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Existence, uniqueness, and finite-time stability of solutions for Ψ-Caputo fractional differential equations with time delay

Naoufel Hatime[1], **Said Melliani**[1], **Ali El Mfadel**[1,2,*], **and M'hamed Elomari**[1]
[1]LMACS Laboratory, Sultan Moulay Slimane University, Beni Mellal, Morocco.
[2]Superior School of Technology, Sultan Moulay Slimane University, Khénifra, Morocco.

Abstract

In this paper, we study the existence, uniqueness, and finite-time stability results for fractional delayed Newton cooling-law equation involving Ψ-Caputo fractional derivatives of order $\alpha \in (0, 1)$. By using Banach fixed point theorem, Henry-Gronwall type retarded integral inequalities, and some techniques of Ψ-Caputo fractional calculus, we establish the existence and uniqueness of solutions for our proposed model. Based on the heat transfer model, a new criterion for finite time stability and some estimated results of solutions with time delay are derived. In addition, we give some specific examples with graphs and numerical experiments to illustrate the obtained results. More importantly, the comparison of model predictions versus experimental data, classical model, and non-delayed model shows the effectiveness of our proposed model with a reasonable precision.

Keywords. Newton's law of cooling equation, Ψ-Caputo fractional derivative, delay, Modelling nature.
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1. Introduction

For decades, ordinary differential equations (ODEs) have been efficiently and frequently used to model many real-world phenomena. For examples in population dynamics, we can find the following approach: maximum sustainable yield model [40, 41], Lotka-Volterra model [33, 44], and SIR (Susceptible, Infectious, and Recovered or Deceased)/SEIR (Susceptible, Exposed, Infectious, and Recovered or Deceased) epidemic model [12, 26], but in biology we can also find drug concentration in the blood model [27, 28] and Michaelis–Menten kinetics model [22, 37, 38]. Using the fact that the classical derivative of a function reveals the actual rate at which that function varies in relation to its independent variable, the obvious question here is: can ODEs always provide us with a precise prediction and a realistic depiction of the real world? At the beginning of the 18^{th} century, Newton [11, 39] studied the phenomena of cooling for a various types of solids, He concluded that an object's rate of change is related to the gap between its and the surrounding air's temperatures. Today this is known as "Newton's Law of Cooling" which is in reality a model of heat transfer [13].
If we denote by $u(s)$ the real-time temperature of a specific object in a room with a constant temperature equal to u_a (called the ambient temperature), then $u(s)$ satisfy the following ODE.

$$\dot{u}(s) = r\big(u(s) - u_a\big), \ s > 0, \tag{1.1}$$

it follows that

$$u(s) \quad = \quad \big(u_0 - u_a\big)e^{rs} + u_a, \ s > 0,$$

where $u_0 := u(0)$ denotes the initial temperature and r an empirical value known as the heat transfer coefficient which measure of the amount of heat transferred quantitatively by convection between a fluid and the wall it flows over. The same model can help forensic pathologists to determine the approximate time of death with a variable ambient

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∗ Corresponding author. Email: a.elmfadel@usms.ma.

temperature [31], such as Marshall and Hoare model [36]. Banks [10] use model (1.1) in another context with $u_a = -\frac{1}{2}$ to analyze the NFL football team results in a duration of 40 years, where it takes a delay duration of around 8 years for a team to reverse for better or worse, for more details about delay differential equations and their applications in real life we suggest [1, 20, 23, 42, 46, 47] and references therein to our readers. Recently Almeida modified [5] the model (1.1) by using ${}^{\mathrm{C}}D_{0+}^{\alpha,\Psi}$ the Ψ-Caputo fractional derivative of lower limit zero and of order α belong to $(0,2)$ as a tool to approximate the instantaneous rate of change, instead of the classical derivatives. Therefore, a fractional derivative with respect to another function known as the Ψ-Caputo derivative was introduced in order to study fractional differential equations in a general manner. For specific selections of Ψ, we can obtain some well-known fractional derivatives, such as Riemann-Liouville [25], Caputo [2], Hilfer [24], Erdelyi-Kober [34], and Hadamard [3], which are dependent on a kernel. From the viewpoint of applications, this approach also seems appropriate. With the help of a good selection of a "trial" function Ψ, the $\Psi-$Caputo fractional derivative allows some measure of control over the modeling of the phenomenon under consideration. As well as reading articles, the reader can see [16–19] and the references therein for more details on Ψ-fractional evolution and differential equations. Therefore, the purpose of this work is to add to the growing body of knowledge in this area. By using this type of kernel Ψ, the classical model (1.1) can be written as follows.

$$ {}^{\mathrm{C}}D_{0+}^{\alpha,\Psi}u(s) = r\big(u(s) - u_a)\big), s > 0, \tag{1.2} $$

where α belong to $(0,1)$ and $s > 0$.In this case, the solution of model (1.2) is given by using the well-known one-parameter Mittag-Leffler function E_α as follows:

$$ u(s) = (u_0 - u_a)E_\alpha(r(\Psi(s) - \Psi(0))^\alpha) + u_a, s > 0. $$

but the previous models do not take into consideration the clear delay effect between the original temperature of the object and taking its measurements in the first time. We assume that we know the temperature of our object in a part of time with duration $\tau > 0$, and we denote it by $\varphi(s)$, this motivate us to study the following generalized fractional Newton's Law of Cooling with delay.

$$ {}^{\mathrm{C}}D_{0+}^{\alpha,\Psi}u(s) = r\big(u(s-\tau) - u_a\big), \ s \in [0,T], \tag{1.3} $$

subject to the initial condition $\varphi \in C^1([-\tau,\ 0],\ \mathbb{R})$ $\big($ i.e $u(s) = \varphi(s)$ for all $s \in [-\tau, 0]$ $\big)$, where α belong to $(0,1)$, $T > 0$ and $\tau > 0$ denote the delay.To the best of the authors' knowledge, no article has studied the existence, uniqueness, and finite time stability results for fractional delayed Newton's law of cooling involving Ψ-Caputo fractional derivatives of order $\alpha \in (0,1)$. We shall then close this deficit. It is crucial to note that the solutions reported in this study are novel and produce a number of novel results as special instances for adequate parameter selection in the relevant issues.

The structure of this article is organized as follows. In section 1, We start the paper with an introduction and the problem statements. In section 2, We give some necessary notations, definitions and lemmas of Ψ-Caputo fractional calculus. In section 3, by employing Banach fixed point theorem, we prove the existence and uniqueness of the solution under a condition on the heat transfer coefficient. In section 4, for a specific class of Ψ, we give the solution via a constructive approach. In section 5, since the real phenomenons studied converge to equilibrium in finite time, based on results of sections 3 and 4, we derive a sufficient condition for finite time stability and some estimation results of our proposed model. In section 6, based on the results of section 3, we propose an accurate numerical scheme to approximate the generalized fractional Newton's Law of Cooling with delay. We conclude this paper by comparing the model prediction versus experimental data and classical model with a non-delayed model in section 7 followed by the conclusion in section 8.

2. Preliminaries

In this section, we introduce the necessary framework that we need to achieve our target, all over this paper, we denote by $C(\Delta,\mathbb{R})$ the space of continuous real-valued functions on Δ equipped with the uniform norm $\|u\|_\Delta = \sup_{s\in\Delta}|u(s)|$. Generally, we designate by $C^m(\Delta,\mathbb{R})$ the space of m-times continuously differentiable real-valued functions on Δ.

We introduce the Almeida kernel space over an interval [a,b] as:

$$Al^n_{a,b}(\Delta) := \{\Psi \in C^n([a,b],\Delta) \ and \ \Psi'(t) > 0 \ for \ all \ t \ \in [a,b]\}.$$

So for an arbitrary $\Psi \in Al^n_{a,b}(\mathbb{R})$, Almeida [4, 6] define the Ψ-Caputo fractional integral of order $\alpha > 0$ as follows:

$$I^{\alpha,\Psi}_{a+}u(s) := \frac{1}{\Gamma(\alpha)}\int_a^s \Psi'(\theta)(\Psi(s)-\Psi(\theta))^{\alpha-1}u(\theta)d\theta.$$

Let $q \in \mathbb{N}^*$ and u be a function in $C^{q-1}([a,b],\mathbb{R})$, we consider the following operator :

$$u^{[q]}_{\Psi}(s) := \left(\frac{1}{\Psi'(s)}\frac{d}{ds}\right)^q u(s) \ .$$

Let $n = [\alpha] + 1 - \chi_{\mathbb{N}}(\alpha)$, where χ denotes the indicator function [ ] the floor function, hence, for an arbitrary u in $C^n([a,b],\mathbb{R})$ the Ψ-Caputo fractional derivative [16, 17] can be defined as

$$^{\mathrm{C}}D^{\alpha,\Psi}_{a+}u(s) = I^{n-\alpha,\Psi}_{a+}u^{[n]}_{\Psi}(s).$$

It follows that

$$^{\mathrm{C}}D^{\alpha,\Psi}_{a+}u(s) := \frac{1-\chi_{\mathbb{N}^*}(\alpha)}{\Gamma(n-\alpha)}\int_a^s \Psi'(\theta)(\Psi(s)-\Psi(\theta))^{n-\alpha-1}u^{[n]}_{\Psi}(\theta)d\theta + \chi_{\mathbb{N}^*}(\alpha)u^{[n]}_{\Psi}(s).$$

In our case, the previous expression can be rewritten as follows

$$^{\mathrm{C}}D^{\alpha,\Psi}_{0+}u(s) = \frac{1}{\Gamma(1-\alpha)}\int_0^s (\Psi(s)-\Psi(\theta))^{-\alpha}u'(\theta)d\theta \ for \ all \ s \ in \ [0,T].$$

Proposition 2.1. *[6, 18] Let $\alpha > 0$ and $n = [\alpha] + 1 - \chi_{\mathbb{N}}(\alpha)$, if $u \in C^{n-1}([0,T],\mathbb{R})$, then for all t in [0,T], we have*

1) $^{C}D^{\alpha,\Psi}_{0+}I^{\alpha,\Psi}_{0+}u(s) = u(s)$.
2) $I^{\alpha,\Psi}_{0+}\ {}^{C}D^{\alpha,\Psi}_{0+}u(s) = u(s) - \sum_{k=0}^{n-1}\frac{u^{[k]}_{\Psi}(0)}{k!}(\Psi(s)-\Psi(0))^k.$

Definition 2.2. [25] We recall that the Mittag-Leffler function is given by

$$E_\alpha(z) = \sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \quad \alpha \in \mathbb{C}, R(\alpha) > 0, z \in \mathbb{C},$$

and the general form is given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \beta,\alpha \in \mathbb{C}, R(\beta) > 0, R(\alpha) > 0, z \in \mathbb{C}.$$

Far away of fractional calculus, we introduce here a definition of finite time stability and a short version of delayed Henry–Gronwall integral inequality (Lemma 2.3 [45]).

Definition 2.3. [14] The model (1.3) is finite time stable with respect to $\{0, V, \tau, \sqrt{\delta}, \sqrt{\beta}\}$ if and only if $\|\varphi\|_{[-\tau,0]} < \delta$ implies that the solution u of (1.3) satisfies $|u(s)| < \beta$ for all $s \in V$ where $\delta < \beta$.

Lemma 2.4. *[45] Assume that $a, b \in C([0,T],\mathbb{R}^+), z \in C([-\tau,0],\mathbb{R}^+)$ and a and z are nondecreasing functions such that $a(0) = z(0)$. If $v \in C([-\tau,T],\mathbb{R}^+)$ and*

$$\begin{cases} v(s) \le a(s) + \int_0^s b(\theta)v(\theta-\tau)d\theta, \ \ s \in [0,T], \\ v(s) \le z(s), \ \ s \in [-\tau,0], \end{cases}$$

then

$$v(s) \leq a(s) \exp\left(\int_0^s b(\theta)d\theta\right), \ s \in [0,T].$$

3. Existence and uniqueness results

Following Almeida approach [7], we give an existence and uniqueness result for our model (1.3).

Proposition 3.1. *A continuous function u on $[\tau, T]$ is a solution to our problem (1.3) if and only if u satisfies the delayed integral equation of the second kind:*

$$u(s) = \begin{cases} \varphi(s) \ , & s \in [-\tau, 0], \\ \varphi(0) + rI_{0+}^{\alpha,\Psi} u(s-\tau) - ru_a \mathbb{A}_{\alpha,\Psi}(s), & s \in [0,T], \end{cases}$$

where

$$\mathbb{A}_{\alpha,\Psi}(s) = \frac{1}{\Gamma(\alpha)} \int_0^s \Psi'(\theta)(\Psi(s)-\Psi(\theta))^{\alpha-1} d\theta.$$

Proof. Let $s \in [0,T]$, we have ${}^{\mathrm{C}}D_{0+}^{\alpha,\Psi} u(s) = r\big(u(s-\tau) - u_a\big)$, by using the definition of Ψ-Caputo fractional integral, we have

$$I_{0+}^{\alpha,\Psi}\ {}^{C}D_{0+}^{\alpha,\Psi} u(s) = rI_{0+}^{\alpha,\Psi} u(s-\tau) - ru_a \mathbb{A}_{\alpha,\Psi}(s).$$

On the other hand, via Proposition 2.1 we have

$$I_{0+}^{\alpha,\Psi}\ {}^{C}D_{0+}^{\alpha,\Psi} u(s) = u(s) - u(0).$$

Consequently, we get the integral form of u.Conversely, we apply the Ψ-Caputo fractional derivative to the both sides of the previous integral equation, Proposition 2.1 leads us to deduce that u is a solution to problem (1.3). □

Theorem 3.2 (Banch fixed point theorem)**.** *[9] Let (U,d) be a complete metric space,then every contraction map $B : U \to U$ has a unique fixed point in U.*

Following the same techniques used by Almeida [7], we have the next result:

Theorem 3.3. *The problem (1.3) has a unique solution if the following inequality holds*

$$|r| < \frac{\Gamma(\alpha+1)}{(\Psi(T)-\Psi(0))^\alpha}.$$

Proof. We consider the following space,

$$U_\Psi := \{u \in C([-\tau,T],\mathbb{R}) : {}^{\mathrm{C}}D_{0+}^{\alpha,\Psi} u \text{ is continuous in } [0,T]\},$$

and $\mathfrak{B}_\Psi : U_\Psi \to U_\Psi$ defined by

$$\mathfrak{B}_\Psi[u](s) := \begin{cases} \varphi(s) \ , & s \in [-\tau, 0], \\ \varphi(0) + rI_{0+}^{\alpha,\Psi} u(s-\tau) - ru_a \mathbb{A}_{\alpha,\Psi}(s), & s \in [0,T]. \end{cases}$$

Lets $u, v \in U_\Psi$ and $s \in [0,T]$, then we have

$$\begin{aligned} \Big|\mathfrak{B}_\Psi[u](s) - \mathfrak{B}_\Psi[v](s)\Big| &= |r| \Big| I_{0+}^{\alpha,\Psi}\Big(u(s-\tau) - v(s-\tau)\Big)\Big| \\ &\leq \frac{|r|}{\Gamma(\alpha)} \int_0^s \Psi'(\theta)(\Psi(t)-\Psi(\theta))^{\alpha-1} \Big|u(s-\tau) - v(s-\tau)\Big| d\theta \\ &\leq \frac{|r|}{\Gamma(\alpha)} \int_0^s \Psi'(\theta)(\Psi(t)-\Psi(\theta))^{\alpha-1} d\theta \ \|u-v\|_{[-\tau,T]} \\ &\leq |r| \frac{(\Psi(s)-\Psi(0))^\alpha}{\Gamma(\alpha+1)} \|u-v\|_{[-\tau,T]}. \end{aligned}$$

We deduce that,

$$\|\mathfrak{B}_{\Psi}[u]-\mathfrak{B}_{\Psi}[v]\|_{[-\tau,T]} \leq |r|\frac{(\Psi(T)-\Psi(0))^{\alpha}}{\Gamma(\alpha+1)}\|u-v\|_{[-\tau,T]}.$$

It follows that $\mathfrak{B}_{\Psi}$ is a contraction, hence by the Banach fixed point theorem problem (1.3) admits a unique solution. □

4. Constructive approach of solutions

Motivated by Khusainov et al works [29, 30], specialy the idea of using the delayed exponential function to construct a continuous solution of linear delay equations and Wang et al [32, 35] works, using Caputo fractional derivative instead of classical derivative in a similar model to (1.1), we adopt the same approach in order the get an explicit solution of model (1.3).

Definition 4.1. Let $J_{\tau,k}=[(k-1)\tau,k\tau]$, we introduce the $\Psi-$Delayed Mittag-Leffler function $\mathrm{M}_{\Psi}^{\tau,\alpha}$ as

$$\mathrm{M}_{\Psi}^{\tau,\alpha}(s)=\begin{cases} 0, & s\in J_{\tau,-\infty},\\ 1, & s\in J_{-\tau,0},\\ 1+r\dfrac{\Psi(s)^{\alpha}}{\Gamma(\alpha+1)}+r^2\dfrac{\Psi(s-\tau)^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots+r^k\dfrac{\Psi(s-(k-1)\tau)^{k\alpha}}{\Gamma(k\alpha+1)}, & s\in J_{\tau,k},\end{cases}$$

where k in $\{1,2,\cdots,K\}$, $T=K\tau$ and $J_{\tau,-\infty}=(-\infty,-\tau)$.

Lemma 4.2. *Let s in $J_{\tau,k}$, $t>0$, and $\Psi(s-t)=\Psi(s)-\Psi(t)$ then we have*

$$\int_{(k-1)\tau}^{s}\Psi'(t-(k-1)\tau)(\Psi(s)-\Psi(t))^{-\alpha}\Psi(t-(k-1)\tau)^{k\alpha-1}dt=\Psi(s-(k-1)\tau)^{(k-1)\alpha}\mathcal{B}\big(1-\alpha,\ k\alpha\big),$$

where $\mathcal{B}$ is the well known Beta function.

Proof. By linearity of Ψ and a double change of variables, we get the result. □

Theorem 4.3. *The $\Psi-$Delayed Mittag-Leffler function $\mathrm{M}_{\Psi}^{\tau,\alpha}$ satisfay the delay equation*

$$^{\mathrm{C}}D_{0+}^{\alpha,\Psi}u(s) \quad = \quad ru(s-\tau),\ \textit{where } s\in[0,T]. \tag{4.1}$$

subject to the initial condition $\varphi(s)=1$ for all $s\in[-\tau,0]$.

Proof. Let $s\in(-\infty,0]$, then $s-\tau\in(-\infty,0]$, hence by the construction of $\mathrm{M}_{\Psi}^{\tau,\alpha}$ the result holds.

We proceed by induction to prove our theorem, for a fixed integer n greater than 1, we denote by P(n) the following mathematical statement:

$$\forall s\in J_{\tau,n}, D_{0+}^{\alpha,\Psi}\mathrm{M}_{\Psi}^{\tau,\alpha}(s)=r+r^2\frac{\Psi(s-\tau)^{\alpha}}{\Gamma(\alpha+1)}+r^3\frac{\Psi(s-2\tau)^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots+r^n\frac{\Psi(s-(n-1)\tau)^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)}.$$

Base Case: n = 1. Let $s\in J_{\tau,1}$, we set $u(s)=\mathrm{M}_{\Psi}^{\tau,\alpha}(s)$, then

$$u(s)=1+r\frac{\Psi(s)^{\alpha}}{\Gamma(\alpha+1)} \text{ and } u'(s)=\alpha r\frac{\Psi'(s)\Psi(s)^{\alpha}}{\Gamma(\alpha+1)}.$$

Thus by applying the Ψ Caputo derivative on u and using Lemma 4.2 we have

$$\begin{aligned}
D_{0+}^{\alpha,\Psi}\mathrm{M}_{\Psi}^{\tau,\alpha}(s) &= \frac{1}{\Gamma(1-\alpha)}\int_0^s (\Psi(s)-\Psi(t))^{-\alpha}u'(t)dt \\
&= \frac{\alpha r}{\Gamma(1-\alpha)\Gamma(\alpha+1)}\int_0^s \Psi'(t)(\Psi(s)-\Psi(t))^{-\alpha}\Psi(t)^{\alpha}dt \\
&= \frac{\alpha r\mathcal{B}(1-\alpha,\alpha)}{\Gamma(1-\alpha)\Gamma(\alpha+1)} \\
&= r \\
&= r\mathrm{M}_{\Psi}^{\tau,\alpha}(s-\tau).
\end{aligned}$$

Induction step: We assume that P(n) is true for some fixed positive integer n,

Let $s\in J_{\tau,n+1}$,and $\theta\in\Big\{1,2,\cdots,n+1\Big\}$ and we set $u_\theta(s)=r^\theta\dfrac{\Psi(s-(\theta-1)\tau)^{\theta\alpha}}{\Gamma(\theta\alpha+1)}$, then

$$u_\theta'(s)=\theta\alpha r^\theta\ \frac{\Psi'(s-(\theta-1)\tau)\Psi(s-(\theta-1)\tau)^{\theta\alpha-1}}{\Gamma(\theta\alpha+1)}.$$

According to Lemma 4.2, we have:

$$\begin{aligned}
\int_{(\theta-1)\tau}^s(\Psi(s)-\Psi(t))^{-\alpha}u_\theta'(t)dt &= \frac{\theta\alpha r^\theta}{\Gamma(\theta\alpha+1)}\Psi(s-(\theta-1)\tau)^{(\theta-1)\alpha}\mathcal{B}\big(1-\alpha,\theta\alpha\big) \\
&= \frac{r^\theta\Gamma(1-\alpha)}{\Gamma((\theta-1)\alpha+1)}\Psi(s-(\theta-1)\tau)^{(\theta-1)\alpha}.
\end{aligned}$$

We deduce that

$$\begin{aligned}
D_{0+}^{\alpha,\Psi}\mathrm{M}_{\Psi}^{\tau,\alpha}(s) &= \frac{1}{\Gamma(1-\alpha)}\int_0^s(\Psi(s)-\Psi(t))^{-\alpha}\Big(\mathrm{M}_{\Psi}^{\tau,\alpha}(t)\Big)'dt \\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{\theta=1}^{n+1}\int_{(\theta-1)\tau}^s(\Psi(s)-\Psi(t))^{-\alpha}u_\theta'(t)dt \\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{\theta=1}^{n+1}\frac{r^\theta\Gamma(1-\alpha)}{\Gamma((\theta-1)\alpha+1)}\Psi(s-(\theta-1)\tau)^{(\theta-1)\alpha} \\
&= r\Big(1+r\frac{\Psi(s-\tau)^\alpha}{\Gamma(\alpha+1)}+r^2\frac{\Psi(s-2\tau)^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots+r^n\frac{\Psi(s-n\tau)^{n\alpha}}{\Gamma(n\alpha+1)}\Big) \\
&= r\mathrm{M}_{\Psi}^{\tau,\alpha}(s-\tau).
\end{aligned}$$

□

Hence, by induction, we prove the result.

Theorem 4.4. *The continuous solution u of model(1.3) can take the following form*

$$u(s)=\mathrm{M}_{\Psi}^{\tau,\alpha}(s)\varphi(-\tau)+\int_{-\tau}^0\mathrm{M}_{\Psi}^{\tau,\alpha}(s-\tau-\theta)\varphi'(\theta)d\theta-\frac{ru_a\big(\Psi(s)-\Psi(0)\big)^\alpha}{\Gamma(\alpha+1)}.$$

Proof. Let w a real constant and $v\in C^1([-\tau,0],\mathbb{R})$, then

$$u_1(s)=\mathrm{M}_{\Psi}^{\tau,\alpha}(s)w+\int_{-\tau}^0\mathrm{M}_{\Psi}^{\tau,\alpha}(s-\tau-\theta)v(\theta)d\theta,$$

is also a solution of equation 4.1 subject to the initial condition $\varphi \in C^1([-\tau,\ 0],\ \mathbb{R})$, it follows that

$$u_1(s) = \mathrm{M}_{\Psi}^{\tau,\alpha}(s)w + \int_{-\tau}^{s} \mathrm{M}_{\Psi}^{\tau,\alpha}(s-\tau-\theta)v(\theta)d\theta + \int_{s}^{0} \mathrm{M}_{\Psi}^{\tau,\alpha}(s-\tau-\theta)v(\theta)d\theta.$$

Since $s \in J_{\tau,0}$, then by definition of $\mathrm{M}_{\Psi}^{\tau,\alpha}$ we get

$$\varphi(s) = w + \int_{-\tau}^{s} v(\theta)d\theta,$$

by taking $s = -\tau$ and differentiating the above expression of φ we deduce that $w = \varphi(-\tau)$ and $v(s) = \varphi'(s)$, hence

$$u_1(s) = \mathrm{M}_{\Psi}^{\tau,\alpha}(s)\varphi(-\tau) + \int_{-\tau}^{0} \mathrm{M}_{\Psi}^{\tau,\alpha}(s-\tau-\theta)\varphi'(\theta)d\theta.$$

On the other hand, we have

$$\mathbb{A}_{\alpha,\Psi}(s) = \frac{\big(\Psi(s)-\Psi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}.$$

Then, by the linearity of differentiation, we deduce that u is the continuous solution of model (1.3). □

5. Finite time stability and estimation results

In this section we give some estimation results of $\mathrm{M}_{\Psi}^{\tau,\alpha}$,that will help us to get a various finite time stability results.

Lemma 5.1. *For all k in $\mathbb{N}$ and s in $J_{\tau,k}$, we have*

$$|\mathrm{M}_{\Psi}^{\tau,\alpha}(s)| \leq \mathrm{E}_{\alpha}(|r||\Psi(s)|^{\alpha}).$$

Proof. Let s in $J_{\tau,k}$, by construction of $\mathrm{M}_{\Psi}^{\tau,\alpha}$ we have

$$\begin{aligned}
|\mathrm{M}_{\Psi}^{\tau,\alpha}(s)| &\leq 1 + |r|\frac{|\Psi(s)|^{\alpha}}{\Gamma(\alpha+1)} + |r|^2\frac{|\Psi(s-\tau)|^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + |r|^k\frac{|\Psi(s-(k-1)\tau)|^{k\alpha}}{\Gamma(k\alpha+1)} \\
&\leq 1 + |r|\frac{|\Psi(s)|^{\alpha}}{\Gamma(\alpha+1)} + |r|^2\frac{|\Psi(s)|^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + |r|^k\frac{|\Psi(s)|^{k\alpha}}{\Gamma(k\alpha+1)} \\
&\leq \sum_{k=0}^{\infty} \frac{|r|^k|\Psi(s)|^{k\alpha}}{\Gamma(k\alpha+1)} \\
&\leq \mathrm{E}_{\alpha}(|r||\Psi(s)|^{\alpha}).
\end{aligned}$$

□

Remark 5.2. By using the fact that Ψ is a strictly increasing function, we have the following estimation:

$$|\mathrm{M}_{\Psi}^{\tau,\alpha}(s)| \leq \mathrm{E}_{\alpha}(|r||\Psi(T)|^{\alpha}).$$

For all k in $\mathbb{N}$ and s in $J_{\tau,k}$. For the rest of this work, we assume that the condition of Theorem 4.3 is fulfilled.

Proposition 5.3. *If the following inequality holds for all s in $V = [0, T]$*

$$\mathrm{E}_{\alpha}(|r||\Psi(s)|^{\alpha}) < \frac{\beta - |u_a|}{C_{\delta,\tau,\varphi}}.$$

Then the model (1.3) is finite time stable with respect to $\{0, V, \tau, \sqrt{\delta}, \sqrt{\beta}\}$ where $C_{\delta,\tau,\varphi} = \delta + \tau\|\varphi'\|_{[-\tau,0]}$ and δ, β are defined in Definition 2.3.

Proof. The continuous solution u of problem (1.3) can take the following form

$$u(s) = \mathrm{M}_{\Psi}^{\tau,\alpha}(s)\varphi(-\tau) + \int_{-\tau}^{0} \mathrm{M}_{\Psi}^{\tau,\alpha}(s-\tau-\theta)\varphi'(\theta)d\theta + \frac{ru_a\big(\Psi(s)-\Psi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}.$$

hence,

$$|u(s)| \leq \mathrm{M}_{|\Psi|}^{\tau,\alpha}(s)|\varphi(-\tau)| + \int_{-\tau}^{0} \mathrm{M}_{|\Psi|}^{\tau,\alpha}(s-\tau-\theta)|\varphi'(\theta)|d\theta + \frac{|ru_a\big(\Psi(s)-\Psi(0)\big)^{\alpha}|}{\Gamma(\alpha+1)}.$$

By using Lemma 4.2 and Theorem 4.3 we deduce that

$$\begin{aligned}
|u(s)| &\leq \mathrm{E}_{\alpha}(|r||\Psi(s)|^{\alpha})\|\varphi\|_{[-\tau,0]} + \mathrm{E}_{\alpha}(|r||\Psi(s)|^{\alpha}) \int_{-\tau}^{0} |\varphi'(\theta)|d\theta + \frac{|ru_a|\big(\Psi(s)-\Psi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} \\
&\leq \mathrm{E}_{\alpha}(|r||\Psi(s)|^{\alpha})\Big(\|\varphi\|_{[-\tau,0]} + \tau\|\varphi'\|_{[-\tau,0]}\Big) + \frac{|ru_a|\big(\Psi(s)-\Psi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} \\
&\leq \mathrm{E}_{\alpha}(|r||\Psi(s)|^{\alpha})\Big(\delta + \tau\|\varphi'\|_{[-\tau,0]}\Big) + \frac{|ru_a|\big(\Psi(s)-\Psi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} \\
&< \mathrm{E}_{\alpha}(|r||\Psi(s)|^{\alpha})C_{\delta,\tau,\varphi} + |u_a| \\
&< \beta.
\end{aligned}$$

On the other hand, for all s in $V=[0,T]$, we have $\mathrm{E}_{\alpha}(|r||\Psi(s)|^{\alpha}) > 1$, it follow that $\beta > \delta$, hence we get our result. □

Corollary 5.4. *If the following inequality holds*

$$\mathrm{E}_{\alpha}(|r||\Psi(T)|^{\alpha}) < \frac{\beta - |u_a|}{C_{\delta,\tau,\varphi}}.$$

Then the model (1.3) *is finite time stable with respect to* $\{0, V, \tau, \sqrt{\delta}, \sqrt{\beta}\}$.

Now, we will weaken the previous results, before we introduce the next result we need to define the following real function $\Omega_{p,\alpha}(x,y) = \max\Big(|x|, |x|^{p\alpha}\Big)\max\Big(|y|, |y|^{p\alpha}\Big)$ for all $(x,y) \in \mathbb{R}^* \times \mathbb{R}^*$ and all $p \in \mathbb{N}$.

Lemma 5.5. *For all k in $\mathbb{N}$ and s in $J_{\tau,k}$, we have*

$$|\mathrm{M}_{\Psi}^{\tau,\alpha}(s)| \leq \mathrm{G}_{k,r}^{\alpha,\Psi}(s),$$

where

$$\mathrm{G}_{k,r}^{\alpha,\Psi}(s) = 1 + \frac{k\Omega_{k,\alpha}(|r|, |\Psi(s)|)}{\Gamma(\alpha+1)}.$$

Proof. By using the definition of the $\Psi-$ Delayed Mittag-Leffler function and the triangle inequality, we have,

$$\begin{aligned}
|\mathrm{M}_{\Psi}^{\tau,\alpha}(s)| &\leq 1 + |r|\frac{|\Psi(s)|^{\alpha}}{\Gamma(\alpha+1)} + |r|^2\frac{|\Psi(s-\tau)|^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + |r|^k\frac{|\Psi(s-(k-1)\tau)|^{k\alpha}}{\Gamma(k\alpha+1)} \\
&\leq 1 + \frac{\max\Big(|r|, |r|^{k\alpha}\Big)}{\Gamma(\alpha+1)}\Big(|\Psi(s)|^{\alpha} + |\Psi(s)|^{2\alpha} + \cdots + |\Psi(s)|^{k\alpha}\Big) \\
&\leq 1 + \frac{k\Omega_{k,\alpha}(|r|, |\Psi(s)|)}{\Gamma(\alpha+1)} \\
&\leq \mathrm{G}_{k,r}^{\alpha,\Psi}(s).
\end{aligned}$$

□

Theorem 5.6. *The model* (1.3) *is finite time stable with respect to* $\{0, V, \tau, \sqrt{\delta}, \sqrt{\beta}\}$ *for all* $\beta > G_{FTS}$ *where* $G_{FTS} = \mathrm{G}_{K,r}^{\alpha,\Psi}(T)C_{\delta,\tau,\varphi} + |u_a|$.

Proof. Let s in $V = [0, T]$, we proceed as in the proof of Proposition 3.1, we have the continuous solution u of problem (1.3) can take the following form

$$u(s) = \mathrm{M}_{\Psi}^{\tau,\alpha}(s)\varphi(-\tau) + \int_{-\tau}^{0} \mathrm{M}_{\Psi}^{\tau,\alpha}(s-\tau-\theta)\varphi'(\theta)d\theta + \frac{ru_a\big(\Psi(s)-\Psi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}.$$

Hence,

$$|u(s)| \leq \mathrm{M}_{|\Psi|}^{\tau,\alpha}(s)|\varphi(-\tau)| + \int_{-\tau}^{0} \mathrm{M}_{|\Psi|}^{\tau,\alpha}(s-\tau-\theta)|\varphi'(\theta)|d\theta + \frac{|ru_a\big(\Psi(s)-\Psi(0)\big)^{\alpha}|}{\Gamma(\alpha+1)}.$$

By using Lemma 5.1 and Theorem 4.4, we deduce that,

$$\begin{aligned}
|u(s)| &\leq \mathrm{G}_{k,r}^{\alpha,\Psi}(s)\|\varphi\|_{[-\tau,0]} + \mathrm{G}_{k,r}^{\alpha,\Psi}(s)\int_{-\tau}^{0}|\varphi'(\theta)|d\theta + \frac{|ru_a\big(\Psi(s)-\Psi(0)\big)^{\alpha}|}{\Gamma(\alpha+1)} \\
&\leq \mathrm{G}_{k,r}^{\alpha,\Psi}(s)\Big(\|\varphi\|_{[-\tau,0]} + \tau\|\varphi'\|_{[-\tau,0]}\Big) + \frac{|ru_a\big(\Psi(s)-\Psi(0)\big)^{\alpha}|}{\Gamma(\alpha+1)} \\
&\leq \mathrm{G}_{k,r}^{\alpha,\Psi}(s)\Big(\delta + \tau\|\varphi'\|_{[-\tau,0]}\Big) + \frac{|ru_a\big(\Psi(s)-\Psi(0)\big)^{\alpha}|}{\Gamma(\alpha+1)} \\
&< \mathrm{G}_{K,r}^{\alpha,\Psi}(T)C_{\delta,\tau,\varphi} + |u_a| \\
&< \beta.
\end{aligned}$$

On the other hand, we have $\mathrm{G}_{k,r}^{\alpha,\Psi}(T) > 1$, it follow that $\beta > \delta$, hence we get our result. $\square$

Now, we get another criterion for finite time stability by using the integral equation of the solution giving in Proposition 3.1, via a Gronwall integral inequality with time delay.

Proposition 5.7. *If the following inequality holds for all s in $V = [0, T]$*

$$a(s)\exp\left(|r|\frac{(\Psi(s)-\Psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right) < \beta.$$

Then the model (1.3) is finite time stable with respect to $\{0, V, \tau, \sqrt{\delta}, \sqrt{\beta}\}$ where $a(s) = \delta + \dfrac{|ru_a|\big(\Psi(s)-\Psi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}$.

Proof. According to Proposition 3.1, for all s in V we have,

$$\begin{aligned}
u(s) &= \varphi(0) + rI_{0+}^{\alpha,\Psi}u(s-\tau) - ru_a\mathbb{A}_{\alpha,\Psi}(s) \\
&= \varphi(0) + \frac{r}{\Gamma(\alpha)}\int_0^s \Psi'(\theta)(\Psi(s)-\Psi(\theta))^{\alpha-1}u(\theta-\tau)d\theta - \frac{ru_a\big(\Psi(s)-\Psi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}$$

Let $v(s) = |u(s)|$, then

$$\begin{aligned}
v(s) &\leq |\varphi(0)| + \frac{|r|}{\Gamma(\alpha)}\int_0^s \Psi'(\theta)(\Psi(s)-\Psi(\theta))^{\alpha-1}v(\theta-\tau)d\theta + \frac{|ru_a|\big(\Psi(s)-\Psi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} \\
&\leq \delta + \frac{|r|}{\Gamma(\alpha)}\int_0^s \Psi'(\theta)(\Psi(s)-\Psi(\theta))^{\alpha-1}v(\theta-\tau)d\theta + \frac{|ru_a|\big(\Psi(s)-\Psi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}$$

We define the following functions:

$$\begin{cases} a(s) = \delta + \dfrac{|ru_a|\big(\Psi(s)-\Psi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}, \ s \in [0,T], \\ b(\theta) = \dfrac{|r|}{\Gamma(\alpha)}\Psi'(\theta)(\Psi(s)-\Psi(\theta))^{\alpha-1}, \ \theta \in [0,s], \ s \in [0,T], \\ z(s) = \delta, \ s \in [0,T]. \end{cases}$$

It is easy to see that a,b, and z satisfy the requirements of Lemma 5.1, thus, for all $s \in [0,T]$ we deduce that,

$$\begin{aligned} |u(s)| &\le \left(\delta + \frac{|ru_a|\big(\Psi(s)-\Psi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\right) \exp\left(\int_0^s \frac{|r|}{\Gamma(\alpha)}\Psi'(\theta)(\Psi(s)-\Psi(\theta))^{\alpha-1}d\theta\right) \\ &\le \left(\delta + \frac{|ru_a|\big(\Psi(s)-\Psi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\right) \exp\left(|r|\frac{(\Psi(s)-\Psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right). \end{aligned}$$

And this is finish the proof. □

Remark 5.8. A general Gronwall's inequality involving Ψ-Caputo fractional derivative without delay was discussed in [43](Theorem 3).

Corollary 5.9. *Under the condition of Theorem 4.4, for all $\varepsilon \ge 0$, the model (1.3) is finite time stable with respect to $\{0, V, \tau, \sqrt{\delta}, \sqrt{\beta_\varepsilon}\}$ where $\beta_\varepsilon = \big(\delta + |u_a|\big)\mathrm{e} + \varepsilon$ and e denote Euler's number.*

Proof. Let s in V, we proceed as in the previous proof, we have

$$|u(s)| \le \left(\delta + \frac{|ru_a|\big(\Psi(s)-\Psi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\right) \exp\left(|r|\frac{(\Psi(s)-\Psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right).$$

The condition of Theorem 4.4 implies that,

$$|u(s)| < \big(\delta + |u_a|\big)\exp(1) \le \beta_\varepsilon.$$

hence we get our result. □

6. Numerical approximations

In this section, we propose a numerical scheme to approximate the $\Psi-$ Caputo integral, according to Proposition 3.1. The solution u satisfy a delayed Volterra integral equation of the second kind .

$$u(s) = g(s) + \int_0^s K_\Psi(s,\theta)u(\theta-\tau)d\theta, \ s \in [0,T].$$

Subject to the initial condition $\varphi \in C([-\tau,0],\mathbb{R})$. Where

$$K_\Psi(s,\theta) = \frac{r}{\Gamma(\alpha)}\Psi'(\theta)(\Psi(s)-\Psi(\theta))^{\alpha-1},$$

and

$$g(s) = \varphi(0) - ru_a\frac{\big(\Psi(s)-\Psi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}.$$

In order to solve the previous equation, so for $N \in \mathbb{N}^*$, we consider the following step $h = \frac{T}{N}$, in this case, we consider a delay $\tau = qh$ where $q \in \mathbb{N}$, we define the $s_n = nh$, $n = -q, -q+1, \cdots, -1, 0, 1 \cdots N$, in this case,

$$u(s_n) = \varphi(s_n) = \varphi(nh), \ where \ n = -q, -q+1, \cdots, -1, 0.$$

On other hand, for $n = 0, 1 \cdots N$,

$$\begin{aligned} u(s_{n+1}) &= g(s_{n+1}) + \int_0^{s_{n+1}} K_\Psi(s_{n+1}, \theta) u(\theta - \tau) d\theta \\ &= g(s_{n+1}) + \frac{r}{\Gamma(\alpha)} \int_0^{s_{n+1}} \Psi'(\theta)(\Psi(s_{n+1}) - \Psi(\theta))^{\alpha-1} u(\theta - \tau) d\theta \\ &= g(s_{n+1}) + \frac{r}{\Gamma(\alpha)} \sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} \Psi'(\theta)(\Psi(s_{n+1}) - \Psi(\theta))^{\alpha-1} u(\theta - \tau) d\theta \\ &= g(s_{n+1}) + \frac{r}{\Gamma(\alpha)} \sum_{i=0}^{n} \int_{ih}^{(i+1)h} \Psi'(\theta)(\Psi(s_{n+1}) - \Psi(\theta))^{\alpha-1} u(\theta - \tau) d\theta. \end{aligned}$$

For a very small values of h, one can see that, for $\theta \in [s_i, s_{i+1}]$ such that $i \in \{0, 1, \cdot, n\}$, then $u(\theta - \tau)$ approximately equal to $u(s_{i-q})$, if we set

$$H_\alpha^\Psi(s, x) = \Big(\Psi(s) - \Psi(x)\Big)^\alpha.$$

We deduce that

$$\begin{aligned} u(s_{n+1}) &= g(s_{n+1}) + \frac{r}{\Gamma(\alpha)} \sum_{i=0}^{n} \int_{ih}^{(i+1)h} \Psi'(\theta)(\Psi(s_{n+1}) - \Psi(\theta))^{\alpha-1} u(s_{i-q}) d\theta \\ &= g(s_{n+1}) + \frac{r}{\Gamma(\alpha+1)} \sum_{i=0}^{n} \Big(H_\alpha^\Psi(s_{n+1}, s_i) - H_\alpha^\Psi(s_{n+1}, s_{i+1})\Big) u(s_{i-q}). \end{aligned}$$

To test the accuracy of the proposed scheme, we consider the case of $q = 0$ ($i.e$ $\tau = 0$), in this scenario, we are dealing with the model (1.2), which has the exact solution:

$$u(s) = (u_0 - u_a) E_\alpha(r(\Psi(s) - \Psi(0))^\alpha) + u_a, \ for \ s > 0.$$

The numerical simulation of the model (1.2) was taking over the interval $[0, 60]$ with the following values in both tests $u_0 = 100$, $u_a = 23$.

Let u be the exact solution of model (1.2) and u_{app} the proposed approximation over $[0, T]$, we refer to $\|u - u_{app}\|_\infty$ as the piecewise error, $\|u - u_{app}\|_2$ as the absolute error and $\dfrac{\|u - u_{app}\|_2}{\|u\|_2}$ as the relative error, where $\|\cdot\|_2$ denotes the well-known ℓ^2 vector norm and $\|\cdot\|_\infty$ the infinity vector norm.

The four tables show the accuracy of the proposed scheme on a large interval, note that the errors decrease depending on the decrease of step h, our approach gives us the reasonable precision.

7. Experimental data and model validity

In this section, we aim to test the validity of our model (1.3) based on real experimental data [21], the numerical analysis is done in Matlab based on the previous section. Gieseking takes 3 beakers (with different capacities: 100 ml, 300 ml, 800 ml) held water at $100°C$ in a room with a constant temperature equal to $23°C$ (i.e the ambient temperature $u_a = 23°C$), she placed the beakers on a granite countertop, and kept a thermometer in each beakers, in order to avoid any temperature lag, the measurements were taken every minute for 35 minutes,and every 5 minutes for the remainder 25 minutes [21], All numerical simulation was taking over the interval $[0, 60]$ with N=153600 (i.e $h = 3.90625e^{-04}$) and $\Psi(s) = s + 1$ with the initial condition $\varphi(s) = \Big(1 - \dfrac{s+\tau}{\tau}\Big)100 + \Big(\dfrac{s+\tau}{\tau}\Big)99.98$ and a delay of the form $\tau = qh$.

TABLE 1. Relative error for model (1.2) where $r = -0.130333$ and $\Psi(s) = s$.

h	N	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
1.0000e-01	600	5.8147e-01	4.5686e-01	7.9673e-01	1.4185e00
5.0000e-02	1200	4.4836e-01	3.0962e-01	5.5597e-01	9.9893e-01
2.5000e-02	2400	3.4517e-01	2.1124e-01	3.8980e-01	7.0474e-01
1.2500e-02	4800	2.6533e-01	1.4501e-01	2.7411e-01	4.9768e-01
6.2500e-03	9600	2.0369e-01	1.0010e-01	1.9314e-01	3.5166e-01
3.1250e-03	19200	1.5617e-01	6.9421e-02	1.3625e-01	2.4856e-01
1.5625e-03	38400	1.1960e-01	4.8340e-02	9.6195e-02	1.7572e-01
7.8125e-04	76800	9.1493e-02	3.3771e-02	6.7952e-02	1.2423e-01

TABLE 2. Piecewise error for model (1.2) where $r = -0.130333$ and $\Psi(s) = s$.

h	N	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
1.0000e-01	600	5.4246e-01	2.1319e-01	7.3320e-02	1.0119e-01
5.0000e-02	1200	4.1519e-01	1.2371e-01	3.2146e-02	5.0298e-02
2.5000e-02	2400	3.1739e-01	7.1613e-02	1.4579e-02	2.5067e-02
1.2500e-02	4800	2.4237e-01	4.1378e-02	7.1897e-03	1.2510e-02
6.2500e-03	9600	1.8491e-01	2.3874e-02	3.5638e-03	6.2488e-03
3.1250e-03	19200	1.4096e-01	1.3760e-02	1.7720e-03	3.1226e-03
1.5625e-03	38400	1.0737e-01	7.9237e-03	8.8281e-04	1.5608e-03
7.8125e-04	76800	8.1736e-02	4.5601e-03	4.4037e-04	7.8025e-04

TABLE 3. Relative error for model (1.2) where $r = -0.0666$ and $\Psi(s) = log(s+1)$.

h	N	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
1.0000e-01	600	1.6216e-01	1.3129e-01	1.7325e-01	2.3956e-01
5.0000e-02	1200	1.2481e-01	8.9125e-02	1.1970e-01	1.6783e-01
2.5000e-02	2400	9.5668e-02	6.0672e-02	8.3180e-02	1.1795e-01
1.2500e-02	4800	7.3158e-02	4.1504e-02	5.8093e-02	8.3077e-02
6.2500e-03	9600	5.5864e-02	2.8545e-02	4.0726e-02	5.8604e-02
3.1250e-03	19200	4.2615e-02	1.9733e-02	2.8629e-02	4.1379e-02
1.5625e-03	38400	3.2484e-02	1.3703e-02	2.0164e-02	2.9234e-02
7.8125e-04	76800	2.4747e-02	9.5519e-03	1.4220e-02	2.0661e-02

TABLE 4. Piecewise errors for model (1.2) where $r = -0.0666$ and $\Psi(s) = log(s+1)$.

h	N	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
1.0000e-01	600	1.4435e-01	5.4722e-02	1.8268e-02	1.1601e-02
5.0000e-02	1200	1.1099e-01	3.2190e-02	8.2080e-03	5.7339e-03
2.5000e-02	2400	8.4901e-02	1.8743e-02	3.6338e-03	2.8457e-03
1.2500e-02	4800	6.4756e-02	1.0851e-02	1.5960e-03	1.4163e-03
6.2500e-03	9600	4.9307e-02	6.2623e-03	7.7357e-04	7.0612e-04
3.1250e-03	19200	3.7505e-02	3.6076e-03	3.8143e-04	3.5246e-04
1.5625e-03	38400	2.8507e-02	2.0761e-03	1.8900e-04	1.7605e-04
7.8125e-04	76800	2.1658e-02	1.1940e-03	9.3942e-05	8.7971e-05

TABLE 5. Comparison between the three models with respect to the first beaker data (100ml).

Model	α	r	Absolute error	Relative error
Classical (1.1)	1.0	-0.067600	30.738975	0.090710
Fractional without delay (1.2)	0.800000	-0.130333	6.371300	0.018802
Fractional with delay (1.3)	0.800000	-0.130333	6.374700	0.018812

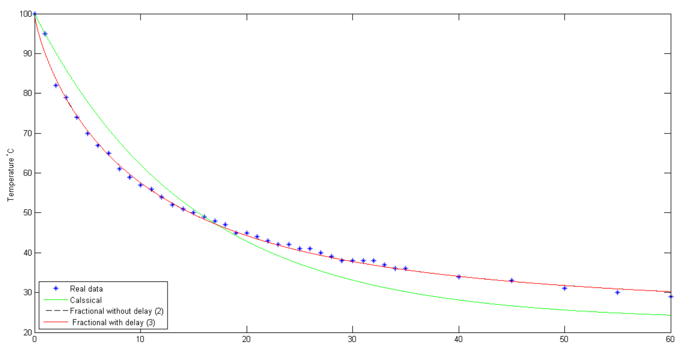

FIGURE 1. Solutions of (1.1), (1.2), and (1.3) models in the description of the real-time temperature of the 100 ml beaker versus experimental data.

TABLE 6. Comparison between the three models with respect to the second beaker data (300ml).

Model	α	r	Absolute error	Relative error
Classical (1.1)	1.0	-0.044700	23.670070	0.059795
Fractional without delay (1.2)	0.820000	-0.075740	3.851464	0.009729
Fractional with delay (1.3)	0.820000	-0.075740	3.847564	0.009719

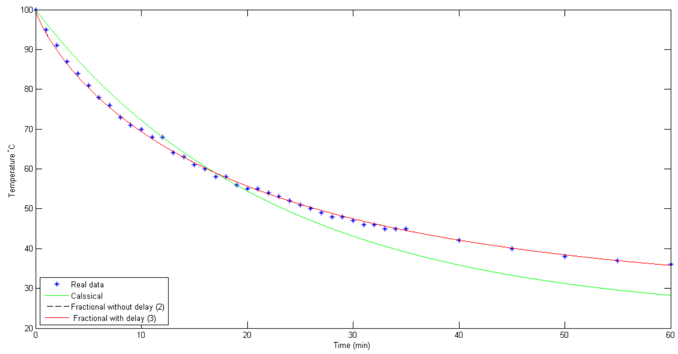

FIGURE 2. Solutions of (1.1), (1.2), and (1.3) models in the description of the real-time temperature of the 300 ml beaker versus experimental data.

TABLE 7. Comparison between the three models with respect to the third beaker data (800ml).

Model	α	r	Absolute error	Relative error
Classical (1.1)	1.0	-0.032700	14.405947	0.033108
Fractional without delay (1.2)	0.870000	-0.0487470	5.6706090	0.013032
Fractional with delay (1.3)	0.870000	-0.0487470	5.6705990	0.013031

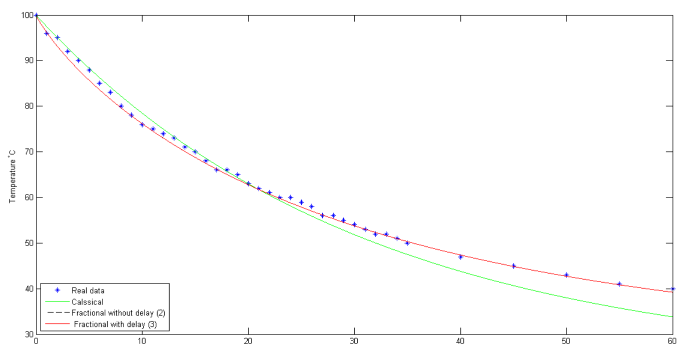

FIGURE 3. Solutions of (1.1), (1.2), and (1.3) models in the description of the real-time temperature of the 800 ml beaker versus experimental data.

On the one hand, empirical data reveal that the water cooled faster in the smaller beakers than in the larger ones, on the other hand, since the beakers were placed on a granite countertop, the heat loss by conduction with the countertop is substantial at the beginning of the investigation (*which we could consider it as a physical delay*) and is greater than later when the surface has warmed [21], that means the environment doesn't maintain a constant temperature at the initial few seconds, which is contrary to the hypotheses on which the first model was built, this explains why the fractional models (1.2) and (1.3) do not match exactly with the data in a few points.

If the countertop is suddenly warmer than the surrounding air, the temperature gradient is not what the initial temperature measurement implied. The numerical analysis shows that the proposed model (1.3) is slightly better than the model (1.2), a more deference between the two models (1.2) and (1.3) can be found in the case of a variable ambient temperature.

Table 8. Experimental data of the three beakers of water (see [21]).

Time (min)	100 ml temperature $^{\circ}C$	300 ml temperature$^{\circ}C$	800 ml temperature $^{\circ}C$
0	100	100	100
1	95	95	96
2	82	91	95
3	79	87	92
4	74	84	90
5	70	81	88
6	67	78	85
7	65	76	83
8	61	73	80
9	59	71	78
10	57	70	76
11	56	68	75
12	54	68	74
13	52	64	73
14	51	63	71
15	50	61	70
16	49	60	68
17	48	58	66
18	47	58	66
19	45	56	65
20	45	55	63
21	44	55	62
22	43	54	61
23	42	53	60
24	42	52	60
25	41	51	59
26	41	50	58
27	40	49	56
28	39	48	56
29	38	48	55
30	38	47	54
31	38	46	53
32	38	46	52
33	37	45	52
34	36	45	51
35	36	45	50
40	34	42	47
45	33	40	45
50	31	38	43
55	30	37	41
60	29	36	40

8. Conclusion

In this paper, we have proved the existence of solutions for fractional delayed Newton's law of cooling involving Ψ-Caputo fractional derivatives of order $\alpha \in (0,1)$. The problem is issued by applying Banach fixed point theorem combined with the Henry–Gronwall inequalities and some basic tools of Ψ-Caputo fractional calculus. In addition, a novel finite time stability criterion and some estimated results of solutions with time delay are established by using the heat transfer model. Finally, the comparison of model predictions versus experimental data, classical model, and non-delayed model shows the effectiveness of our proposed model with a reasonable precision.

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Data Availability

The data used to support the findings of this study are included in table 8. and in the references within the article.

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