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Multi-Soliton Solutions to the Generalized Boussinesq Equation of Tenth Order

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Abstract

In the recent literature, many researchers are interested to apply standard computational methods for exact or numerical solutions of many classical nonlinear partial differential equations. Some leading methods are based on Lie group analysis, Painleve Analysis, G'/G expansion techniques, homotopy perturbation methods, and so on. The equations include complicated Navier-Stokes equation, Schrödinger equation, KdV-like equations, and so on. As a result, the glory of nonlinear dynamics can be witnessed through its applications in many fields namely: ocean engineering, plasma physics, optical communications, fluid dynamics, and much more. One of the significant observations is that whatever may be the order of nonlinear PDE, as far as the soliton and multi-solitons of KdV like equation or Boussinesq equation are concerned Hirota's method and $\tanh-\coth$ method play a crucial role. The main result of the paper demonstrates that the above novel theme works well with the generalized Boussinesq equation of 10^{th} order. In this paper, the Boussinesq equation of order ten is derived and its multi-soliton solutions are deduced by the Hirota's method. The one soliton solution is reconfirmed using the tanh method.

Keywords. Higher order Boussinesq equation, The Hirota bilinear method, The tanh method.
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1. Introduction

Boussinesq equation is one of the important nonlinear PDEs belonging to the class of equations with linear terms $u_{tt}-u_{xx}$ and with nonlinear terms containing as high as fourth order partial derivative with respect to the $x-$ variable. It belongs to the family of completely integrable equations and hence $N-$solitons solution exists [1, 10, 13, 32]. It is an interesting fact that the equation has the bilinear form of exactly the fourth order. The standard method applied to obtain a multi-soliton solution is Hirota's method. For one soliton solution, many methods are applicable such as Inverse Scattering Method, the tanh method, the tanh- coth method, Adomian decomposition method, G'/G method, Homotopy perturbation method, Collocation methods, Lie symmetry method, Similarity analysis methods, and other novel methods [2–7, 9–12, 15, 17–21, 25, 26, 29, 30]. The Boussinesq equation is applicable to study shallow water waves, nonlinear lattice waves, and nonlinear string vibrations[16, 32].

The spearhead researchers R. Hirota, W. Malfliet, J. Hietarinta, R. S. Johnson, A. M. Wazwaz, and many more have given new insights and dimensions to solitary solutions through their works [12, 13, 16, 32]. The Hirota's method is a tool to compute multi-soliton solutions for nonlinear PDE by expressing the given equation in its bilinear form. Any bilinear form admits one and two soliton solutions [13, 14]. The essence of Hirota's direct method is the perturbation supported by a transformation. The tanh method is also one of the popular methods which express the solitary wave solution through a finite series expansion[8].

In this work, we derive the generalized tenth order Boussinesq equation (tB) and deduce its soliton solutions. The work is carried out sequentially as follows: In section 2, we derive tB using Hirota Operators. In section 3, we obtain its multi-soliton solutions using the Hirota's Direct method. In section 4, Hirota's one-soliton solution is reconfirmed

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through the tanh method. In the final section, some computational work is done and a discussion about the scope for future works is mentioned. The results of section 2 to section 5, clearly demonstrate that the Hirota's bilinear method is one of the powerful methods for computing one soliton and multi-soliton solutions of generalized tB equation. However, the existence of multi-soliton will not ensure the integrability of the equation [27]. We have mentioned it as an open problem for future work.

2. Derivation of the Generalized Tenth Order Boussinesq Equation

In this section, we derive the generalized Boussinesq equation of tenth order by using Hirota's operators. The $(k, m)-$ order bilinear partial derivative of a function $f(x,t)\cdot f(x,t)$ is defined in the literature [12] as follows:

$$D_t^k D_x^m(f(x,t)\cdot f(x,t)) = \left[\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^k\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m(f(x,t)\cdot f(x,t))\right]_{x'=x,t'=t}.$$

(i) For $m = 1$ and $k = 1$, we compute that

$$\begin{aligned}D_tD_x(f\cdot f) &= [f_{xt}(x,t)f(x',t') - f_t(x,t)f_{x'}(x',t') - f_x(x,t)f_{t'}(x',t') + f(x,t)f_{x't'}(x',t')]_{x'=x,t'=t}\\ &= 2\,[ff_{xt} - f_xf_t]\,.\end{aligned}$$

(ii) For $k = 0, m = 2n-1, n = 1, 2\ldots$

$$D_x^{2n-1}(f\cdot f) = 0.$$

(iii) For $k = 0, m = 2n, n = 1, 2\ldots$

$$D_x^{2n}(f\cdot f) = 2\left[ff_{2nx} - \binom{2n}{1}f_xf_{(2n-1)x} + \binom{2n}{2}f_{xx}f_{(2n-2)x} - + \cdots + (-1)^n\frac{1}{2}\binom{2n}{n}f_{nx}^2\right].$$

We need the following formulas

(iii)(a) $D_t^2(f\cdot f) = 2\left[ff_{tt} - f_t^2\right].$
(iii)(b) $D_x^2(f\cdot f) = 2\left[ff_{xx} - f_x^2\right].$
(iii)(c) $D_x^4(f\cdot f) = 2\left[ff_{4x} - 4f_xf_{3x} + 3f_{xx}^2\right].$
(iii)(d) $D_x^6(f\cdot f) = 2\left[ff_{6x} - 6f_xf_{5x} + 15f_{xx}f_{4x} - 10f_{xxx}^2\right].$
(iii)(e) $D_x^8(f\cdot f) = 2\left[ff_{8x} - 8f_xf_{7x} + 28f_{xx}f_{6x} - 56f_{3x}f_{5x} + 35f_{4x}^2\right].$
(iii)(f) $D_x^{10}(f\cdot f) = 2\left[ff_{10x} - 10f_xf_{9x} + 45f_{xx}f_{8x} - 120f_{3x}f_{7x} + 210f_{4x}f_{6x} - 126f_{5x}^2\right].$

If $u = w_x = 2\left(\frac{f_x}{f}\right)_x$, then it satisfies the following results [24, 31]

(a) $log(f^2)_{xt} = \dfrac{D_xD_t(f\cdot f)}{f^2} = w_t.$

(b) $\int\int u_{tt}\; dxdx = \dfrac{D_t^2(f\cdot f)}{f^2}.$

(c) $\dfrac{D_x^2(f\cdot f)}{f^2} = u = w_x.$

(d) $\dfrac{D_x^4(f\cdot f)}{f^2} = u_{xx} + 3u^2 = w_{3x} + 3w_x^2.$

(e) $\dfrac{D_x^6(f\cdot f)}{f^2} = u_{4x} + 15uu_{xx} + 15u^3 = w_{5x} + 15w_xw_{3x} + 15w_x^3.$

(f) $\dfrac{D_x^8(f\cdot f)}{f^2} = u_{6x} + 28uu_{4x} + 35u_{xx}^2 + 210u^2u_{xx} + 105u^4$
$= w_{7x} + 28w_xw_{5x} + 35w_{3x}^2 + 210w_x^2w_{3x} + 105w_x^4.$

(g) $\dfrac{D_x^{10}(f\cdot f)}{f^2} = u_{8x} + 45uu_{6x} + 210u_{xx}u_{4x} + 630u^2u_{4x} + 1575uu_{xx}^2 + 3150u^3u_{xx} + 945u^5$
$= w_{9x} + 45w_xw_{7x} + 210w_{3x}w_{5x} + 630w_x^2w_{5x} + 1575w_xw_{3x}^2 + 3150w_x^3w_{3x} + 945w_x^5.$

Consider the Boussinesq equation [32]

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0. \tag{2.1}$$

Integrating (2.1) twice with respect to x, we obtain

$$\int\int u_{tt}\ dx\ dx - u - 3u^2 - u_{xx} = 0. \tag{2.2}$$

Using Hirota's bilinear form, (2.2) can be expressed as

$$\left(D_t^2 - D_x^2 - D_x^4\right)(f\cdot f) = 0,\ \text{where } u = 2\frac{\partial^2}{\partial x^2}\log f(x,t). \tag{2.3}$$

We observe that the Boussinesq equation (2.1) has order 4 and its corresponding bilinear form also has order 4 as given in (2.3).In order to study the effect of increasing the order of bilinearity on multi-soliton, we generalize to 10^{th} order as follows:

$$\left(D_t^2 - D_x^2 - D_x^4 - \alpha D_x^6 - \beta D_x^8 - \gamma D_x^{10}\right)(f\cdot f) = 0, \tag{2.4}$$

where α, β and γ are real constants. Using the results of Hirota operators from (iii)(a) - (iii)(f) in (2.4) we obtain,

$$\begin{aligned} &2\Big[\left(ff_{tt} - f_t^2\right) - \left(ff_{xx} - f_x^2\right) - \left(f_{4x}f - 4f_{3x}f_x + 3f_{xx}^2\right) - \alpha\big(f_{6x}f - 6f_{5x}f_x \\ &+ 15f_{4x}f_{2x} - 10f_{xxx}^2\big) - \beta\big(ff_{8x} - 8f_{7x}f_x + 28f_{6x}f_{xx} - 56f_{5x}f_{3x} + 35f_{4x}^2\big) \\ &- \gamma\big(ff_{10x} - 10f_{9x}f_x + 45f_{8x}f_{2x} - 120f_{7x}f_{3x} + 210f_{6x}f_{4x} - 126f_{5x}^2\big)\Big] = 0. \end{aligned} \tag{2.5}$$

Substituting the Hirota operators given in the section 2 to (2.4) and by differentiating it twice with respect to x we get the corresponding PDE for tB equation

$$\begin{aligned} &u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{4x} - \alpha\big(15uu_{4x} + 30u_xu_{3x} + 15(u_{xx})^2 + 90uu_x^2 + \\ &45u^2u_{xx} + u_{6x}\big) - \beta\big(28uu_{6x} + 56u_{5x}u_x + 98u_{xx}u_{4x} + 70u_{3x}^2 + 210u^2u_{4x} \\ &+ 840u_xu_{3x} + 420uu_{xx}^2 + 420u^3u_{xx} + 1260u^2u_x^2 + 420u_{xx}u_x^2 + u_{8x}\big) \\ &- \gamma\big(u_{10x} + 45uu_{8x} + 90u_{7x}u_x + 255u_{6x}u_{xx} + 420u_{5x}u_{3x} + 210u_{4x}^2 \\ &+ 4410uu_{xx}u_{4x} + 3150uu_{3x}^2 + 6300u_xu_{xx}u_{3x} + 1575u_{xx}^3 + 630u^2u_{6x} \\ &+ 2520uu_xu_{5x} + 1260u_{4x}u_x^2 + 3150u^3u_{4x} + 18900u^2u_xu_{3x} + 9450u^2u_{xx}^2 \\ &+ 18900uu_x^2u_{xx} + 18900u^3u_x^2 + 4725u^4u_{xx}\big) = 0. \end{aligned} \tag{2.6}$$

3. Multi-Soliton Solutions Generalised tenth Order Boussinesq equation

In this section, we compute multi-soliton solutions of (2.5) using Hirota's direct method. Although there are traditional methods that can be used to obtain soliton solutions. It is very interesting to work out the solutions using Hirota's method or Direct method resides in the essence of analysis and perturbation.

In order to deduce the soliton solution, the unknown function f has to be determined, where the solution u and $f(x,t)$ are related by the logarithmic transformation, $u = 2\dfrac{\partial^2}{\partial x^2}\log f(x,t,y)$. And we assume that $f = 1 + \sum_{n=1}^{\infty}\epsilon^n f_n$, where $f_1, f_2\ldots$, are yet to be found. By equating the bilinear form to zero and using the result that for $N-$ soliton solution $f_N = 0$ [11, 12]. We obtain the following expressions for f in terms of $f_1, f_2, f_3\ldots$.

For one soliton solution ; $f(x,t) = 1 + \epsilon f_1$.

For two soliton solution ; $f(x,t) = 1 + \epsilon f_1 + \epsilon^2 f_2$.

For three soliton solution ; $f(x,t) = 1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3$ and so on.
For more details, refer [8, 11–14, 32].
Here we rely on the following facts that are available in the literature. Any bilinear form admits two soliton solution. Also, the three soliton solution existence is assured by expressing the unknown constant b_{123} in terms of a_{12}, a_{23} and a_{31} where b_{123}, a_{12}, a_{23} and a_{31} are coupling constants [11–14, 16, 28, 32].

3.1. **One soliton solution.** In this subsection, we deduce the one soliton solution by considering the auxiliary function $f = 1 + \epsilon f_1 = 1 + \epsilon e^{\theta}$ where ϵ, k , c are real constants and $\theta = k(x - ct)$ in (2.5) to get,

$$\epsilon k^2 f_1[c^2 - (1 + k^2 + \alpha k^4 + \beta k^6 + \gamma k^8)] = 0.$$

This implies

$$c = \pm\sqrt{1 + k^2 + \alpha k^4 + \beta k^6 + \gamma k^8}. \tag{3.1}$$

Thus , we have $f = 1 + \epsilon\ exp(kx \mp \sqrt{1 + k^2 + \alpha k^4 + \beta k^6 + \gamma k^8})$. Using the above f in $u = 2\frac{\partial^2}{\partial x^2}\log f(x,t)$ is the one soliton solution to (2.4).

3.2. **Two Soliton Solution.** In this subsection, we obtain two soliton solution. For two soliton solution we consider the auxiliary function

$$f = 1 + \epsilon f_1 + \epsilon^2 f_2, \tag{3.2}$$

where

$$\begin{aligned} &f_1 = e^{\theta_1} + e^{\theta_2},\\ &f_2 = a_{12}e^{\theta_1+\theta_2}, \theta_i = k_i(x - c_i t), (i = 1, 2), \end{aligned} \tag{3.3}$$

k_i, c_i are real constants and the coupling constant a_{12} to be determined. By equating the coefficients of ϵ^2 in $(D_t^2 - D_x^2 - D_x^4 - \alpha D_x^6 - \beta D_x^8 - \gamma D_x^{10})(f \cdot f) = 0$, we obtain

$$\left(D_t^2 - D_x^2 - D_x^4 - \alpha D_x^6 - \beta D_x^8 - \gamma D_x^{10}\right)(2(f_2 \cdot 1) + (f_1 \cdot f_1)) = 0.$$

By substituting values of f_1, f_2 to the above Bilinear form, it reduces to the following

$$\begin{aligned} 2\Big[a_{12}\big[(k_1c_1 + k_2c_2)^2 - (k_1 + k_2)^2 - (k_1 + k_2)^4 - \alpha(k_1 + k_2)^6 - \beta(k_1 + k_2)^8\\ \gamma(k_1 + k_2)^{10}\big] + (k_1c_1 - k_2c_2)^2 - (k_1 - k_2)^2 - (k_1 - k_2)^4\\ - \alpha(k_1 - k_2)^6 - \beta(k_1 - k_2)^8 - \gamma(k_1 - k_2)^{10}\Big] = 0. \end{aligned}$$

After simplification, we get

$$a_{12} = -\frac{(k_1c_1 - k_2c_2)^2 - (k_1 - k_2)^2 - (k_1 - k_2)^4 - \alpha(k_1 - k_2)^6 - \beta(k_1 - k_2)^8 - \gamma(k_1 - k_2)^{10}}{(k_1c_1 + k_2c_2)^2 - (k_1 + k_2)^2 - (k_1 + k_2)^4 - \alpha(k_1 + k_2)^6 - \beta(k_1 + k_2)^8 - \gamma(k_1 + k_2)^{10}}. \tag{3.4}$$

Thus,

$$\begin{aligned} f &= 1 + \epsilon\big(e^{\theta_1} + e^{\theta_2}\big) + \epsilon^2 a_{12}e^{\theta_1+\theta_2}\\ &= 1 + \epsilon\big(e^{\theta_1} + e^{\theta_2}\big)\\ &- \epsilon^2\frac{(k_1c_1 - k_2c_2)^2 - (k_1 - k_2)^2 - (k_1 - k_2)^4 - \alpha(k_1 - k_2)^6 - \beta(k_1 - k_2)^8 - \gamma(k_1 - k_2)^{10}}{(k_1c_1 + k_2c_2)^2 - (k_1 + k_2)^2 - (k_1 + k_2)^4 - \alpha(k_1 + k_2)^6 - \beta(k_1 + k_2)^8 - \gamma(k_1 + k_2)^{10}}e^{\theta_1+\theta_2}. \end{aligned}$$

Using the above f in $u = 2\frac{\partial^2}{\partial x^2}\log f(x,t)$ we obtain the two soliton solution of (2.4).

3.3. **Three Soliton solution.** In this subsection, we obtain three soliton solution. For three soliton solutions, we consider the auxiliary function

$$f = 1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3, \tag{3.5}$$

where

$$\begin{aligned} &f_1 = e^{\theta_1} + e^{\theta_2} + e^{\theta_3},\\ &f_2 = a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3},\\ &f_3 = b_{123}\ e^{\theta_1+\theta_2+\theta_3}, \theta_i = k_i(x - c_i t), i = 1, 2, 3, \end{aligned} \tag{3.6}$$

and b_{123} is a constant need to be determined. We compute that

$$a_{ij} = -\frac{(k_ic_i - k_jc_j)^2 - (k_i - k_j)^2 - (k_i - k_j)^4 - \alpha(k_i - k_j)^6 - \beta(k_i - k_j)^8 - \gamma(k_i - k_j)^{10}}{(k_ic_i + k_jc_j)^2 - (k_i + k_j)^2 - (k_i + k_j)^4 - \alpha(k_i + k_j)^6 - \beta(k_i + k_j)^8 - \gamma(k_i + k_j)^{10}}, 1 \le i < j \le 3. \tag{3.7}$$

where k_i, k_j, c_i and c_j are real constants.
By equating the the coefficients of ϵ^3 in $\left(D_t^2 - D_x^2 - D_x^4 - \alpha D_x^6 - \beta D_x^8 - \gamma D_x^{10}\right)(f \cdot f) = 0$, we obtain

$$\left(D_t^2 - D_x^2 - D_x^4 - \alpha D_x^6 - \beta D_x^8 - \gamma D_x^{10}\right)(1 \cdot f_3 + f_1 \cdot f_2 + f_2 \cdot f_1 + f_3 \cdot 1) = 0. \tag{3.8}$$

After substituting the values of f_1, f_2 and f_3 from (3.6) and (3.7) into (3.8) and using the fact that for N-soliton solution $f_n = 0$ for $n > N$ [11], we obtain

$$b_{123} = a_{12}a_{13}a_{23}. \tag{3.9}$$

Thus

$$f = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} + a_{12}a_{13}a_{23}\ e^{\theta_1+\theta_2+\theta_3}.$$

Using the above f in $u = 2\dfrac{\partial^2}{\partial x^2}\log f(x,t)$ we obtain the three soliton solution to (2.4).

4. The Tanh method

The tanh method is one of the splendid methods in soliton theory as it describes the solutions to the nonlinear PDE by truncating series expansion in terms of tanh. The stage of truncation can be found by equating the powers of higher order derivative and the highest power of nonlinear term. Before treating the nonlinear PDE with *tanh* method we suppress the time and spatial variable into a single variable by introducing a new variable $z = x - ct$ and hence we solve the transformed ODE [21–23, 31–33].

Once we introduce the variable $z = x - ct$ and denoting $u(x,t) = U(z)$ then the PDE (2.6) transforms to the following ODE

$$\begin{aligned} &(c^2-1)U_{zz} - \{U_{4z} + 6UU_{zz} + 6U_z^2\} - \alpha\{U_{6z} + 15UU_{4z} + 30U_zU_{3z} + 15U_{zz}^2\\ &+ 45U^2U_{zz} + 90UU_z^2\} - \beta\{U_{8z} + 28UU_{6z} + 56U_zU_{5z} + 98U_{2z}U_{4z} + 70U_{3z}^2\\ &+ 210U^2U_{4z} + 840UU_zU_{3z} + 420UU_{2z}^2 + 420U_{2z}U_z^2 + 420U^3U_{2z} + 1260U^2U_z^2\}\\ &- \gamma\{U_{10z} + 45UU_{8z} + 90U_zU_{7z} + 255U_{2z}U_{6z} + 420U_{3z}U_{5z} + 210U_{4z}^2 + 630U^2U_{6z}\\ &+ 2520UU_zU_{5z} + 4410UU_{2z}U_{4z} + 1260U_z^2U_{4z} + 3150UU_{3z}^2 + 6300U_zU_{2z}U_{3z}\\ &+ 3150U^3U_{4z} + 18900U^2U_zU_{3z} + 9450U^2U_{2z}^2 + 18900UU_{2z}U_z^2 + 4725U^4U_{2z}\\ &+ 18900U^3U_z^2 + 1575U_{2z}^3\} = 0, \end{aligned} \tag{4.1}$$

where $U_z = D_zU$.

Now we solve (4.1) using the tanh method. As noted earlier this method admits the solution $U(Y)$ in the finite series of tanh. Hence, we consider

$$U(Y) = \sum_{k=0}^{M} a_k Y^k \text{ where } Y = \tanh\frac{z}{2},\ M \in \mathbb{N} \text{ need to be determined.}$$

By balancing the highest order U_{10z} and the power of nonlinear term $U^3U_z^2$,

$$3M + 2(M+1) = M + 10.$$

We get $M = 2$. Hence, we have

$$u(x,t) = U(Y) = a_0 + a_1 Y + a_2 Y^2.$$

Using the above in the ODE (4.1) we get $a_1 = 0$ and by fixing $a_0 = \frac{k^2}{2}$ and $a_2 = -\frac{k^2}{2}$ the solution is

$$\begin{aligned} U &= \frac{k^2}{2}\left[1 - \tanh^2\left(\frac{z}{2}\right)\right] \\ &= \frac{k^2}{2}\,(1 - Y^2). \end{aligned}$$

Now, we obtain the expression for c from the ODE (4.1) which matches with c in Hirota's one soliton. We compute the terms of the ODE (4.1) as follows

$$(c^2-1)U_{zz} - (U_{4z} + 6UU_{zz} + 6U_z^2) = \frac{k^4}{8}(c^2 - 1 - k^2)(6Y^2 - 2)(1 - Y^2),$$

$$U_{6z} + 15UU_{4z} + 30U_zU_{3z} + 15U_{zz}^2 + 45U^2U_{zz} + 90UU_z^2 = \frac{k^8}{8}(6Y^2 - 2)(1 - Y^2),$$

$$\begin{aligned} &U_{8z} + 28UU_{6z} + 56U_zU_{5z} + 98U_{2z}U_{4z} + 70U_{3z}^2 + 210U^2U_{4z} + 840UU_zU_{3z} + 420UU_{2z}^2 + 420U_{2z}U_z^2 + 420U^3U_{2z} \\ &\qquad + 1260U^2U_z^2 = \frac{k^{10}}{8}(6Y^2 - 2)(1 - Y^2), \\ &U_{10z} + 45UU_{8z} + 90U_zU_{7z} + 255U_{2z}U_{6z} + 420U_{3z}U_{5z} + 210U_{4z}^2 + 630U^2U_{6z} + 2520UU_zU_{5z} + 4410UU_{2z}U_{4z} \\ &+ 1260U_z^2U_{4z} + 3150UU_{3z}^2 + 6300U_zU_{2z}U_{3z} + 3150U^3U_{4z} + 18900U^2U_zU_{3z} + 9450U^2U_{2z}^2 + 18900UU_{2z}U_z^2 \\ &\qquad + 4725U^4U_{2z} + 18900U^3U_z^2 + 1575U_{2z}^3 = \frac{k^{12}}{8}(6Y^2 - 2)(1 - Y^2). \end{aligned}$$

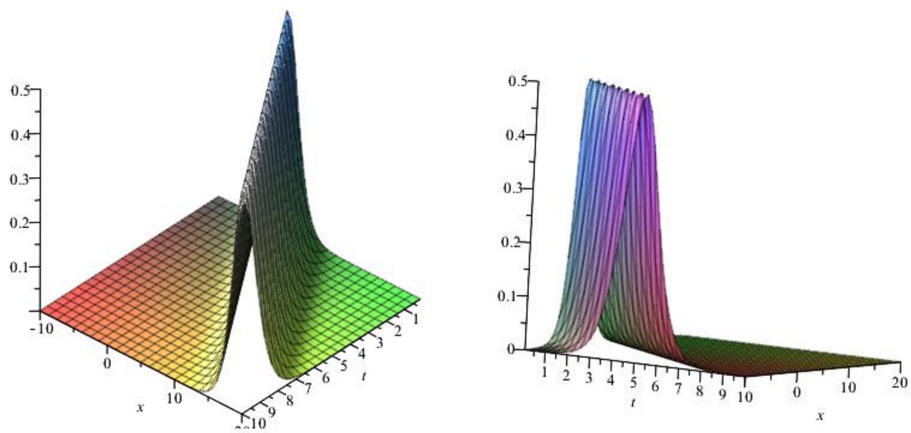
Using the above simplification in the ODE (4.1), we obtain

$$c = \pm\sqrt{1 + k^2 + \alpha k^4 + \beta k^6 + \gamma k^8}.$$

Hence, the solution to (2.6) is

$$u = \frac{k^2}{2}\sec h^2\left(\frac{k}{2}(x \mp ct)\right), \tag{4.2}$$

with $c = \sqrt{1 + k^2 + \alpha k^4 + \beta k^6 + \gamma k^8}$, which agrees with the one-soliton solution in section 3.1. Thus, we have examined the one soliton solution of the PDE (2.4) by two efficient methods namely the Hirota's method and the tanh method.



(a) (b)

Figure 1. (a) stands for the case $(x-2t)$, (b) stands for the case $(x+2t)$.

5. Some Computational Results and Discussions

In this section, we visualize the one soliton solution and two soliton solutions that are derived in the previous sections. Graphs are plotted using mathematical package maple and the command is given for each plot.

(a) **3D Plots for one soliton of equation (2.4) by Hirota's method.**

For one soliton solution of (2.4), we have made the following choices:
$\epsilon = 1, k = 1, \alpha + \beta + \gamma = 2$ then $f = 1 + exp(x \mp 2t)$.
Maple code :
$> plot3d((1/2) * sech(1/2 * (x - 2 * t))^2, t = .1..10, x = -10..20)$

$> plot3d((1/2) * sech(1/2 * (x + 2 * t))^2, t = .1..10, x = -10..20)$

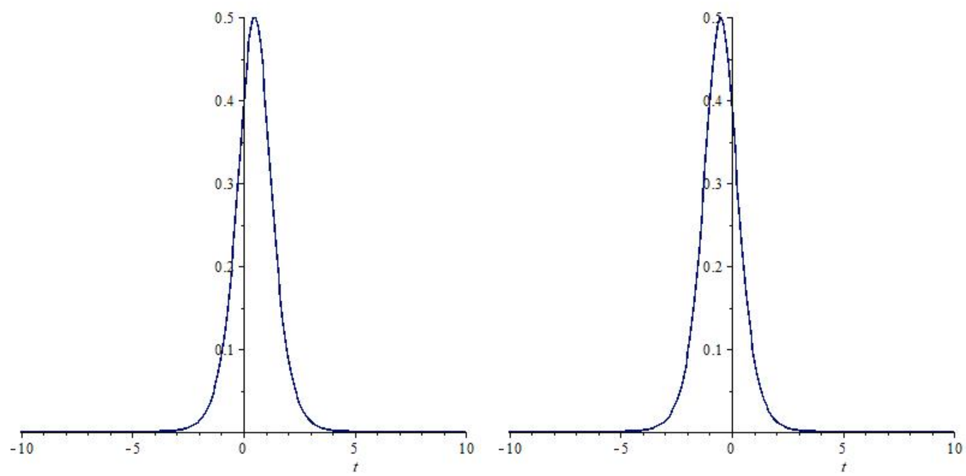
(b) **2D Plot for one soliton of equation (4.1) by** tanh **method.**

For one soliton solution given by (4.2), we have made the following choices $k = 1, x = 1$,
$\alpha + \beta + \gamma = 2, -10 \leq t \leq 10$.
Maple code :
$> plot((1/2) * sech(1/2 * (1 - 2 * t))^2, t = -10..10)$

$> plot((1/2) * sech(1/2 * (1 + 2 * t))^2, t = -10..10)$
These are slices of Figure 1(a) and Figure 1(b) respectively when $x = 1$.

(c) **3D Plot for two soliton of equation given by (3.2)-(3.4) by Hirota's method with** $\alpha = 0, \beta = 0$ **and** $\gamma = 0$.

For two soliton of equation given by (3.2)-(3.4), we have made the following choices.
$\epsilon = 1, k_1 = 1, k_2 = 2, \alpha = 0, \beta = 0, \gamma = 0, c_1 = \sqrt{2}, c_2 = \sqrt{5}, a_{12} \approx .132$,
Maple code :
$plot3d(2*(exp(x-1.41*t)+4*exp(2*x-4.472*t)+(.132*9)*exp(3*x-5.886*t))/(1+exp(x-1.41*t)+exp(2*x-4.472*t)+.132*exp(3*x-5.886*t))-2*(exp(x-1.41*t)+2*exp(2*x-4.472*t)+(.132*3)*exp(3*x-5.886*t))^2/(1+exp(x-1.41*t)+exp(2*x-4.472*t)+.132*exp(3*x-5.886*t))^2, x=-50..500, t=20..400)$



(a) (b)

FIGURE 2. (a) stand in the case $z = 1 - 2t$, (b) stand for the case $z = 1 + 2t$.

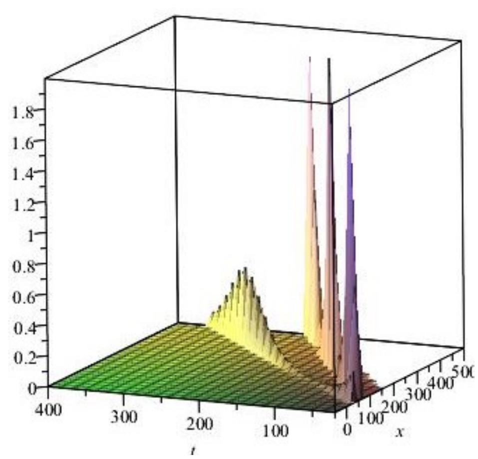


FIGURE 3. stands for 2-soliton of Boussinesq equation.

(d) **3D Plot for two soliton of equation (3.2)-(3.4) by Hirota's method with** $\alpha = 0, \beta = 0$ **and** $\gamma = 1$.

For two soliton of equation given by (3.2)-(3.4), we have made the following choices.
$\epsilon = 1, k_1 = 1, k_2 = 2, \alpha = 0, \beta = 0, \gamma = 1, c_1 = \sqrt{3}, c_2 = \sqrt{26}, a_{12} \approx .0168,$
Maple code:
$> plot3d(2*(exp(x-1.732*t)+4*exp(2*x-32.31*t)+(0.168e-1*9)*exp(3*x-34.043*t))/(1+exp(x-1.732*t)+exp(2*x-32.31*t)+0.168e-1*exp(3*x-34.043*t))-2*(exp(x-1.732*t)+2*exp(2*x-32.31*t)+(0.168e-1*3)*exp(3*x-34.043*t))^2/(1+exp(x-1.732*t)+exp(2*x-32.31*t)+0.168e-1*exp(3*x-34.043*t))^2, x=-200..600, t=1..100)$

(e) **3D Plot for two soliton of equation (3.2)-(3.4) by Hirota's method with** $\alpha = 0, \beta = 1$ **and** $\gamma = 1$.

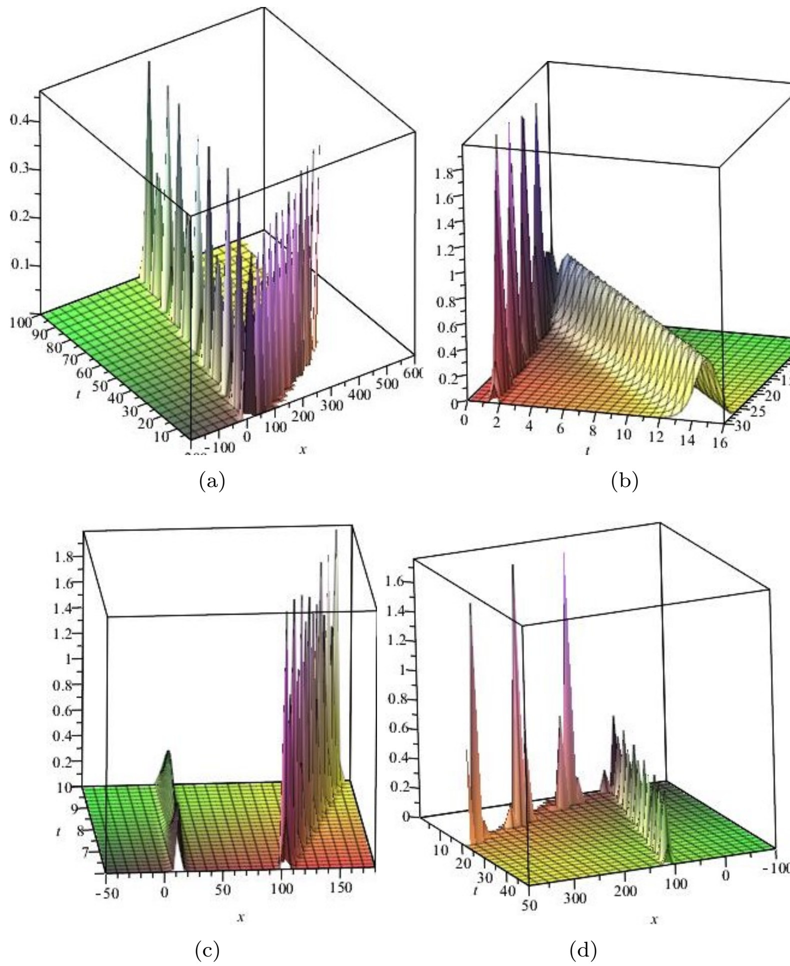


FIGURE 4. Typical 2-solitons of (2.4).

For two soliton of equation given by (3.2)-(3.4), we have made the following choices.
$\epsilon = 1, k_1 = 1, k_2 = 2, \alpha = 0, \beta = 1, \gamma = 1, c_1 = 2, c_2 = \sqrt{325}, a_{12} \approx .02,$
Maple code:
$> plot3d(2*(exp(x-2*t)+4*exp(2*x-(36*4)*t)+(0.2e-1*9)*exp(3*x-38.055*t))/(1+exp(x-2*t)+exp(2*x-36.04*t)+0.2e-1*exp(3*x-38.055*t))-2*(exp(x-2*t)+2*exp(2*x-36.04*t)+(0.2e-1*3)*exp(3*x-38.055*t))^2/(1+exp(x-2*t)+exp(2*x-36.04*t)+0.2e-1*exp(3*x-38.055*t))^2, x = 2..30, t = 0..16)$

(f) **3D Plot for two soliton of equation (3.2)-(3.4) by Hirota's method with $\alpha = 1, \beta = 0$ and $\gamma = 1$.**

For two soliton of equation given by (3.2)-(3.4), we have made the following choices.
$\epsilon = 1, k_1 = 1, k_2 = 2, \alpha = 1, \beta = 0, \gamma = 1, c_1 = 2, c_2 = \sqrt{277}, a_{12} \approx .0162,$
Maple code:
$> plot3d(2*(exp(x-2*t)+4*exp(2*x-(33*28)*t)+(0.162e-1*9)*exp(3*x-35.28*t))/(1+exp(x-2*t)+exp(2*x-33.28*t)+0.162e-1*exp(3*x-35.28*t))-2*(exp(x-2*t)+2*exp(2*x-33.28*t)+(0.162e-1*3)*exp(3*x-35.28*t))^2/(1+exp(x-2*t)+exp(2*x-33.28*t)+0.162e-1*exp(3*x-35.28*t))^2, x = -50..180, t = 6..10)$

(g) **3D Plot for two soliton of equation (3.2)-(3.4) by Hirota's method with** $\alpha = 1, \beta = 1$ **and** $\gamma = 1$.

For two soliton of equation given by (3.2)-(3.4), we have made the following choices.
$\epsilon = 1, k_1 = 1, k_2 = 2, \alpha = 1, \beta = 1, \gamma = 1, c_1 = \sqrt{5}, c_2 = \sqrt{341}, a_{12} \approx .0184,$
Maple code:
$> plot3d(2*(exp(x-2.23*t)+4*exp(2*x-(36*932)*t)+(0.18e-1*9)*exp(3*x-39.168*t))/(1+exp(x-2.23*t)+exp(2*x-36.932*t)+0.18e-1*exp(3*x-39.168*t))-2*(exp(x-2.23*t)+2*exp(2*x-36.932*t)+(0.18e-1*3)*exp(3*x-39.168*t))^2/(1+exp(x-2.23*t)+exp(2*x-36.932*t)+0.18e-1*exp(3*x-39.168*t))^2, x = -100..390, t = 1..50)$

In the present paper, we have derived generalized tenth-order Boussinesq equation and deduced its *multi-soliton solutions* by applying Hirota's Direct method. The tanh method is applied to obtain a *soliton solution* that agreed with the one-soliton of Hirota's method. Their structures are simulated in the above plots.

6. Conclusion

As scope for further work, we would like to have the following discussions. In our work, we have computed multi-soliton solutions to (2.4). One can attempt to compute rational solutions, singular solutions, shock solutions, and periodic solutions to the derived tB equation [10, 20, 32]. Also, the existence of a multi-soliton solution is only a necessary condition for integrability [14, 27]. So, deducing the integrability property of tB equation is still an open problem.

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