Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 11, No. 4, 2023, pp. 822-833
DOI:10.22034/cmde.2023.52094.2178

A second order numerical scheme for solving mixed type boundary value problems involving singular perturbation

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Abstract

A class of singularly perturbed mixed type boundary value problems is considered here in this work. The domain is partitioned into two subdomains. Convection-diffusion and reaction-diffusion problems are posed on the first and second subdomain, respectively. To approximate the problem, a hybrid scheme which consists of a second-order central difference scheme and a midpoint upwind scheme is constructed on Shishkin-type meshes. We have shown that the proposed scheme is second-order convergent in the maximum norm which is independent of the perturbation parameter. Numerical results are illustrated to support the theoretical findings.

Keywords. Singular perturbation, Mixed problem, Bakhvalov-Shishkin mesh, Hybrid scheme, Uniform convergence.
2010 Mathematics Subject Classification. 65L10, 65L12.

1. Introduction

Singularly perturbed problems (SPPs) are more often found while modeling different phenomena in applied sciences, particularly in fluid dynamics, elasticity, chemical reactor theory, etc. Generally, the presence of a small positive parameter at the highest derivative term makes the problem singularly perturbed. A number of articles are devoted to solving SPPs with integral boundary conditions [1, 5, 12]. For SPPs, the continuous solution has boundary or interior layers. It is well known that standard numerical methods are facing several computational difficulties, due to the multi-scale behavior like rapid variations of the solutions in some regions [6, 7]. As a consequence, finding solutions for SPPs has become the most challenging and interesting task [8, 23, 24].

Here in this work, we consider the following model SPP of mixed type:

$$\begin{cases} L_\varepsilon^- \mathcal{Y}(t) \equiv -\varepsilon \mathcal{Y}''(t) + p(t)\mathcal{Y}'(t) + q(t)\mathcal{Y}(t) = f(t), \quad t \in \Omega^-, \\ L_\varepsilon^+ \mathcal{Y}(t) \equiv -\varepsilon \mathcal{Y}''(t) + r(t)\mathcal{Y}(t) = f(t), \quad t \in \Omega^+, \\ \mathcal{Y}(0) = A,\ [\mathcal{Y}(d)] = \mathcal{Y}(d+0) - \mathcal{Y}(d-0) = 0, [\mathcal{Y}'(d)] = 0,\ \mathcal{Y}(1) = B, \end{cases} \tag{1.1}$$

where $0 < \varepsilon \ll 1$ is the perturbation parameter and A, B are given constants. The functions $p(t), q(t)$ and $r(t)$ are sufficiently smooth on $\Omega^- = (0,\ d)$ and $\Omega^+ = (d,\ 1)$ respectively, with $0 < \alpha \leq p(t), 0 \leq q(t), 0 < \beta \leq r(t)$. The function f is smooth in Ω where $\Omega = \Omega^- \cup \Omega^+ \cup \{0, 1\}$ and has a simple discontinuity at $t = d$. Clearly, the solution y doesn't possess a continuous second derivative at $t = d$, that is, $\mathcal{Y}$ doesn't belong to $C^2(\Omega)$. Under the above assumptions, the solution of (1.1) has a unique solution in $C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^+ \cup \Omega^-)$ [4].

SPPs of convection-diffusion and reaction-diffusion type with smooth data have been studied extensively [11, 15, 19]. Recently, various kinds of adaptive meshes are used for solving different class of SPPs [9, 13, 21, 25]. However, only a few results for SPPs having nonsmooth data are reported in the literature [10, 17]. As we know, discontinuity at

Received: 14 June 2022 ; Accepted: 03 April 2023.
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one or more points in the interior domain leads to an interior layer [3, 18]. Miller et al. [16] solved a SPPs with discontinuous source term by Schwarz method on a Shishkin mesh (S-mesh) and shown it to be first order. In [22], the Galerkin method was used on a Bakhvalov-Shishkin mesh (B-S mesh) and proved second-order convergent for the SPPs with discontinuous source terms. In [4], the author have analyzed an inverse-monotone finite volume method on S mesh for an elliptic SPP with a discontinuous source term. Priyadharshini et.al [20] presented two types of hybrid scheme on S mesh and got almost second-order convergence.

Since f is discontinuous in (1.1) at the interface point, it leads to severe numerical difficulty for constructing high accurate schemes. Our main objective in this work is to propose a second-order numerical scheme for SPPSs of type (1.1). To serve our purpose, a proper hybrid scheme is constructed here which consists of second-order central difference operator and the midpoint upwind scheme on Shishkin-type meshes namely S mesh and B-S mesh. We prove that our proposed scheme is uniformly convergent with respect to ε and has an accuracy of second order.

Here, $C > 0$ denotes a generic constant independent of perturbation and mesh parameters. But C is not necessarily the same at each occurrence while the subscripted C is a fixed constant. The simple discontinuity of the function $s(t)$ at $t = d \in \Omega$ is denoted by $[s](d) = s(d+) - s(d-)$. For any continuous function $g(t)$, we define the supremum norm, by $\|g\|_{\overline{\Omega}} = \sup_{t\in\overline{\Omega}} |g(t)|$.

2. Properties of the solutions

Lemma 2.1. *(Maximum Principle) For any smooth function $Z \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^+ \cup \Omega^-)$ with $Z(0) \geq 0$, $Z(1) \geq 0$ with $L_\varepsilon^- Z(t) \geq 0,\ t \in \Omega^-$, $L_\varepsilon^+ Z(t) \geq 0,\ t \in \Omega^+$ and $[Z'(d)] \leq 0$, then $Z(t) \geq 0, \forall t \in \overline{\Omega}$.*

Proof. Suppose there exists t^* with $Z(t^*) = \min_{t\in\overline{\Omega}} Z(t)$ and $Z(t^*) < 0$. Clearly, $t^* \in \Omega^+ \cup \Omega^-$ or $t^* = d$. If $t^* \in \Omega^+ \cup \Omega^-$ then $Z''(t^*) \geq 0$ and $Z'(t^*) = 0$. Then $L_\varepsilon^- Z(t) \leq 0,\ t \in \Omega^-$, $L_\varepsilon^+ Z(t) \leq 0,\ t \in \Omega^+$, which contradict our assumptions. Now for $t^* = d$, there are two cases.
Case 1: $Z(t)$ is not differentiable at t^*. Since Z attains minimum at t^*, then $Z'(t^* - 0) \leq 0, Z'(t^* + 0) \geq 0$ and $[Z'(d)] > 0$, which is a contradiction.
Case 2: $Z(t)$ is differentiable at t^*. Then $Z'(t^*) = 0$ and there exists a subinterval $\delta_t = (t^* - \delta, t^*)$ with $Z(t) \leq 0$, $Z(t^*) \leq Z(t)$, $t \in \delta_t$. Let $t_1 \in \delta_t$, then there exists $t_2 \in \delta_t$ such that

$$Z'(t_2) = \frac{Z(t_1) - Z(t^*)}{t_1 - t^*} > 0,$$

and $t_2 \in \delta_t$ such that

$$Z''(t_3) = \frac{Z'(t_2) - Z'(t^*)}{t_2 - t^*} > 0.$$

Then $t_3 \in \Omega^-$ and $L_\varepsilon^- Z(t_3) \leq 0$, which again contradict our assumption. Hence, we prove that $Z(t) \geq 0, \forall t \in \overline{\Omega}$. □

Lemma 2.2. *If $\mathcal{Y} \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^+ \cup \Omega^-)$, then $\|\mathcal{Y}\|_{\overline{\Omega}} \leq C \max\left\{|\mathcal{Y}(0)|, |\mathcal{Y}(1)|, |L_\varepsilon^- \mathcal{Y}|, |L_\varepsilon^+ \mathcal{Y}|\right\}$.*

Proof. Refer [4] for the proof. □

Decomposing the solution as $\mathcal{Y} = \mathcal{V} + \mathcal{W}$, with $\mathcal{V} = \mathcal{V}_0 + \varepsilon\mathcal{V}_1 + \varepsilon^2\mathcal{V}_2 + \varepsilon^3\mathcal{V}_3$. The regular component $\mathcal{V} \in C^0(\Omega)$ is the solution of

$$\begin{cases} L_\varepsilon^- \mathcal{V}(t) = f(t), \quad t \in \Omega^-, \\ L_\varepsilon^+ \mathcal{V}(t) = f(t), \quad t \in \Omega^+, \\ \mathcal{V}(0) = y(0),\ [\mathcal{V}'(d)] = [\mathcal{V}_0'(d)] + \varepsilon[\mathcal{V}_1'(d)] + \varepsilon^2\mathcal{V}[\mathcal{V}_2'(d)] = 0, \mathcal{V}(1) = 0. \end{cases} \tag{2.1}$$

Now the layer component $\mathcal{W} \in C^0(\Omega)$ satisfies

$$\begin{cases} L_\varepsilon^- \mathcal{W}(t) = 0, \quad t \in \Omega^-, \\ L_\varepsilon^+ \mathcal{W}(t) = 0, \quad t \in \Omega^+, \\ \mathcal{W}(0) = 0,\ [\mathcal{W}'(d)] = -[\mathcal{W}'(d)], \mathcal{W}(1) = \mathcal{Y}(1) - \mathcal{V}(1). \end{cases} \tag{2.2}$$

Further, we decompose $\mathcal{W}$ as $\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2$, where $\mathcal{W}_1$ and $\mathcal{W}_2$ satisfy:

$$\begin{cases} \mathcal{W}_1(t) = 0, t \in \Omega^-, \\ L_\varepsilon^+ \mathcal{W}_1(t) = 0, \quad t \in \Omega^+, \mathcal{W}_1(d) = -[\mathcal{V}(d)], \mathcal{W}_1(1) = \mathcal{Y}(1) - \mathcal{V}(1), \end{cases} \tag{2.3}$$

$$\begin{cases} L_\varepsilon^- \mathcal{W}_2(t) = 0, \quad t \in \Omega^-, \mathcal{W}_2(0) = 0, [\mathcal{W}_2'(d)] = -[\mathcal{V}'(d)] - [\mathcal{W}_1'(d)], \\ L_\varepsilon^+ \mathcal{W}_2(t) = 0, \quad t \in \Omega^+, \mathcal{W}_2(1) = 0. \end{cases} \tag{2.4}$$

Lemma 2.3. *For $0 \le l \le 4$, we have*

$$\begin{aligned} &\left|\mathcal{V}^l(t)\right| \leqslant C(1 + \varepsilon^{-3-l}), \\ &\left|\mathcal{W}^l(t)\right| \leqslant C\varepsilon^{-l+1/2} e^{-(d-t)\alpha/\varepsilon}, \ t \in \Omega^-, \\ &\left|\mathcal{W}^l(t)\right| \leqslant C\varepsilon^{-l/2} (e^{-(t-d)\sqrt{\beta/\varepsilon}} + e^{-(1-t)\sqrt{\beta/\varepsilon}}), \ t \in \Omega^+. \end{aligned} \tag{2.5}$$

Proof. Refer [4]. □

3. Discrete problem

The construction and generalization of Shishkin meshes have gained much attention from researchers [6]. However, in the analysis and construction of the Bakhvalov mesh, a rare contribution is there to date. Here, the hybrid difference schemes to approximate (1.1) on Shishkin-type meshes are characterized by the choice of transition points. The domain Ω^- is subdivided into $[0, d - \tau_1]$ and $[\tau_1, d]$ for some τ_1. Similarly Ω^+ is subdivided into $[d, d + \tau_2]$ $[d + \tau_2, 1 - \tau_2]$ and $[1 - \tau_2, 1]$ for some τ_2. Here

$$\tau_1 = \min\Big(\frac{d}{2}, \frac{2}{\theta_1}\ln \mathcal{N}\Big), \quad \text{and } \tau_2 = \min\Big(\frac{1-d}{4}, \frac{2}{\theta_2}\ln \mathcal{N}\Big),$$

where $\mathcal{N} \ge 4$ is the number of mesh intervals, $\theta_1 = \dfrac{\alpha}{\varepsilon}, \theta_2 = \sqrt{\dfrac{\varepsilon}{\beta}}$ and $t_m = d$. Now, the S mesh is given by

$$t_i = \begin{cases} \dfrac{2(d-\tau_1)i}{m}, & \text{if} \quad 0 \le i \le m/2, \\ (d-\tau_1) + (i - \dfrac{m}{2})\dfrac{2\tau_1}{m}, & \text{if} \quad m/2 \le i \le m, \\ d + (i - \dfrac{m}{4})\dfrac{4\tau_2}{m}, & \text{if} \quad m \le i \le 5m/4, \\ (d+\tau_2) + (i - \dfrac{m}{2})\dfrac{2(1-d-2\tau_2)}{m}, & \text{if} \quad 5m/4 \le i \le 7m/4, \\ 1 - \tau_2 + (i - \dfrac{m}{4})\dfrac{4\tau_2}{m}, & \text{if} \quad 7m/4 \le i \le \mathcal{N}. \end{cases} \tag{3.1}$$

The B-S mesh which is an alteration of S-mesh condensed in the layer region by effectively inverting the boundary layer terms using the idea of Bakhvalov [2]. On Ω^-, we consider that, in $[0, d - \tau_1]$ the mesh is equidistant with $\mathcal{N}/4$ subintervals having the width $\dfrac{4}{\mathcal{N}}(d - \tau_1)$ and in $[d - \tau_1, d]$ is subdivided into $\mathcal{N}/4$ graded subintervals by inverting $e^{-\theta_1(d-t)/2}$ linearly in it. That is

$$e^{-\theta_1(d-t_i)/2} = C_1 i + C_2, \quad i = \mathcal{N}/4, \cdots, \mathcal{N}/2, \tag{3.2}$$

with $t_{N/4} = d - \tau_1$ and $t_{\mathcal{N}/2} = d$. Now putting the value of $i = \mathcal{N}/4$ in (3.2) and substituting the value $t_{\mathcal{N}/4} = d - \tau_1$, we get

$$e^{-\theta_1(d-t_{\mathcal{N}/4})/2} = C_1\frac{\mathcal{N}}{4} + C_2 \Rightarrow \frac{2}{\mathcal{N}} = C_1\frac{\mathcal{N}}{4} + C_2. \tag{3.3}$$

Similarly, by putting $i = \mathcal{N}/2$ in (3.2), and substituting the value $t_{\mathcal{N}/2} = d$ we have

$$e^{-\theta_1(d-t_{\mathcal{N}/2})/2} = C_1\frac{\mathcal{N}}{2} + C_2 \Rightarrow 1 = C_1\frac{\mathcal{N}}{2} + C_2, \tag{3.4}$$

Now, solving (3.3) and (3.4), we get $C_1 = \frac{4}{\mathcal{N}} - \frac{8}{\mathcal{N}^2}$ and $C_2 = \frac{4}{\mathcal{N}} - 1$. Hence,

$$t_i = \frac{1}{2} + \frac{2}{\theta_1}\log\left(\left(4 - \frac{8}{\mathcal{N}}\right)\frac{i}{\mathcal{N}} + \left(\frac{4}{\mathcal{N}} - 1\right)\right), \text{ if } \quad \mathcal{N}/4 \leq i < N/2.$$

On Ω^+, the interval $[d, d+\tau_2]$ is subdivided into $\mathcal{N}/8$ graded subintervals by inverting $e^{-\theta_2 t/2}$. That is

$$e^{-\theta_2 t_i/2} = C_3 i + C_4, \quad i = \mathcal{N}/2, \cdots, 5\mathcal{N}/8,$$

with $t_{\mathcal{N}/2} = d$ and $t_{5\mathcal{N}/8} = \tau_2$. After solving the above, we obtain

$$t_i = -\frac{1}{2} - \frac{2}{\theta_2}\log\left(\left(\frac{16}{\mathcal{N}} - 8\right)\frac{i}{\mathcal{N}} + 1\right), \text{ if } \quad \mathcal{N}/2 \leq i < 5\mathcal{N}/8.$$

The subintervals $[d+\tau_2, 1-\tau_2]$ is divided into $\mathcal{N}/4$ subintervals with length $\frac{4}{\mathcal{N}}(1-d-2\tau_2)$. Now, in the interval $[1-\tau_2, 1]$, we invert the function $e^{-\theta_2(1-t)/2}$ to obtain the mesh point in it. For $i = 7\mathcal{N}/2, \cdots, \mathcal{N}$,

$$e^{-\theta_2(1-t_i)/2} = C_5 i + C_6,$$

with $t_{7\mathcal{N}/8} = 1 - \tau_2$ and $t_{\mathcal{N}} = 1$. After solving the above, we obtain

$$t_i = 1 + \frac{2}{\theta_2}\log\left(\left(8 - \frac{16}{N}\right)\frac{i}{\mathcal{N}} + \left(\frac{16}{\mathcal{N}} - 7\right)\right), \text{ if } \quad 7\mathcal{N}/8 \leq i \leq \mathcal{N}.$$

Now, the B-S mesh is given by

$$t_i = \begin{cases} 4(d-\tau_1)\frac{i}{\mathcal{N}}, & \text{if } \quad 0 \leq i \leq \mathcal{N}/4, \\ \frac{1}{2} + \frac{2}{\theta_1}\log\left(\left(4 - \frac{8}{\mathcal{N}}\right)\frac{i}{\mathcal{N}} + \left(\frac{4}{N} - 1\right)\right), & \text{if } \quad \mathcal{N}/4 \leq i \leq \mathcal{N}/2, \\ -\frac{1}{2} - \frac{2}{\theta_2}\log\left(\left(\frac{16}{\mathcal{N}} - 8\right)\frac{i}{\mathcal{N}} + 1\right), & \text{if } \quad \mathcal{N}/2 \leq i \leq 5\mathcal{N}/8, \\ (d+\tau_2) + 4(1-d-2\tau_2)\left(\frac{i}{\mathcal{N}} - \frac{1}{2}\right), & \text{if } \quad 5\mathcal{N}/8 \leq i \leq 7\mathcal{N}/8, \\ 1 + \frac{2}{\theta_2}\log\left(\left(8 - \frac{16}{\mathcal{N}}\right)\frac{i}{\mathcal{N}} + \left(\frac{16}{\mathcal{N}} - 7\right)\right), & \text{if } \quad 7\mathcal{N}/8 \leq i \leq \mathcal{N}. \end{cases} \tag{3.5}$$

The above mesh points in terms of mesh generating function can be written as

$$\phi(s) = \begin{cases} 4(d-\tau_1)s, & s = \frac{i}{\mathcal{N}}, \quad 0 \leq i \leq \mathcal{N}/4, \\ \frac{1}{2} + \frac{2}{\theta_1}\phi_1(s), & s = \frac{i}{\mathcal{N}}, \quad \mathcal{N}/4 \leq i \leq \mathcal{N}/2, \\ -\frac{1}{2} - \frac{2}{\theta_2}\phi_2(s), & s = \frac{i}{\mathcal{N}}, \quad N/2 \leq i \leq 5\mathcal{N}/8, \\ (d+\tau_2) + 4(1-d-2\tau_2)\left(s - \frac{1}{2}\right), & s = \frac{i}{\mathcal{N}}, \quad 5\mathcal{N}/8 \leq i \leq 7\mathcal{N}/8, \\ 1 + \frac{2}{\theta_2}\phi_3(s), & s = \frac{i}{\mathcal{N}}, \quad 7\mathcal{N}/8 \leq i \leq \mathcal{N}, \end{cases}$$

where s is uniform in $[0,1]$ and ϕ_1, ϕ_2 and ϕ_3 are monotonically increasing functions. The mesh generating functions are given by

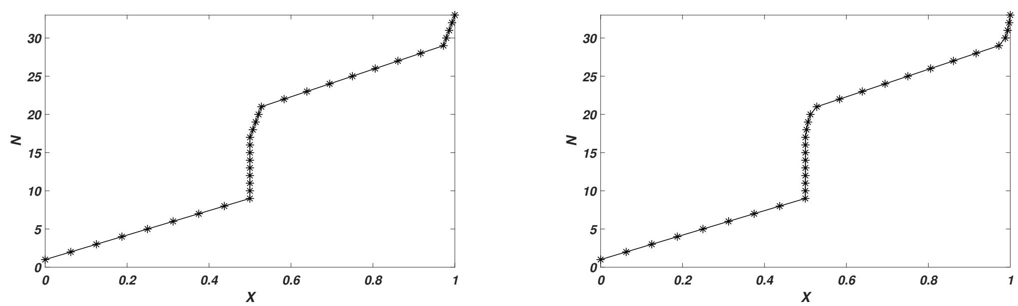
$$\begin{aligned}
\phi_1(s) &= -\log\left(\left(4-\frac{8}{N}\right)\frac{i}{\mathcal{N}}+\left(\frac{4}{\mathcal{N}}-1\right)\right), \quad s\in[1/4,1/2] \text{ and } \phi_1(1/4)=\log(\mathcal{N}/2), \phi_1(1/2)=0,\\
\phi_2(s) &= -\log\left(\left(\frac{16}{\mathcal{N}}-8\right)\frac{i}{\mathcal{N}}+1\right), \quad s\in[1/2,3/4] \text{ and } \phi_2(1/2)=0, \phi_2(1/2)=\log(\mathcal{N}/2),\\
\phi_3(s) &= -\log\left(\left(8-\frac{16}{\mathcal{N}}\right)\frac{i}{\mathcal{N}}+\left(\frac{16}{\mathcal{N}}-7\right)\right), \quad s\in[3/4,1] \text{ and } \phi_3(3/4)=\log(\mathcal{N}/2), \phi_3(1)=0.
\end{aligned}$$

Also the corresponding characterizing functions φ_1, φ_2 and φ_3 are given by

$\varphi_1 = e^{-\phi_1}, \varphi_2 = e^{-\phi_2}, \varphi_3 = e^{-\phi_3}$.

The difference operators D^-, D^0, δ^2 are defined as:

$$D^-Y_i = \frac{Y_i - Y_{i-1}}{h_i}, \quad D^0Y_i = \frac{Y_{i+1}-Y_{i-1}}{h_{i+1}+h_i}, \quad \delta^2 Y_i = \frac{2}{h_i+h_{i+1}}\left(\frac{Y_{i+1}-Y_i}{h_{i+1}} - \frac{Y_{i+1}-Y_i}{h_i}\right).$$



(a) Shishkin mesh *(b) Bakhvalov mesh*

FIGURE 1. Mesh construction with $\varepsilon = 10^{-4}, N = 32$.

Using the above, the proposed scheme on $\overline{\Omega}^N$ takes the form

$$L^N_{hs}Y_i = f_i, \text{ for } i = 1, \cdots, N-1, \tag{3.6}$$

where

$$L^N_{hs}Y_i = \begin{cases} L^N_{mus}Y_i, \text{ for } i = 1,\cdots, m/2,\\ L^{-N}_{cds}Y_i, \text{ for } i = m/2,\cdots, m-1,\\ L^N_t Y_i, \text{ for } i = \mathcal{N}/2,\\ L^{+N}_{cds}Y_i, \text{ for } i = m+1,\cdots,\mathcal{N}-1,\end{cases} \tag{3.7}$$

and

$$f_i = \begin{cases} f_{i-1/2}, & i = 1,\cdots, m/2,\\ f_i, & i = m/2,\cdots,m-1,\\ \frac{h_2}{-2\varepsilon - h_2 p_{m-1}} f_{m-1} - \frac{h_3}{2\varepsilon}p_{m+1}, & i = \mathcal{N}/2,\\ f_i, & i = m+1,\cdots,\mathcal{N}-1.\end{cases} \tag{3.8}$$

Explicitly using the notation of [26], we have

$$\begin{aligned} L^N_{mus} Y_i &= -\varepsilon\delta^2 Y_i + p_{i-1/2} D^- Y_i - q_{i-1/2} Y_i = f_{i-1/2}, \\ L^{-N}_{cds} Y_i &= -\varepsilon\delta^2 Y_i + p_i D^- Y_i - q_i Y_i = f_i, \\ L^{+N}_{cds} Y_i &= -\varepsilon\delta^2 Y_i + r_i Y_i = f_i, \end{aligned}$$

and

$$L^N_t Y_i = \frac{-Y_{m+2} + 4Y_{m+1} - 3Y_m}{2h_{m+1}} - \frac{Y_{m-2} - 4Y_{m-1} + 3Y_m}{2h_{m-1}} = 0.$$

Lemma 3.1. *Assume that* $\frac{\mathcal{N}}{\ln \mathcal{N}} \geq \frac{4\|p\|}{\alpha}$. *Also, if* $Z(t_0) \geq 0, Z(t_N) \geq 0$ *and* $L^N_{hs}(t) \geq 0$ *for* $i = i = 0, \cdots, \mathcal{N}$, *then* $Z(t_i) > 0$ *for* $i = i = 0, \cdots, \mathcal{N}$.

Proof. One can refer to [20] for the proof. □

4. Error estimates

We shall analyze the errors in Ω^-, Ω^+, and $t = d$ in this section. Let us rewrite the hybrid scheme (3.7) as:

$$[L^N_{hs} Y^N] = \frac{[A^N Y^N]_{i+1} - [A^N Y^N]_i}{h_{\sigma,i}} = 0, \text{ for } i = 1, \cdots, \mathcal{N} - 1,$$

where $[A^N Y^N]_i = \varepsilon \frac{Y_i - Y_{i-1}}{h_i} + \sigma p_i Y_i + (1-\sigma) p_{i-1} Y_{i-1} - \sum_{j=1}^{i-1} q_{i-1/2} Y_{i-1/2}$ and $h_{\sigma,i} = (1-\sigma)h_i + \sigma h_{i+1}$. Note that for $\sigma = 1/2$, we recover the central difference scheme, while for $\sigma = 1$ the midpoint scheme is obtained.

Now, we provide the error associated with the scheme (3.7) using the S mesh and the B-S mesh which is the main result of this work.

Theorem 4.1. *The error associated with the hybrid scheme (3.7) satisfies the following bounds:*

$$\begin{aligned} \left|\left|\mathcal{Y} - Y^N\right|\right| &\leq C\,\mathcal{N}^{-2} \ln^2 \mathcal{N} \quad \textit{on S-mesh}, \\ \left|\left|\mathcal{Y} - Y^N\right|\right| &\leq C\mathcal{N}^{-2}, \qquad \textit{on B-S mesh}. \end{aligned} \tag{4.1}$$

Proof. The error associated for S-mesh satisfies the bound $\left|\left|\mathcal{Y} - Y^N\right|\right| \leq C\mathcal{N}^{-2} \ln^2 \mathcal{N}$. The proof is given in Theorem 5.1 of [4]. For the proof of B-S mesh, we consider the following cases.

Case 1: For the first domain Ω^-: Let us integrate (1.1) over $[t_j, t_{j+1}]$, we have

$$(S\mathcal{Y})(t_{j+1}) - (S\mathcal{Y})(t_j) - \int_{t_j}^{t_{j+1}} (p(t)\mathcal{Y}(t) - f(t))dt = 0,$$

where $(S\mathcal{Y})(t) = \varepsilon\mathcal{Y}'(t) - b(t)\mathcal{Y}(t)$. Now we introduce the notation

$$[S_{hs}\mathcal{V}] = \varepsilon \frac{\mathcal{V}_i - \mathcal{V}_{i-1}}{h_i} - \sigma p_i \mathcal{V}_i - (1-\sigma) p_{i-1} \mathcal{V}_{i-1}.$$

For any arbitrary mesh function V_i and W_i there exists an h_σ (see [14]) such that

$$\left|\left|V - W\right|\right| \leq C \max_{i=1,\cdots,\mathcal{N}-1} \left| \sum_{j=i}^{\mathcal{N}-1} h_{\sigma,j} [L^N_{hs} V^N - L^N_{hs} W^N]_j \right|. \tag{4.2}$$

Using the above bound (4.2), we have

$$\begin{aligned} ||\mathcal{Y} - Y^N|| &\leq C \max_{i=1,\cdots,N-1} \left| \sum_{j=i}^{N-1} [S_{hs}Y^N] - S\mathcal{Y} \right| \\ &+C \max_{i=1,\cdots,N-1} \left| \sum_{j=i}^{N-1} \int_{t_j}^{t_{j+1}} (p(t)\mathcal{Y}(t) - f(t))dt - (h_j + h_j + 1)(p_j - f_j)/2 \right|. \end{aligned}$$

For finding the bounds for the first term we take two cases: $\sigma = 1$ and $\sigma = 1/2$. For $\sigma = 1$, we have

$$[S_{hs}Y^N] - S\mathcal{Y} = \varepsilon \Big\{ \frac{\mathcal{Y}_i - \mathcal{Y}_{i-1}}{h_i} - \mathcal{Y}' \Big\} = \frac{\varepsilon}{h_i} \int_{t_{j-1}}^{t_j} \mathcal{Y}''(z)(z - t_{i-1})dz,$$

by Taylor's expansion of y about t_j and using $2\varepsilon < \beta^* h_i$, we have

$$\Big| [S_{hs}Y^N] - S\mathcal{Y} \Big| \leq C \int_{t_{j-1}}^{x_j} (1 + \varepsilon^{-2} e^{\beta z/\varepsilon})(z - t_{i-1})dz.$$

Now for $\sigma = 1/2$, we have

$$[S_{hs}Y^N] - Sy = \varepsilon \Big\{ \frac{\mathcal{Y}_i - \mathcal{Y}_{i-1}}{h_i} - \mathcal{Y}'_{i-1/2} \Big\} + \frac{p_i\mathcal{Y}_i + p_{i-1}\mathcal{Y}_{i-1}}{2} - p_{i-1/2}\mathcal{Y}_{i-1/2}.$$

Using Taylor's expansion for $\mathcal{Y}$ and $\mathcal{Y}'$ about t_j, we have

$$\varepsilon \Big| \Big\{ \frac{\mathcal{Y}_i - \mathcal{Y}_{i-1}}{h_i} - \mathcal{Y}'_{i-1/2} \Big\} \Big| \leq \frac{3\varepsilon}{2} \int_{t_{j-1}}^{t_j} |\mathcal{Y}'''(t)|(z - t_{i-1})dz,$$

and

$$\Big| \frac{p_i\mathcal{Y}_i + p_{i-1}\mathcal{Y}_{i-1}}{2} - p_{i-1/2}\mathcal{Y}_{i-1/2} \Big| \leq \frac{3}{2} \int_{t_{j-1}}^{x_j} |(p''\mathcal{Y}'')(t)|(z - t_{i-1})dz.$$

So

$$\Big| [S_{hs}Y^N] - S\mathcal{Y} \Big| \leq C \int_{t_{j-1}}^{t_j} (1 + \varepsilon^{-2} e^{\beta z/\varepsilon})(z - t_{i-1})dz.$$

Finally, for the right side of (4.3) use the Taylor series expansions of $\mathcal{Y}$ and q about x_{j+1} to obtain

$$\Big| \int_{t_j}^{t_{j+1}} (p(t)\mathcal{Y}(t) - f(t))dx - (h_j + h_j + 1)(p_j - f_j)/2 \Big| \leq C(h_j + h_j + 1) \int_{t_j}^{t_{j+1}} (1 + \varepsilon^{-2} e^{\beta z/\varepsilon})(z - t_{i-1})dz.$$

Combining the above estimates we get

$$\begin{aligned} ||\mathcal{Y} - Y^N||_{\Omega^-} &\leq C \max_{i=1,\cdots,N-1} \int_{t_i}^{t_{i+1}} (1 + \varepsilon^{-2} e^{\beta z/\varepsilon})(z - t_{i-1})dz \\ &\leq \frac{C}{2} \max_{i=1,\cdots,N-1} \Big(\int_{t_i}^{t_{i+1}} (1 + \varepsilon^{-2} e^{\beta z/\varepsilon})dz \Big)^2 \leq C\mathcal{N}^{-2}. \end{aligned} \tag{4.3}$$

Case 2: For the second domain Ω^+: On the side, we discretize the problem by the central difference scheme [14]. So by using a similar process we get

$$||y - Y^N||_{\Omega^+} \leq C\mathcal{N}^{-2}. \tag{4.4}$$

Case 3: At the point of interface $t = d$:

$$\begin{aligned} \left|L_t^N Y^N(t_m) - L\mathcal{Y}(t_m)\right| &\leq \left|L_t^N Y^N(t_m) - \frac{h_2}{-2\varepsilon - h_2 p_{m-1}} f_{m-1} - \frac{h_3}{2\varepsilon} f_{m+1}\right| \\ &\leq Cm^{-2}\Big(\tau_1^2 \varepsilon^{-5/2} + \tau_1^2 \varepsilon^{-1}\Big). \end{aligned} \tag{4.5}$$

Now, If we consider the barrier function as $\phi^{\pm}(t_i) = \varphi(t_i) \pm \left|Y^N - y\right|$, where

$$\varphi(t_i) = \begin{cases} Cm^{-2} + Cm^{-2}\tau_1^2\varepsilon^{-5/2}(t_i - d + \tau_1) & t_i \in \Omega^-, \\ Cm^{-2} + Cm^{-2}\tau_2^2\varepsilon^{-1}(1 - t_i), & t_i \in \Omega^+. \end{cases} \tag{4.6}$$

and applying Lemma 3.1, we get

$$\left|\left|\mathcal{Y} - Y^N\right|\right|_{t=d} \leq C\mathcal{N}^{-2}. \tag{4.7}$$

This completes the proof. ☐

5. Numerical results

Example 5.1. *Consider the test problem:*

$$\begin{cases} -\varepsilon\mathcal{Y}''(t) + (1 + t^2)\mathcal{Y}'(t) = 2, & t \in (0,\ 0.5), \\ -\varepsilon\mathcal{Y}''(t) + (4tx^3)\mathcal{Y}(t) = 1.8t, & t \in (0.5,\ 1), \\ \mathcal{Y}(0) = \mathcal{Y}(1) = [\mathcal{Y}(0.5)] = [\mathcal{Y}'(0.5)] = 0. \end{cases} \tag{5.1}$$

Example 5.2. *Consider the following model:*

$$\begin{cases} -\varepsilon\mathcal{Y}''(t) + (1 + \cos(\pi x))\mathcal{Y}'(t) + (1 + \sin(\frac{\pi}{2}t))\mathcal{Y}(t) = 1 + \sin(\pi t)\cos(\pi t),\ t \in (0,\ 0.5), \\ -\varepsilon\mathcal{Y}''(t) + (1 + \cos(\frac{\pi}{2}x))\mathcal{Y}(t) = 3 + 2\cos(\frac{\pi}{2}t)\sin(\frac{\pi}{2}t)),\ t \in (0.5,\ 1), \\ \mathcal{Y}(0) = \mathcal{Y}(1) = [\mathcal{Y}(0.5)] = [\mathcal{Y}'(0.5)] = 0. \end{cases} \tag{5.2}$$

Since the exact solutions are not available, so we use the idea of a double mesh principle. That is, the solution is computed on a mesh that is twice as fine keeping the transition parameter fixed [15]. The maximum pointwise error is defined as follows:

$$E_\varepsilon^N = \|\mathcal{Y}_j - \widetilde{Y_j}\|_{\Omega^N},$$

where $\widetilde{Y_j}$ is the interpolation of Y_j, on Ω^{2N} to Ω^N. The corresponding rate is given by

$$R_\varepsilon^N = \log_2\left(\frac{E_\varepsilon^N}{E_\varepsilon^{2N}}\right).$$

Tables 1 and 2 represent E_ε^N and R_ε^N of the hybrid scheme for Example 5.1 and Example 5.2 respectively. In Table 3, we compare E_ε^N generated by the proposed scheme for Example 5.2 with the results given in [4]. The log-log plots of the maximum pointwise error on S mesh and B-S mesh are shown in Figure 3. The use of B-S mesh produces more accurate results as compared to S mesh which is already proved theoretically. Further from these tables and figures, one can notice the parameter uniform nature and the second-order convergence of the proposed scheme.

Conclusions

This paper studies the numerical solution for a class of mixed type SPPs of type (1.1). A hybrid scheme on the Shishkin-type meshes are constructed and second-order convergent error estimates are derived. Numerical results are presented which are in agreement with the theoretical findings.

Table 1. E_ε^N *and* R_ε^N *of the proposed scheme for Example 5.1*

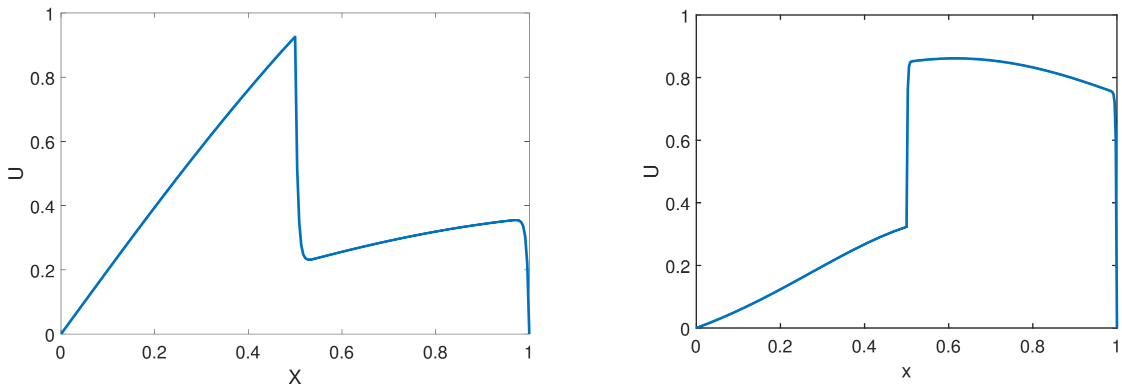
	$\varepsilon = 10^{-6}$		$\varepsilon = 10^{-8}$		$\varepsilon = 10^{-10}$	
N	S mesh	B-S mesh	S mesh	B-S mesh	S mesh	B-S mesh
	5.5390e-3	4.7216e-3	5.5035e-3	4.7083e-3	5.5000e-3	4.7069e-3
64	1.3560	1.8849	1.3561	1.8845	1.3561	1.8845
	2.1639e-3	1.2785e-3	2.1499e-3	1.2752e-3	2.1485e-3	1.2748e-3
128	1.5184	1.9626	1.5221	1.9586	1.5225	1.9582
	7.5537-4	3.2801e-4	7.4852e-4	3.2808e-4	7.4784e-4	3.2809e-4
256	1.5930	1.9869	1.5924	1.9776	1.5923	1.9767
	2.5039e-4	8.2753e-5	2.4823e-4	8.3301e-5	2.4802e-4	8.3357e-5
512	1.6496	2.0147	1.6515	1.9954	1.6517	1.9933
	7.9809e-5	2.0478e-5	7.9017e-5	2.0892e-5	7.8937e-5	2.0936e-5
1024	1.6193	2.0413	1.6918	2.0009	1.6919	1.9972

Table 2. E_ε^N *and* R_ε^N *of the proposed scheme for Example 5.2*

	$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-7}$		$\varepsilon = 10^{-9}$	
N	S mesh	B-S mesh	S mesh	B-S mesh	S mesh	B-S mesh
	1.0544e-2	7.0687e-3	8.1949e-3	7.0850e-3	8.1954e-3	7.0853e-3
64	1.1272	1.8883	1.5273	1.8769	1.5273	1.8769
	4.8271e-3	1.9242e-3	2.8430e-3	1.9290e-3	2.8432e-3	1.9291e-3
128	1.3833	1.8772	1.6041	1.9455	1.6041	1.9455
	1.8505e-3	5.4629e-4	9.3516e-4	5.0082e-4	9.3523e-4	4.9969e-4
256	1.5348	1.8165	1.6574	1.7025	1.6884	1.7025
	6.3865e-4	1.7114e-4	2.9646e-4	1.5338e-4	2.9016e-4	1.5345e-4
512	1.6652	1.7745	1.5472	1.4092	1.5472	1.4092
	1.8789e-4	6.6522e-5	1.0144e-4	5.7540e-5	9.7024e-5	5.7785e-5
1024	1.6193	1.7632	1.6918	2.0009	1.6919	1.9972

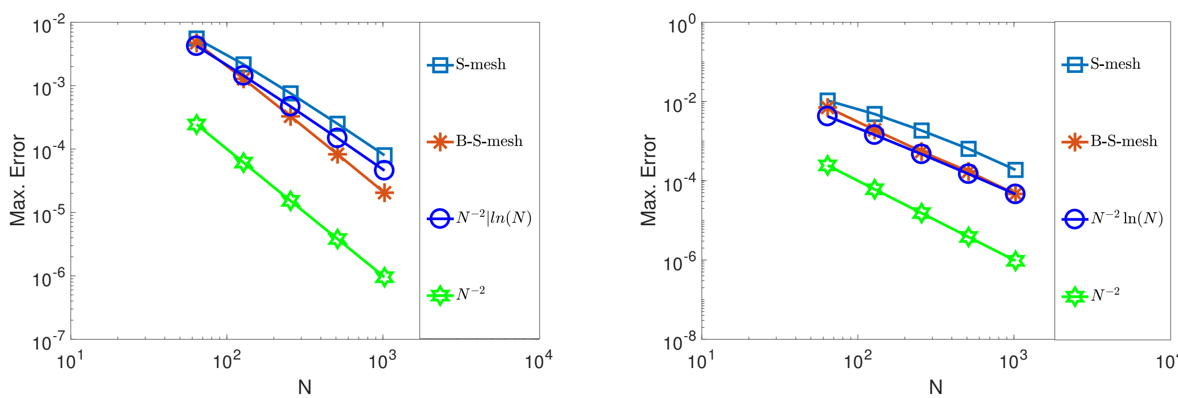
Table 3. *Comparison of* E_ε^N *for Example 5.2*

	$\varepsilon = 2^{-10}$		$\varepsilon = 2^{-18}$	
N	Results in [4]	Our results	Results in [4]	Our results
32	1.80e-2	1.95e-2	1.80e-2	1.73e-2
64	7.94e-3	5.04e-3	7.99e-3	8.19e-3
128	2.79e-3	1.98e-3	3.03e-3	2.93e-3
256	7.09e-4	9.41e-4	1.04e-3	1.04e-3
512	1.78e-4	3.44e-4	3.41e-4	3.64e-4
1024	4.44e-5	1.05e-4	1.07e-4	1.29e-4



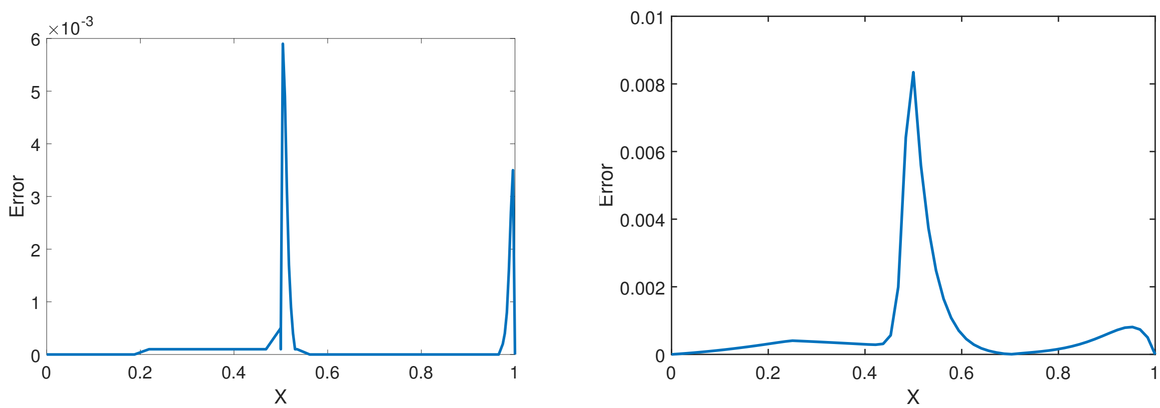
(a) Example 5.1 *(b) Example 5.2*

FIGURE 2. Solution plots with $N = 64, \varepsilon = 10^{-4}$.



(a) Example 5.1 *(b) Example 5.2*

FIGURE 3. *Loglog plots of maximum pointwise error.*



(a) Example 5.1 *(b) Example 5.2*

FIGURE 4. Error plots with $N = 64, \varepsilon = 10^{-4}$.

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