right], \tag{2.3}$$

$$\boldsymbol{a} \vee \boldsymbol{b} := \sup_{\subseteq} \{\boldsymbol{a}, \boldsymbol{b}\} = \left[\min\{\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}\}, \max\{\overline{\boldsymbol{a}}, \overline{\boldsymbol{b}}\} \right]. \tag{2.4}$$

Since these operations with respect to $\mathbb{IR}$, in a sense, are "not closed", for example,

$$[0, 1] \wedge [2, 3] = [2, 1],$$

is not defined in the classical interval arithmetic, so we need to the interval arithmetic where $[\lambda, \mu]$ such that $\lambda > \mu$, it is meaningful. The elements of the complete interval arithmetic $\mathbb{KR}$ are pairs of real numbers $[\eta, \xi]$, not necessarily connected by the relation $\eta \leq \xi$. Therefore, $\mathbb{KR}$ is obtained by appending improper intervals $[\eta, \xi]$, $\eta > \xi$ to the set $\mathbb{IR} = \{[\eta, \xi] \mid \eta, \xi \in \mathbb{R}, \eta \leq \xi\}$ of proper intervals and real numbers (identified with degenerate intervals of zero width). A detailed description of Kaucher interval arithmetic $\mathbb{KR}$ can be found in [11]. For each two proper and improper intervals, the dualization of the interval $\boldsymbol{a} = [\underline{\boldsymbol{a}}, \overline{\boldsymbol{a}}]$ defined as follow

$$\operatorname{dual} \boldsymbol{a} := [\overline{\boldsymbol{a}}, \underline{\boldsymbol{a}}].$$

Proper *projection* of an interval $\boldsymbol{a}$ is the value

$$\operatorname{pro} \boldsymbol{a} := \begin{cases} \boldsymbol{a}, & \text{if } \boldsymbol{a} \text{ is proper,} \\ \operatorname{dual} \boldsymbol{a}, & \text{otherwise.} \end{cases}$$

Similar to the classical interval arithmetic $\mathbb{IR}$, the"inclusion" of one interval to another is defined in $\mathbb{KR}$ as follows [33]:

$$\boldsymbol{a} \subseteq \boldsymbol{b} \iff \underline{\boldsymbol{a}} \geq \underline{\boldsymbol{b}}, \overline{\boldsymbol{a}} \leq \overline{\boldsymbol{b}}. \tag{2.5}$$

For example, $[2, 0] \subseteq [1, 1] = 1 \in \mathbb{R}$. As a consequence, the operations of taking the minimum (2.3) and the maximum (2.4) keep their definitions unchanged in $\mathbb{KR}$, but now they are always possible due to the presence of improper intervals. In particular, $[0, 1] \wedge [2, 3] = [2, 1]$. Therefore, the extension of $\mathbb{IR}$ to $\mathbb{KR}$ makes the set of intervals a lattice, and even a conditionally complete lattice with respect to the inclusion ordering (2.5). In addition to the set-theoretic inclusion on the set of intervals $\mathbb{KR}$, there is another partial ordering, which naturally generalizes the linear order "$\leq$" on the real axis [3]:

Definition 2.2. For the intervals $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{KR}$, we say that $\boldsymbol{a}$ does not exceed $\boldsymbol{b}$ and write "$\boldsymbol{a} \leq \boldsymbol{b}$" if and only if $\underline{\boldsymbol{a}} \leq \underline{\boldsymbol{b}}$ and $\overline{\boldsymbol{a}} \leq \overline{\boldsymbol{b}}$.

The semigroup of all proper intervals with the operation of addition is fairly simple: the addition of intervals is divided into independent operations of addition of the left and right endpoints of the operands. As a consequence, the

extension of addition from $\mathbb{IR}$ to $\mathbb{KR}$ is easy, and it is defined in $\mathbb{KR}$ in exactly the same way as in classical interval arithmetic:

$$\boldsymbol{a}+\boldsymbol{b} \;:=\; \left[\,\underline{\boldsymbol{a}}+\underline{\boldsymbol{b}}\;,\;\overline{\boldsymbol{a}}+\overline{\boldsymbol{b}}\,\right].$$

But now it follows from the existence of improper intervals that each element $\boldsymbol{a}$ of $\mathbb{KR}$ has a unique inverse with respect to addition (also called opposite), denoted by "opp $\boldsymbol{a}$", and the equality $\boldsymbol{a}+\mathrm{opp}\boldsymbol{a}=0$ implies that

$$\mathrm{opp}\;\boldsymbol{a} \;:=\; [-\underline{\boldsymbol{a}},-\overline{\boldsymbol{a}}]\,. \tag{2.6}$$

With respect to addition, the arithmetic $\mathbb{KR}$ is thus a commutative group that is isomorphic to the additive group of the standard linear space $\mathbb{R}^2$. For brevity, we denote by "$\ominus$" an operation that is the inverse of addition, and it will be called *interval subtraction* in $\mathbb{KR}$ (or *algebraic subtraction*). Then

$$\boldsymbol{a}\ominus\boldsymbol{b} \;:=\; \boldsymbol{a}+\mathrm{opp}\;\boldsymbol{b}=\left[\,\underline{\boldsymbol{a}}-\underline{\boldsymbol{b}},\overline{\boldsymbol{a}}-\overline{\boldsymbol{b}}\,\right].$$

Now we extend the definition of multiplication to the entire set $\mathbb{KR}$. The ability of algebra to do this has already been exhausted, and we need to involve considerations concerning the inclusion ordering in $\mathbb{KR}$ and the related properties of arithmetic operations. Using the maximum with respect to inclusion (2.4), the fundamental property (2.1), which defines the operations of classical interval arithmetic, can be rewritten in the following equivalent form:

$$\boldsymbol{a}*\boldsymbol{b} \;=\; \{a*b \mid a\in\boldsymbol{a}, b\in\boldsymbol{b}\} \tag{2.7}$$

$$\;=\; \left[\min_{a\in\boldsymbol{a}}\min_{b\in\boldsymbol{b}}(a*b),\;\max_{a\in\boldsymbol{a}}\max_{b\in\boldsymbol{b}}(a*b)\right] \;=\; \bigvee_{a\in\boldsymbol{a}}\bigvee_{b\in\boldsymbol{b}}(a*b), \tag{2.8}$$

where $*\in\{+,-,\cdot,/\}$. It is easily seen that the addition extended to the entire set of proper and improper intervals $\mathbb{KR}$, as well as the multiplication defined for intervals that do not contain zero and are not contained in zero can be represented in a similar way through the operations (2.4) and (2.3) of taking minimum and maximum with respect to inclusion. If both operands $\boldsymbol{a}$ and $\boldsymbol{b}$ are improper, then

$$\boldsymbol{a}*\boldsymbol{b} \;=\; \bigwedge_{a\,\in\,\mathrm{pro}\,\boldsymbol{a}}\;\bigwedge_{b\,\in\,\mathrm{pro}\,\boldsymbol{b}}(a*b),$$

where $*\in\{+,\cdot\}$. The lower arguments of the operations "$\wedge$" should have proper projections pro $\boldsymbol{a}$ and pro $\boldsymbol{b}$, since improper intervals themselves are contained in points due to the definition of inclusion in $\mathbb{KR}$. We introduce the so-called conditional operation of taking the extremum with respect to inclusion:

$$\mathop{И}_{x}^{\boldsymbol{a}} \;:=\; \begin{cases} \bigvee\limits_{x\in\boldsymbol{a}}, & \text{if } \boldsymbol{a} \text{ is proper,}\\[2ex] \bigwedge\limits_{x\in\mathrm{dual}\;\boldsymbol{a}}, & \text{if } \boldsymbol{a} \text{ is improper.}\end{cases}$$

This is an operation that depends on the interval parameter $\boldsymbol{a}$ standing as its upper index. The operation is either maximum or minimum with respect to the inclusion "$\subseteq$", depending on whether $\boldsymbol{a}$ is proper or improper. This extremum is taken over all x from the proper projection of the interval $\boldsymbol{a}$. Note that any interval $\boldsymbol{a}\in\mathbb{KR}$ can be represented as

$$\boldsymbol{a}=\mathop{И}_{x}^{\boldsymbol{a}}\,x,$$

(instead of x, any letter can be used in the formula). Anyway, for $*\in\{+\,,\,\cdot\}$, the following relation is valid:

$$\boldsymbol{a}*\boldsymbol{b}=\mathop{И}_{a}^{\boldsymbol{a}}\;\mathop{И}_{b}^{\boldsymbol{b}}\,(a*b). \tag{2.9}$$

This representation, first introduced in [9], expresses the relationship between the result of the interval operation $\boldsymbol{a}*\boldsymbol{b}$ and the results of the point operations $a*b$ for $a\in\mathrm{pro}\,\boldsymbol{a}$, $b\in\mathrm{pro}\,\boldsymbol{b}$. It can be taken as a basis for the definition of arithmetic operations in the complete interval arithmetic $\mathbb{KR}$.

It is not hard to derive, from (2.9), the monotonicity of the interval arithmetic operations with respect to inclusion.

In order to write out explicit formulas for multiplication in complete interval arithmetic, we select the following subsets $\mathcal{P}$,$\mathcal{Z}$, $-\mathcal{P}$, and dual$\mathcal{Z}$ respectively non-negative intervals, zero-containing intervals, non-positive intervals, intervals contained in zero in $\mathbb{KR}$:

$$\mathcal{P} := \{\boldsymbol{a} \in \mathbb{KR} \mid (\underline{\boldsymbol{a}} \geq 0) \,\&\, (\overline{\boldsymbol{a}} \geq 0)\},$$

$$\mathcal{Z} := \{\boldsymbol{a} \in \mathbb{KR} \mid \underline{\boldsymbol{a}} \leq 0 \leq \overline{\boldsymbol{a}}\},$$

$$-\mathcal{P} := \{\boldsymbol{a} \in \mathbb{KR} \mid -\boldsymbol{a} \in \mathcal{P}\},$$

$$\text{dual}\,\mathcal{Z} := \{\boldsymbol{a} \in \mathbb{KR} \mid \text{dual}\,\boldsymbol{a} \in \mathcal{Z}\}.$$

Overall, $\mathbb{KR} = \mathcal{P} \cup \mathcal{Z} \cup (-\mathcal{P}) \cup (\text{dual}\,\mathcal{Z})$. Then the multiplication in Kaucher interval arithmetic can be described by Table 1 [11], the cells of which are obtained as an outcome of detailed writing out the particular cases of applying formula (2.9) and our previous results. A remarkable fact is that this table is the supplement of a similar table for multiplication in classical interval arithmetic with one more row and one more column that correspond to the case of operands from the set dual $\mathcal{Z}$.

TABLE 1. Multiplication in Kaucher complete interval arithmetic.

$\cdot$	$\boldsymbol{b} \in \mathcal{P}$	$\boldsymbol{b} \in \mathcal{Z}$	$\boldsymbol{b} \in -\mathcal{P}$	$\boldsymbol{b} \in \text{dual}\,\mathcal{Z}$
$\boldsymbol{a} \in \mathcal{P}$	$[\underline{\boldsymbol{a}}\,\underline{\boldsymbol{b}},\ \overline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}]$	$[\overline{\boldsymbol{a}}\,\underline{\boldsymbol{b}},\ \overline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}]$	$[\overline{\boldsymbol{a}}\,\underline{\boldsymbol{b}},\ \underline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}]$	$[\underline{\boldsymbol{a}}\ \underline{\boldsymbol{b}},\ \underline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}]$
$\boldsymbol{a} \in \mathcal{Z}$	$[\underline{\boldsymbol{a}}\,\overline{\boldsymbol{b}},\ \overline{\boldsymbol{a}}\ \overline{\boldsymbol{b}}]$	$[\min\{\overline{\boldsymbol{a}}\,\overline{\boldsymbol{b}},\ \overline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}\},$ $\max\{\underline{\boldsymbol{a}}\,\underline{\boldsymbol{b}},\ \overline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}\}]$	$[\overline{\boldsymbol{a}}\,\underline{\boldsymbol{b}},\ \underline{\boldsymbol{a}}\ \underline{\boldsymbol{b}}]$	0
$\boldsymbol{a} \in -\mathcal{P}$	$[\underline{\boldsymbol{a}}\,\overline{\boldsymbol{b}},\ \overline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}]$	$[\underline{\boldsymbol{a}}\,\overline{\boldsymbol{b}},\ \underline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}]$	$[\overline{\boldsymbol{a}}\,\overline{\boldsymbol{b}},\ \underline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}]$	$[\overline{\boldsymbol{a}}\,\overline{\boldsymbol{b}},\ \overline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}]$
$\boldsymbol{a} \in \text{dual}\,\mathcal{Z}$	$[\underline{\boldsymbol{a}}\,\underline{\boldsymbol{b}},\ \overline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}]$	0	$[\overline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}\ ,\ \underline{\boldsymbol{a}}\ \overline{\boldsymbol{b}}]$	$[\max\{\underline{\boldsymbol{a}}\,\underline{\boldsymbol{b}},\ \overline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}\},$ $\min\{\underline{\boldsymbol{a}}\,\overline{\boldsymbol{b}},\ \overline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}\}]$

As is clear from the table above, the multiplication in Kaucher arithmetic admits non-trivial zero divisors. For example, $[1,-2]\cdot[-4,3]=0$. The interval multiplication in Kaucher arithmetic turns out to be commutative and associative [10], but the multiplication group in $\mathbb{KR}$ is formed only by intervals $\boldsymbol{a}$ for which $\underline{\boldsymbol{a}}\,\overline{\boldsymbol{a}} > 0$ (or, otherwise, $0 \notin \text{pro}\,\boldsymbol{a}$), because no any wider subset of $\mathbb{KR}$ satisfies the so-called "cancellation law":

$$\boldsymbol{ab} = \boldsymbol{ac} \Rightarrow \boldsymbol{b} = \boldsymbol{c}.$$

This is the algebraic condition that a semigroup can be embedded into a group. Therefore, for any interval $\boldsymbol{a}$ of $\mathbb{KR}$ that does not contain zero and is not contained in zero itself, there is a single inverse element with respect to multiplication, which we will denote by "inv". From the equality $\boldsymbol{a}\cdot\text{inv}\boldsymbol{a}=1$, it follows that

$$\text{inv}\,\boldsymbol{a} := [1/\underline{\boldsymbol{a}}, 1/\overline{\boldsymbol{a}}].$$

For brevity, we will denote the inverse operation of the multiplication, the so-called interval (algebraic) division in $\mathbb{KR}$, by $\oslash$, so that

$$\boldsymbol{a} \oslash \boldsymbol{b} := \boldsymbol{a} \cdot \text{inv}\,\boldsymbol{b} = \boldsymbol{a} \cdot [1/\underline{\boldsymbol{b}},\ 1/\overline{\boldsymbol{b}}] \qquad \text{for } 0 \notin \text{pro}\,\boldsymbol{b}.$$

The above table of explicit formulas for multiplication in the complete interval arithmetic is convenient for computer implementation that we will use these some multiplications in our computing.

3. Theory and algorithm

Let us consider first the traditional non-interval case. We suppose that a system of linear algebraic equations is given,

$$Ax = b, \tag{3.1}$$

where $A \in \mathbb{R}^{n\times n}$, $x, b \in \mathbb{R}^n$. Assuming that the matrix A is nonsingular, i. e. $\det A \neq 0$, we can guarantee that the solution to the system (3.1) is unique. One of the main ideas used in the construction of stationary iterative methods for numerical solutions of system (3.1) is the use of the so-called splitting of A, that is, the representation

$$A = M - N,$$

where M is a nonsingular matrix [36, 38]. Then the original system (3.1) can be equivalently rewritten as

$$x = Cx + d, \tag{3.2}$$

$C = M^{-1}N$ and $d = M^{-1}b$. From (3.2) a *linear stationary single-step iterative method*, completely consistent with (3.1), is constructed by

$$\tilde{x}^{(m+1)} = C\tilde{x}^{(m)} + d, \quad m = 0, 1, 2, \dots \tag{3.3}$$

where $\tilde{x}^{(0)} \in \mathbb{R}^n$ is arbitrary. One of the well-known sufficient and necessary conditions for the sequence $\{\tilde{x}^{(m)}\}$ to converge to the solution $\tilde{x} = A^{-1}b = (I - C)^{-1}d$ for any $\tilde{x}^{(0)} \in \mathbb{R}^n$ is $\rho(C) < 1$. For any iterative method of the form (3.3), an *extrapolated method* can be associated by replacing, at each step m, x^{m+1} by the extrapolated value $\beta\tilde{x}^{(m+1)} + (1 - \beta)x^{(m)}$ [1, 34, 35]:

$$x^{(m+1)} \leftarrow \beta\tilde{x}^{(m+1)} + (1 - \beta)x^{(m)}, \qquad \beta \neq 0, \quad \beta \in \mathbb{R}. \tag{3.4}$$

Using simple transformations, the resulting method can also be given the canonical form (3.3). Namely, we have

$$x^{(m+1)} = C(\beta)x^{(m)} + \beta d, \quad m = 0, 1, 2, \dots, \tag{3.5}$$

with iteration matrix $C(\beta) = \beta C + (1 - \beta)I$. The iteration (3.5) is called β-extrapolation of the scheme (3.3). The main idea of the extrapolation methods is thus very close to the idea of popular overrelaxation iterative methods advanced by Young and Frankel (see [38] and their earlier works).

The comparison theorem for extrapolated methods presented in [1] shows that the convergence of method (3.5) and the best value of the parameter β depends on the eigenvalues of iteration matrix C.

Theorem 3.1. *[1] Let C be the iteration matrix of the iterative scheme (3.3) and $r = \rho(C)$ be its spectral radius.*

- *Let (3.3) converge ($r < 1$). Then its β-extrapolated (3.5) converges asymptotically fast for some $\beta = \beta_0$, if all eigenvalues λ of C satisfy, exclusively, either* (1) $\mathrm{Re}\lambda < r^2$ *or* (2) $\mathrm{Re}\lambda > r^2$*; in case* (1) *we have $\beta_0 < 1$ and in case* (2) $\beta_0 > 1$.
- *Let (3.3) diverge ($r \geqslant 1$). Then its β-extrapolated (3.5) converges for some $\beta = \beta_0$ with $|\beta| < 1$ if and only if all eigenvalues λ of matrix C satisfy, exclusively, either* (3) $\mathrm{Re}\lambda < 1$ *or* (4) $\mathrm{Re}\lambda > 1$*; in case* (3) *we have $\beta_0 > 0$ and in case* (4) $\beta_0 < 0$.

An important feature of the extrapolation technique is the possibility to easily accelerate the convergence of original iterative methods (3.3). This fact motivated our study of iterative methods for interval linear systems of equations. In an interval analysis, an analogue of the usual solutions to equations and systems of equations is the so-called formal (algebraic) solutions, and this is why we apply the extrapolation technique for the iterative numerical methods computing formal solutions to interval system of linear equations. The following is an introduction to basic interval concepts. For the convenience of the reader, we repeat some relevant material from [31, 33] and describe our purpose using slight additions.

We remind the reader that in interval analysis, the direct analogue of the usual notion of the solution to an equation is various *solution sets* for interval equations and systems of equations, which are composed of solutions to individual

equations or systems of equations that form these interval systems. In particular, for the interval linear system of equations

$$\boldsymbol{A}x = \boldsymbol{b}, \tag{3.6}$$

with $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ and $\boldsymbol{b} \in \mathbb{IR}^n$, the most common is the so-called united solution set

$$\Xi(\boldsymbol{A}, \boldsymbol{b}) := \{\, x \in \mathbb{R}^n \mid (\exists A \in \boldsymbol{A})\,(\exists b \in \boldsymbol{b})\,(Ax = b)\,\}, \tag{3.7}$$

formed by all solutions to point systems $Ax = b$ for $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$. There are also other solution sets that may be more appropriate for specific practical situations, see [26, 28].

The structure of the solution sets is very complex in general, so their direct description is typically impossible. On the other hand, it is not really necessary for practitioners. As a rule, it is sufficient to know some simple estimates for the solution sets, and the most popular and demanded problem is that of computing outer estimates of solution sets, when we need to find the most narrow interval vector (box) containing the solution set, i. e. giving its outer coordinate-wise evaluation. Turning to the interval linear systems of equations (3.6), we know that the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$ is a polyhedral set, but it need not to be convex in general [21]. Its best outer estimate is the interval hull of the solution set, that is, $\square\, \Xi(\boldsymbol{A}, \boldsymbol{b})$, and computing interval vectors as close as possible to it is the primary goal of our efforts.

Several numerical methods to find enclosures for the solution set (3.7) of (3.6) have been proposed, such as the interval Gauss-Seidel iteration, Krawczyk method, interval Gauss method [18, 21], and some others. But if we take any of these methods as a basis to implement the extrapolated process

$$\boldsymbol{x}^{(m+1)} \leftarrow C(\beta)\boldsymbol{x}^{(m)} + \beta d, \qquad \beta \neq 0, \quad \beta \in \mathbb{R}, \tag{3.8}$$

it mostly diverges for arbitrary value of the parameter β. In other words, due to the weaknesses of classical interval computations, the solution sets obtained from extrapolation (3.8) were either too large or too small so they were invalid for the solution sets of the interval system of linear equations. To clarify this issue, we consider an example in the following.

Example 3.2. *We consider* 2×2*-system* $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ *where*

$$\boldsymbol{A} = \begin{pmatrix} [-4,-2] & [8,10] \\ [2,4] & [4,6] \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} [-6,4] \\ [-10,2] \end{pmatrix}. \tag{3.9}$$

By applying the interval Gauss-Seidel iteration [21]

$$\boldsymbol{y}_i := \big(\boldsymbol{b}_i - \sum_{k<i} \boldsymbol{a}_{ik}\boldsymbol{y}_k - \sum_{k>i} \boldsymbol{a}_{ik}\boldsymbol{x}_k\big)/\boldsymbol{a}_{ii} \cap \boldsymbol{x}_i, \qquad (i = 1, 2), \tag{3.10}$$

for the interval system (3.9) *with use*

$$\boldsymbol{x}^{(0)} = \begin{pmatrix} [-100, 100] \\ [-100, 100] \end{pmatrix}, \tag{3.11}$$

as an initial approximation obtains the interval box

$$\boldsymbol{x} \subseteq \boldsymbol{y} = \begin{pmatrix} [-5.013, 2.7988] \\ [-2.1689, +1.3355] \end{pmatrix}, \tag{3.12}$$

which is contain the solution set $\boldsymbol{x}$ *of interval system* (3.9) *that shown in the Figure 1.*

But when the extrapolated iteration version is implemented on the interval Gauss-Seidel iterations (3.10)*, solution sets are obtained for some different values of* $\beta\,(0<\beta<2)$*:*

$$\beta=0.3\ :\ \boldsymbol{y}_1^*=\begin{pmatrix}[-2.9119,1.3619]\\ [-1.079,0.4956]\end{pmatrix},\qquad \beta=0.7\ :\ \boldsymbol{y}_2^*=\begin{pmatrix}[-1.3748,0.71042]\\ [-0.56039,0.31039]\end{pmatrix},$$

$$\beta=1.3\ :\ \boldsymbol{y}_3^*=\begin{pmatrix}[-1.0101,1.6744]\\ [-0.51279,0.76279]\end{pmatrix},\qquad \beta=1.7\ :\ \boldsymbol{y}_4^*=\begin{pmatrix}[-3.1174,4.6674]\\ [-1.6725,2.2558],\end{pmatrix},$$

which none of the obtained solutions $\boldsymbol{y}_i^*$, $i=1,\ldots,4$ *are fully containing of the solution set* $\boldsymbol{x}$ *of interval system* (3.9).

As illustrated in Figure 1, for example, solution $\boldsymbol{y}_2^*$ with $\beta=0.7$ does not include all of the solution set $\Xi(\boldsymbol{A},\boldsymbol{b})$ of interval system (3.9) and represents an invalid estimation. In this example, we experimented with some different values of β in the extrapolated value $\beta\tilde{\boldsymbol{x}}^{(m+1)}+(1-\beta)\boldsymbol{x}^{(m)}$ that are not good solutions for the system (3.9). This phenomenon can due to the lack of some algebraic properties of classical interval arithmetic. Therefore, to correct this drawback, we propose the extrapolated method for which single-step iterative methods that finding the formal solution of the interval system of linear equations using the Kaucher interval arithmetic. In the following, we discuss this issue.

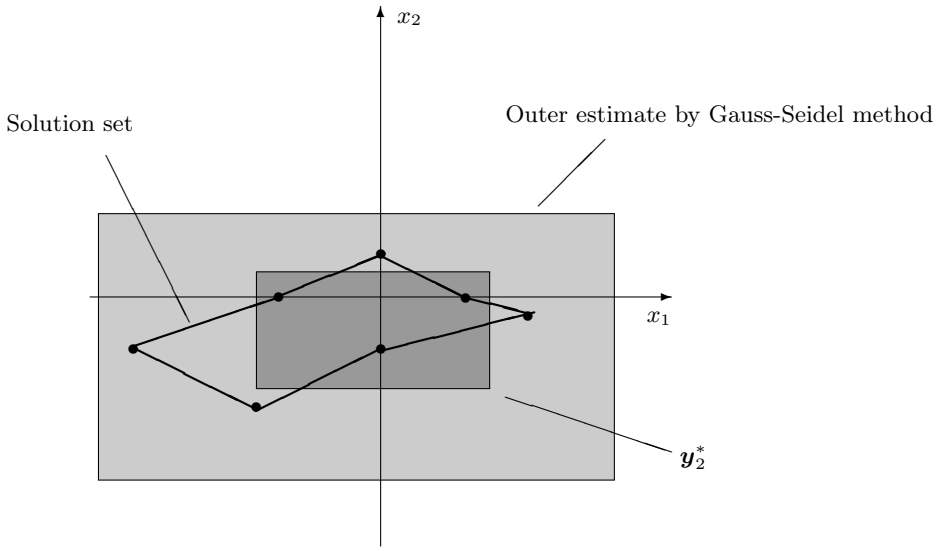

FIGURE 1. Solution set, outer estimate by GS method, extrapolated solution "$\boldsymbol{y}_2^*$".

3.1. **Extrapolated method to find formal solution based on triangular splitting.** In this paper, our goal present an extrapolated method on the formal solution obtained from iterative methods based on splitting matrix of the interval system of linear algebraic equations.

The main object of our study is the interval system of linear equations having the form

$$\begin{cases}\boldsymbol{a}_{11}x_1+\boldsymbol{a}_{12}x_2+\ldots+\boldsymbol{a}_{1n}x_n=\boldsymbol{b}_1,\\ \boldsymbol{a}_{21}x_1+\boldsymbol{a}_{22}x_2+\ldots+\boldsymbol{a}_{2n}x_n=\boldsymbol{b}_2,\\ \quad\vdots\qquad\quad\vdots\qquad\ddots\qquad\vdots\qquad\quad\vdots\\ \boldsymbol{a}_{n1}x_1+\boldsymbol{a}_{n2}x_2+\ldots+\boldsymbol{a}_{nn}x_n=\boldsymbol{b}_n,\end{cases}\tag{3.13}$$

with intervals $\boldsymbol{a}_{ij}$, $\boldsymbol{b}_i$, which will be considered as elements of Kaucher complete interval arithmetic $\mathbb{KR}$. Equivalent matrix form follows

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \tag{3.14}$$

where $\boldsymbol{A} = (\boldsymbol{a}_{ij})$ is an interval $n \times n$-matrix and $\boldsymbol{b} = (\boldsymbol{b}_i)$ is an interval n-vector. The general scheme of stationary single-step iterative methods is reducing the original equation (3.13)-(3.14) to a fixed-point form

$$\boldsymbol{x} = \boldsymbol{T}\boldsymbol{x}, \tag{3.15}$$

where $\boldsymbol{T} : \mathbb{KR}^n \longrightarrow \mathbb{KR}^n$ is an operator. Then, after choosing some initial approximation $\boldsymbol{x}^{(0)}$, iterations start:

$$\boldsymbol{x}^{(k+1)} \longleftarrow \boldsymbol{T}\boldsymbol{x}^{(k)}, \qquad k = 0, 1, 2, \dots .$$

So, we have to reformulate the original interval system (3.14) in the form (3.15). This is not a trivial task in general, since we have two occurrences of the unknown variable $\boldsymbol{x}$ in equation (3.15), while the original system (3.14) has only one occurrence of $\boldsymbol{x}$. At the same time, in both classical interval arithmetic and Kaucher complete interval arithmetic, we have $(\boldsymbol{g}+\boldsymbol{h})\boldsymbol{x} \neq \boldsymbol{g}\boldsymbol{x}+\boldsymbol{h}\boldsymbol{x}$ in the general case, that is, the distributivity of the addition with respect to multiplication does not hold. How to split the unknown variable $\boldsymbol{x}$ into two copies?

We can recall various cases of distributivity, when additional restrictions are imposed on the variables $\boldsymbol{g}$, $\boldsymbol{h}$, and $\boldsymbol{x}$. On the other hand, in a multidimensional situation, when the variables are vectors or matrices, distributivity can also be performed under some additional circumstances, which do not severely limit $\boldsymbol{g}$, $\boldsymbol{h}$ and $\boldsymbol{x}$. For example, if

$$\boldsymbol{g} = (\boldsymbol{g}_1, 0), \qquad \boldsymbol{h} = (0, \boldsymbol{h}_2), \qquad \boldsymbol{x} = \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix},$$

then

$$(\boldsymbol{g} + \boldsymbol{h})\boldsymbol{x} = \boldsymbol{g}\boldsymbol{x} + \boldsymbol{h}\boldsymbol{x}, \tag{3.16}$$

for any $\boldsymbol{g}_1$, $\boldsymbol{h}_2 \in \mathbb{KR}$ and $\boldsymbol{x} \in \mathbb{KR}^2$. The above example can be generalized, and its idea gives us the key to solving our "splitting question". Namely, let $\boldsymbol{g}$, $\boldsymbol{h}$, and $\boldsymbol{x}$ be such interval vectors that, in $\boldsymbol{g}$ and $\boldsymbol{h}$, the zero and nonzero elements are mutually complementary to each other, i. e. if, on the i-th place in $\boldsymbol{g}$, there is a nonzero element, then, in $\boldsymbol{h}$, zero is on the i-th place, and vice versa. Then the distributivity relation (3.16) holds.

Following [30, 31], we will say that the matrix $\boldsymbol{A} = (\boldsymbol{a}_{ij})$ has *disjoint splitting*, if it is represented in the form $\boldsymbol{A} = \boldsymbol{G} + \boldsymbol{H}$, where $\boldsymbol{G} = (\boldsymbol{g}_{ij})$ and $\boldsymbol{H} = (\boldsymbol{h}_{ij})$ are matrices of the same size as $\boldsymbol{A}$ in which nonzero elements in $\boldsymbol{G}$ and $\boldsymbol{H}$ are mutually exclusive, that is, for any indices i, j from their respective ranges,

$$\text{either } \boldsymbol{a}_{ij} = \boldsymbol{g}_{ij},\ \boldsymbol{h}_{ij} = 0,$$

$$\text{or } \boldsymbol{a}_{ij} = \boldsymbol{h}_{ij},\ \boldsymbol{g}_{ij} = 0.$$

Then, obviously, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{G}\boldsymbol{x} + \boldsymbol{H}\boldsymbol{x}$ for any $\boldsymbol{x} \in \mathbb{KR}^n$.

For a given matrix, many different disjoint splittings can be constructed, but it will be most convenient for us to work with some special splittings in which the operator of multiplication by the matrix $\boldsymbol{G}$ is easily invertible [31]. This is necessary in order to obtain the fixed-point form (3.15) and, based on it, organize stationary single-step iterations.

Next, we will use the so-called *triangular disjoint splitting* of the matrix $\boldsymbol{A}$ (or simply *triangular splitting*) also proposed in [30, 31], in which $\boldsymbol{A} = \boldsymbol{G} + \boldsymbol{H}$ and $\boldsymbol{G} = (\boldsymbol{g}_{ij})$ is the upper triangular matrix and $\boldsymbol{H} = (\boldsymbol{h}_{ij})$ is the strictly lower triangular matrix made up of elements of $\boldsymbol{A}$, and their nonzero elements are mutually complement. Specifically, the elements of $\boldsymbol{G}$ and $\boldsymbol{H}$ are defined as follows:

$$\boldsymbol{g}_{ij} = \begin{cases} \boldsymbol{a}_{ij}, & \text{for } i \leq j, \\ 0, & \text{for } i > j, \end{cases} \qquad \boldsymbol{h}_{ij} = \begin{cases} \boldsymbol{a}_{ij}, & \text{for } i > j, \\ 0, & \text{for } i \leq j. \end{cases}$$

Given the triangular splitting of the interval matrix $\boldsymbol{A}$, we can rewrite the initial system (3.14) in the equivalent form

$$\boldsymbol{G}\boldsymbol{x} + \boldsymbol{H}\boldsymbol{x} = \boldsymbol{b}. \tag{3.17}$$

If, for the operator of multiplication by the matrix $\boldsymbol{G}$ we construct the inverse operator $\mathcal{T}$, then from the equation

$$\boldsymbol{G}\boldsymbol{x} = \boldsymbol{b} \ominus \boldsymbol{H}\boldsymbol{x},$$

equivalent to (3.14), we get

$$\boldsymbol{x} = \mathcal{T}(\boldsymbol{b} \ominus \boldsymbol{H}\boldsymbol{x}),$$

and the iterative process can be organized by the formulas

$$\begin{aligned} &\tilde{\boldsymbol{x}}^{(k+1)} = \mathcal{T}(\boldsymbol{b} \ominus \boldsymbol{H}\tilde{\boldsymbol{x}}^{(k)}), \\ &\boldsymbol{x}^{(k+1)} \leftarrow \beta\tilde{\boldsymbol{x}}^{(k+1)} + (1-\beta)\boldsymbol{x}^{(k)}, \end{aligned} \tag{3.18}$$

which obtains the extrapolated formal solution for the interval system (3.14) with a "method based on triangular splitting". It is worth noting that the "method based on triangular splitting" is presented in [31] called `TrnSplit` so we call this process (3.18) the `ETrnSplit` (has the form presented in Table 2), where "$\oslash$" is an internal division in $\mathbb{KR}$, i.e. multiplication by the inverse interval.

A range for the parameter β can be obtained in a way that `ETrnSplit` is faster than `TrnSplit`, and it will be studied in the future. It is noted that the optimal value of β in this paper is found experimentally.

4. Numerical tests

In this section, we present numerical tests for comparing the extrapolation method with other numerical methods.

We show results demonstrating the work of the algorithm `ETrnSplit` on a number of test problems. The `ETrnSplit` method was implemented using the IntelliJ IDEA Community package under JAVA [4] on a laptop computer with Intel® Core i5-3337U CPU at 1.8 GHz and 6 GB RAM.

Example 4.1. *Let us consider an interval* 5×5*-system with matrix*

$$\begin{pmatrix} [1.8,2.2] & [-1.1,-0.9] & 0 & 0 & 0 \\ [-1.1,-0.9] & [1.8,2.2] & [-1.1,-0.9] & 0 & 0 \\ 0 & [-1.1,-0.9] & [1.8,2.2] & [-1.1,-0.9] & 0 \\ 0 & 0 & [-1.1,-0.9] & [1.8,2.2] & [-1.1,-0.9] \\ 0 & 0 & 0 & [-1.1,-0.9] & [1.8,2.2] \end{pmatrix}_{5\times 5}, \tag{4.1}$$

and the right-hand side vector

$$\begin{pmatrix} [0.9,1.1] \\ [1.8,2.2] \\ [2.7,3.3] \\ [3.6,4.4] \\ [4.5,5.5] \end{pmatrix}_{5\times 1}. \tag{4.2}$$

The matrix (4.1) *is obtained from a popular tridiagonal matrix approximating the second derivative on a uniform grid by 10% broadening of the elements, and the right-hand side* (4.2) *of the system is obtained by the same broadening of the vector* $(1,2,3,4,5)^\top$ *[31]. The formal solution of this system is an interval vector dual to the following box*

$$\begin{pmatrix} [6.4259259254, 5.3484848484] \\ [11.8518518518, 9.6969696962] \\ [14.8333333321, 12.40909090910] \\ [14.8148148148, 12.12121212020] \\ [9.9074074068, 8.5606060603] \end{pmatrix},$$

TABLE 2. Algorithm `ETrnSplit` for computing Extrapolated formal solution.

Input

An interval linear algebraic system $\boldsymbol{A}x = \boldsymbol{b}$.
A triangular splitting of the matrix $\boldsymbol{A}$ of the system
to interval matrices $\boldsymbol{G} = (\boldsymbol{g}_{ij})$ and $\boldsymbol{H} = (\boldsymbol{h}_{ij})$.
A specified accuracy ϵ.
A constant $\beta > 0$.

Output

An approximation to formal solution of the system
$\boldsymbol{A}x = \boldsymbol{b}$.

Algorithm

$q \leftarrow +\infty$;
assign an interval value to the vector $\boldsymbol{x}$;
`DO WHILE` ($q \geq \epsilon$)
  $\boldsymbol{p}_1 \leftarrow \boldsymbol{b}_1$;
  `DO FOR` $i = 2$ `TO` n
    $$\boldsymbol{p}_i \leftarrow \boldsymbol{b}_i \ominus \sum_{j=1}^{i-1} \boldsymbol{h}_{ij}\boldsymbol{x}_j$$
  `END DO`
  $\tilde{\boldsymbol{x}} \leftarrow \boldsymbol{p}_n \oslash \boldsymbol{g}_{nn}$;
  `DO FOR` $i = n-1$ `TO` 1 `STEP` (-1)
    $$\tilde{\boldsymbol{x}}_i \leftarrow \left(\boldsymbol{p}_i \ominus \sum_{j=1}^{i-1} \boldsymbol{g}_{ij}\tilde{\boldsymbol{x}}_j \right) \oslash \boldsymbol{g}_{ii}$$
  `END DO`
  $\tilde{\boldsymbol{x}} \leftarrow \beta\tilde{\boldsymbol{x}} + (1-\beta)\boldsymbol{x}$;
  $q \leftarrow$ distance between the vectors $\boldsymbol{x}$ and $\tilde{\boldsymbol{x}}$;
  $\boldsymbol{x} \leftarrow \tilde{\boldsymbol{x}}$;
`END DO`

with an accuracy of about 10^{-9}.

In Table 3, we list the iteration numbers (Iter) and the CPU times in seconds with respect to ***`TrnSplit`*** *and* ***`ETrnSplit`*** *methods. The number of iterations in the* ***`TrnSplit`*** *algorithm is 81 while by experimentally selecting the optimal value of* $\beta \in [1.46, 1.5]$ *the number of iterations with the* ***`ETrnSplit`*** *algorithm is 51.*

Example 4.2. *Following the work [15], let us consider a system of linear algebraic equations* $Ax = b$ *with block-tridiagonal matrix of the form*

$$A = diag(G, D, G),$$

TABLE 3. Iter, CPU and β for `TrnSplit` and `ETrnSplit` for Example .4.1.

Parameters	*Iter*	*CPU*	β
TrnSplit	*81*	*5.5*	*1*
	73	*4.62*	*1.1*
	65	*4.60*	*1.2*
	59	*4.40*	*1.3*
ETrnSplit	*54*	*4.31*	*1.4*
	51	*4.22*	*[1.46,1.5]*
	71	*4.53*	*1.6*
	103	*5.83*	*1.7*

where $D = diag(-4, 20, -4)$ *and* $G = diag(-1, -4, -1)$ *are the tridiagonal matrices. We take the block size of the matrix* A *as* q *by* q*, and the sizes of the matrices* D *and* G *are equal* p *by* p*. The matrix* A *thus obtained is the nine-point finite-difference approximation, on a uniform mesh, of the differential operator from the 2D Poisson equation*

$$-\Delta u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2}\right) = f(x, y), \tag{4.3}$$

with $(x, y) \in \Omega$ *and* $\Omega = [0, 1] \times [0, 1]$ *being a square region. To determine a solution to* (4.3) *uniquely, a boundary condition is usually added to the problem statement, which specify the unknown function* u *at the boundary* $\partial\Omega$ *of the region* Ω*. Both the boundary condition and the function* f *from equation* (4.3) *jointly form the right-hand side vector* b *of the equations system* $Ax = b$*.*

When testing our algorithms, the matrix A *of the system is most important, while the vector* b *of the right-hand side can be quite arbitrary. For this reason, we do not pay much attention to the choice of the function* f *and the boundary condition for equation* (4.3)*. In our test, the right-hand side vector* b *is chosen so that* $b = Ae$ *with* e *being the vector of all ones. Also, we set* $p = q = 20$ *in our example.*

Finally, the interval system of linear algebraic equations $\boldsymbol{A}x = b$*, to which we apply our extrapolation algorithm, is obtained by intervalizing the entry at the place* $(1, 1)$ *in the matrix* A*. In Table 4, we report the results of the algorithms* `TrnSplit` *and* `ETrnSplit`*. One can see that the number of iterations for* `TrnSplit` *is 136, while the number of iterations with the extrapolation method is 85, when* β *belongs to the real interval* $[1.54, 1.56]$*.*

5. CONCLUSION

The extrapolation approach, which accelerates convergence, is presented in the study for the formal solutions derived from iterative methods to the interval system of linear algebraic equations. The main idea of the extrapolation method is obtained on the system of linear equations in iterative methods. However, unlike a system of linear equations, this method could not be used to implement classical interval arithmetic due to some peculiar characteristics of interval computing. In order to give an extrapolation approach to the formula answers, we choose the triangular splitting method to overcome this issue.

Empirically, the best values for the extrapolation parameter β in an actual interval are found. In order for the `ETrnSplit` to be faster than the `TrnSplit`, an interval for the parameter β can be found. Further research can look at the discovery of an interval for β and its ideal interval.

TABLE 4. Iter, CPU and β for `TrnSplit` and `ETrnSplit` for Example 4.2.

Parameters	*Iter*	*CPU*	β
TrnSplit	*136*	*5.19*	*1*
ETrnSplit	*120*	*4.76*	*1.1*
	112	*4.68*	*1.2*
	102	*4.40*	*1.3*
	95	*4.31*	*1.4*
	88	*4.25*	*1.5*
	85	*4.16*	*[1.54, 1.56]*
	95	*4.35*	*1.6*

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