Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 11, No. 4, 2023, pp. 803-810
DOI:10.22034/cmde.2023.53967.2262

A finite generating set of differential invariants for Lie symmetry group of the fifth-order KdV types

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Abstract

In this paper, we study the algebraic structure of differential invariants of a fifth-order KdV equation, known as the Kawahara KdV equation. Using the moving frames method, we locate a finite generating set of differential invariants, recurrence relations, and syzygies among the differential invariants generators of the equation. We prove that the differential invariant algebra of the equation can be generated by two first-order differential invariants.

Keywords. Differential invariants, Moving frames, KdV equations, Kawaharara equation.
2010 Mathematics Subject Classification. 58J70, 35B06, 53A55.

1. Introduction

The KdV types of equations are well-known nonlinear evolution equations (NLEEs) which are a model for many physical phenomena. Many works have been done on investigating KdV equations, especially on the third-order KdV equation $u_t + \alpha u u_x + \beta u_{xxx} = 0$ (e.g. [2, 26]). However, in some cases, for instance, when the coefficient of the equation becomes very small or zero, the third-order KdV equation can not present a good model for dispersive waves. Kawahara proposes a fifth-order KdV type of equation that describes dispersive waves for these cases [19]. Since then, the fifth-order KdV types are investigated in several papers (e.g. [1, 18, 39–41]). The general form of the KdV types of equations is written as follows

$$u_t + \beta u_{xxx} - \mu u_{xxxxx} = \frac{\partial}{\partial x} f(u, u_x, u_{xx}), \tag{1.1}$$

where β, μ are constant with $\mu \neq 0$ and $f(u, u_x, u_{xx})$ is a smooth function.

In this paper, we consider the fifth-order KdV type of equation as follows:

$$u_t + \alpha u u_x + \beta u_{xxx} - \mu u_{xxxxx} = 0, \tag{1.2}$$

where α,β, and $\mu > 0$ are constant.

The equation (1.2) is known as the Kawahara KdV equation [9]. Since the value of β does not affect the differential invariants of equation (1.2), without disturbing the generality, we set β equal to one and consider the equation:

$$u_t + \alpha u u_x + u_{xxx} - \mu u_{xxxxx} = 0. \tag{1.3}$$

Using the moving frames method, we study the structure of algebras of differential invariants of the equation (1.3). As far as we know, a complete structure of algebras of differential invariants of the equation (1.3) is not obtained so far.

The concept of differential invariants comes to literature by Halphen and is studied in great detail by Lie [15, 28]. In a program titled Erlangen [20], Felix Klein presented a method to consider various geometries. The Felix Klein's

Received: 30 October 2022 ; Accepted: 03 April 2023.
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method is based on differential invariants. In 1894, Tresse proposed a theorem and extended the Lie's work to pseudo-group actions [22, 38]. Later, Cartan continued the consideration of differential invariants and developed a method known as Cartan's equivalence problem [28]. The Cartan's equivalence problem is an enhancement of Felix Klein's method [21]. Cartan continued his research and developed Cartan's moving frame approach [8]. Cartan's main effort was to obtain the equivalent geometric objects and to classify the objects using differential invariants. A modern version of Cartan's moving frames approach can be found in Fels and Olver works [11, 12]. In recent years, the moving frames method is developed by a number of researchers [3, 4, 32, 35] and is used in several papers [6, 10, 37].

The idea of differential invariants is to study a geometric object by acting as a group on the object and study the differential invariants of the group action. Indeed, to analyze geometric objects, we look for systems that remain invariant by a group transformation.

Differential invariants have many applications. The most prominent one is obtaining and studying equivalent objects. Other applications include finding similarity solutions for PDEs, and calculation of invariant variational problems.

There are many methods to study the structure of differential invariants of a system of equations. In this paragraph, we consider three approaches that are the most efficient and popular methods among researchers. We regard the advantages and disadvantages of these methods and compare them. The most famous one is the Lie method. The Lie method is well described in several monographs [17, 29] and is used in several papers [5, 14, 16, 27, 42]. Two main drawbacks of the Lie method are (1) the method needs effort for integration and (2) the method does not give complete information about the algebraic structure of differential invariants. The second method is the Cartan equivalence method [8, 28]. The Cartan's method gives complete information about the algebraic structure of differential invariants. In the Cartan method, one has to express the problem in terms of a coframe. The main disadvantage of this method is the complexity of the calculation. The third method is the moving frames method. In the moving frames approach, the structure of differential invariant algebras of the equation is yielded from knowledge of the infinitesimal determining equations and setting an appropriate cross-section normalization. It just contains simple calculations. Moreover, it gives a complete algebraic structure of differential invariants. However, in the moving frames method, the freeness of action is required. Fortunately, one can make the action free by prolonging the action.

This paper is arranged as follows: In section 2.1, we review essential preliminaries of the moving frames method. In section 3, using the moving frames method, we locate a set of generating differential invariants for equation (1.3) and obtain recurrence relations and syzygies among the generation differential invariants.

2. Preliminaries

We define coordinates $z = (x, u)$ on M, where the first α components $x = (x^1, ..., x^\alpha)$ is considered as independent variables, and the latter $\beta = m - \alpha$ components $u = (u^1, ..., u^\beta)$ is considered as independent variables. We denote the induced coordinates on the jet space J^n by $z^{(n)} = (x, u^{(n)})$, where it is consisting of independent variables x^i, dependent variables u^μ, and their derivatives u^μ_J, of order $\#J \leq n$.

The function $I : J^n \to \mathbb{R}$ is a differential invariant for the group G, if the infinitesimal generator prolongation of the group G annihilates the function I everywhere, so $X^{(n)}(I(z^{(n)})) = 0$, where $X^{(n)}$ is the n-th prolongation of the infinitesimal generator.

There is a finite number of low order invariants that generates all the differential invariants by repeated invariant differentiation [24, 38]. Indeed, there exists a finite system of differential invariants $I_1, ...I_\ell$, and α invariant differential operators $\mathcal{D}_1, ..., \mathcal{D}_\alpha$ that preserve the differential invariant algebra, such that every differential invariant can be locally expressed as a function of the invariants generators and their invariant derivatives, namely $\mathcal{D}_J I_k = \mathcal{D}_{j_1}\mathcal{D}_{j_2} \cdot \mathcal{D}_{j_k} I_k$ for $k = \#J \geq 0$. The order of differentiation is important, since generally the invariant differential operators need not commute. Moreover, the differentiated invariants are generally not functionally independent, however they have certain functional relations. Indeed, these relations are *syzygies* $H(\ldots \mathcal{D}_J I_k \ldots) \equiv 0$ [35].

2.1. **Moving Frames Method.** Cartan proposed the moving frames method to solve the equivalence problem [13]. Later, Fels and Olver [11, 12] expand the moving frames method to a completely general, algorithmic, equivariant framework which enables us to find and classify equivalence submanifolds, differential invariants, and their syzygies [30–32].

In the moving frames method, for a given system, we consider an action of a group on the system and prolong the group action until it becomes a free action. Freeness enables us to construct moving frames Next, we choose a cross-section to the prolonged group orbits and construct a moving frame. After finding a moving frame, we substitute the obtained moving frame into the prolonged group action. This substitution is recognized as invariantazation process which is used to construct complete systems of differential invariants, invariant differential operators, and invariant differential forms.

A *cross-section* is an embedded submanifold $\mathcal{K}^n \subset J^n$, that intersect the prolonged group orbits transversally. A right moving frame associates to each $z^{(n)} \in J^n$ is the unique group element $g = \rho^{(n)}(z^{(n)}) \in G$ that maps $z^{(n)}$ to the cross-section $g.z^n = \rho^{(n)}(z^{(n)}).z^{(n)} \in \mathcal{K}$ [32]. A *coordinate cross-section* is specified by setting the $r = \dim G$ coordinates to aproprate constants (i.e. $\mathcal{K} = \{z_1 = c_1, \dots, z_r = c_r\}$).

The right moving frame $g = \rho^{(n)}(z^{(n)})$ associated with the coordinate cross-section

$$\mathcal{K} = \{z_1 = c_1, \dots, z_r = c_r\}, \tag{2.1}$$

is yielded by solving

$$Z_1 = g_1.z_1 = c_1, \quad \dots \quad , Z_r = g_r.z_r = c_r, \tag{2.2}$$

where $g = (g_1, \dots, g_r)$ are group parameters. Equations (2.2) are called *normalization equations*.

A complete system of functionally independent invariants is found by replacing the moving frame for the group parameters with the last coordinate expressions [36].

$$I(z^{(n)}) = g.z^{(n)} = \rho^{(n)}(z^{(n)}).z^{(n)}. \tag{2.3}$$

The normalization components (2.2), which are constants, are called the *phantom differential invariants*. Other components (2.2) are called *fundamental invariants*.

In particular,

$$H^i(z, u^{(n)}) = \iota(x^i), \quad I^\alpha_J(z, u^{(n)}) = \iota(u^\alpha_J), \tag{2.4}$$

denotes the normalized differential invariants.

Replacing the group parameters by their moving frame $\rho^{(n)}(z, u^{(n)})$ [35], the invariantization process

$$\iota : F(z, u^{(n)}) \to I(z, u^{(n)}) = F(\rho^{(n)}(z, u^{(n)}).(z, u^{(n)})), \tag{2.5}$$

transforms the differential function F to their differential invariant counterparts $I = \iota(F)$ [35].

Even though invariantization respect all algebraic operators, it does not respect differentiation. i.e. $D[\iota(F)] \neq \iota[D(F)]$. However, the missed expression can be found by the *recurrence formula* [12]. Let $F(x, u^{(n)})$ be a differential function and $\iota(F)$ its moving frame invariantization. Then

$$D_i[\iota(F)] = \iota[D_i(F)] + \sum_{\kappa=1}^{r} R^\kappa_i \iota[X^{(n)}_\kappa(F)], \tag{2.6}$$

where R^κ_i are the Maurer-Cartan invariants and $X^{(n)}_\kappa$ are the nth prolongations of the infinitesimal generators X_κ [36].

The invariant differential operators $\mathcal{D}_i$ transform differential invariants to differential invariants. In general, they do not commute, but they satisfy in linear commutation relations in form of

$$[\mathcal{D}_i, \mathcal{D}_j] = \sum_{k=1}^{p} Y^k_{ij} \mathcal{D}_k, \quad i, j = 1, \dots, p, \tag{2.7}$$

where the coefficients Y^k_{ij} are certain differential invariants called the *commutator invariants* [35].

3. Differential Invariants of The Fifth-Order KdV Equation (1.3)

We start by considering the infinitesimal Lie transformations for equation (1.3), which are of the form:

$$x \to x + \lambda \xi^x(x,t,u), \qquad t \to t + \lambda \xi^t(x,t,u), \qquad u \to u + \lambda \varphi(x,t,u), \tag{3.1}$$

with the symmetry generator

$$X = \xi^t(x,t,u)\frac{\partial}{\partial t} + \xi^x(x,t,u)\frac{\partial}{\partial x} + \varphi(x,t,u)\frac{\partial}{\partial u}. \tag{3.2}$$

The Lie algebra of the symmetries is generated by the following three vector fields [9, 25]:

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial t}, \qquad X_3 = \alpha t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}. \tag{3.3}$$

The action of equation (1.3) symmetry group on M is obtained by composing the flows of the symmetry algebra basis (3.3), which is given by

$$(X,T,U) \quad = \exp(\lambda_1 X_1) \circ \exp(\lambda_2 X_2) \circ \exp(\lambda_3 X_3), \tag{3.4}$$

where $\lambda_1, \ldots, \lambda_3$ are the group parameters. Calculating (3.4) lead to

$$X = \alpha \lambda_3 \, t + x + \lambda_1, \qquad T = t + \lambda_2, \qquad U = u + \lambda_3.$$

The dual implicit differential operators are defined as follows [31]

$$D_{X^i} = \sum_{j=1}^{p} W_j^i D_{x^j}, \quad W_i^j = (D_{x^j} X^i)^{-1}, \tag{3.5}$$

where D_{x^i} are total derivatives and are

$$D_{x^i} = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{J} u^{\alpha}_{J,j} \frac{\partial}{\partial u^{\alpha}_J}. \tag{3.6}$$

From (3.5),

$$D_X = D_x, \qquad D_T = -\alpha \lambda_3 \, D_x + D_t. \tag{3.7}$$

We choose the coordinate cross-section that determined by the three normalization equations

$$X = 0, \qquad T = 0, \qquad U = 1, \tag{3.8}$$

By solving the normalization equations for the group parameters,

$$\lambda_1 = \alpha t \, u - \alpha \, t - x, \qquad \lambda_2 = -t, \qquad \lambda_3 = -u + 1. \tag{3.9}$$

Substituting (3.9) into (3.7), we obtain invariant differential operators

$$\mathcal{D}_1 = D_x, \qquad \mathcal{D}_2 = \alpha(u-1)D_x + D_t, \tag{3.10}$$

where $\mathcal{D}_1 = \iota(D_x)$ and $\mathcal{D}_2 = \iota(D_t)$.

Invariantization the last coordinate functions provides a complete system of fundamental differential invariants:

$$\begin{aligned} I_{00} &= \iota(u) = 1, \\ I_{10} &= \iota(u_x) = \mathcal{D}_1(u) = u_x, \\ I_{01} &= \iota(u_t) = \mathcal{D}_2(u) = \alpha(u-1)u_x + u_t, \\ I_{30} &= \iota(u_{xxx}) = u_{xxx}, \\ I_{50} &= \iota(u_{xxxxx}) = u_{xxxxx}, \\ &\vdots \end{aligned} \tag{3.11}$$

where

$$I_{ij} = \iota(u_{i,j}) = \iota\left(\frac{\partial^{i+j}\, u}{\partial x^i \partial t^j}\right). \tag{3.12}$$

Performing the invariantization process, we can express equation (1.3) in terms of the differential invariants as

$$I_{01} + \alpha I_{00} I_{10} + I_{30} - \mu\, I_{50} = 0. \tag{3.13}$$

Our next step is to locate a finite generating set of differential invariants for the equation (1.3). Differential invariants are produced order by order by applying the invariant differential operators to the differential invariants.

According to (2.6), the recurrence formula for the differential invariants are

$$\begin{aligned} \mathcal{D}_i H^j &= \delta_i^j + \sum_{\kappa=1}^{r} \iota(\xi_\kappa^j) R_i^\kappa, \\ \mathcal{D}_1 I_{jk} &= I_{j+1,k} + \sum_{\kappa=1}^{r} \iota(\varphi_\kappa^{jk}) R_1^\kappa, \\ \mathcal{D}_2 I_{jk} &= I_{j,k+1} + \sum_{\kappa=1}^{r} \iota(\varphi_\kappa^{jk}) R_2^\kappa, \end{aligned} \tag{3.14}$$

where R_1^κ and R_2^κ are the Maurer-Cartan invariants, and ξ^j and φ_κ^{jk} are the coefficients of $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial u_{jk}}$ in the infinitesimal generator prolongation of X_κ respectively [36]. Solving the resulting phantom recurrence formula produces the Maurer-Cartan invariants

$$\begin{aligned} R_1^1 = -1,\ R_1^2 = 0,\ R_1^3 = -I_{10}, \\ R_2^1 = 0,\ R_2^2 = -1,\ R_2^3 = -I_{01}, \end{aligned} \tag{3.15}$$

Substituting these expressions back into (3.14) yields the recurrence formula that specifies the form of the differential invariant algebra completely.

$$\begin{aligned}
&\mathcal{D}_1 I_{10} = I_{20}, && \mathcal{D}_2 I_{10} = I_{11}, \\
&\mathcal{D}_1 I_{01} = I_{11} + \alpha\, I_{10}^2, && \mathcal{D}_2 I_{01} = I_{02} + I_{01} I_{01}, \\
&\mathcal{D}_1 I_{11} = I_{21} + \alpha\, I_{20} I_{10}, && \mathcal{D}_2 I_{11} = I_{21} + \alpha\, I_{20} I_{01}, \\
\\
&\mathcal{D}_1 I_{20} = I_{30}, && \mathcal{D}_2 I_{20} = I_{21}, \\
&\mathcal{D}_1 I_{02} = I_{12} + 2\alpha\, I_{11} I_{10}, && \mathcal{D}_2 I_{02} = I_{03} + 2\alpha\, I_{11} I_{01}, \\
&\mathcal{D}_1 I_{21} = I_{31} + \alpha\, I_{30} I_{10}, && \mathcal{D}_2 I_{02} = I_{03} + \alpha\, I_{30} I_{01}, \\
&\mathcal{D}_1 I_{12} = I_{22} + 2\alpha\, I_{21} I_{10}, && \mathcal{D}_2 I_{02} = I_{03} + 2\alpha\, I_{21} I_{01}, \\
\\
&\mathcal{D}_1 I_{30} = I_{40}, && \mathcal{D}_2 I_{30} = I_{31}, \\
&\mathcal{D}_1 I_{03} = I_{13} + 3\alpha\, I_{12} I_{10}, && \mathcal{D}_2 I_{03} = I_{04} + 3\alpha\, I_{12} I_{01}, \\
\\
&\mathcal{D}_1 I_{40} = I_{50}, && \mathcal{D}_2 I_{40} = I_{41}, \\
&\mathcal{D}_1 I_{31} = I_{41} + \alpha\, I_{40} I_{10}, && \mathcal{D}_2 I_{31} = I_{32} + \alpha\, I_{40} I_{01}, \\
&\mathcal{D}_1 I_{22} = I_{32} + 2\alpha\, I_{31} I_{10}, && \mathcal{D}_2 I_{22} = I_{23} + 2\alpha\, I_{31} I_{01}, \\
&\mathcal{D}_1 I_{13} = I_{23} + 3\alpha\, I_{22} I_{10}, && \mathcal{D}_2 I_{13} = I_{23} + 3\alpha\, I_{22} I_{01}, \\
&\mathcal{D}_1 I_{04} = I_{14} + 4\alpha\, I_{13} I_{10}, && \mathcal{D}_2 I_{04} = I_{05} + 4\alpha\, I_{13} I_{01},
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
&\mathcal{D}_1 I_{50} = I_{60}, && \mathcal{D}_2 I_{50} = I_{51},\\
&\mathcal{D}_1 I_{41} = I_{51} + \alpha\, I_{50} I_{10}, && \mathcal{D}_2 I_{04} = I_{05} + \alpha\, I_{50} I_{01},\\
&\mathcal{D}_1 I_{32} = I_{42} + 2\alpha\, I_{41} I_{10}, && \mathcal{D}_2 I_{32} = I_{33} + 2\alpha\, I_{41} I_{01},\\
&\mathcal{D}_1 I_{23} = I_{33} + 3\alpha\, I_{32} I_{10}, && \mathcal{D}_2 I_{23} = I_{24} + 3\alpha\, I_{32} I_{01},\\
&\mathcal{D}_1 I_{14} = I_{24} + 4\alpha\, I_{23} I_{10}, && \mathcal{D}_2 I_{14} = I_{15} + 4\alpha\, I_{23} I_{01},\\
&\mathcal{D}_1 I_{05} = I_{15} + 5\alpha\, I_{14} I_{10}, && \mathcal{D}_2 I_{05} = I_{06} + 5\alpha\, I_{14} I_{01},
\end{aligned}$$

The sixth orders are

$$\begin{aligned}
&\mathcal{D}_1 I_{60} = I_{70}, && \mathcal{D}_2 I_{60} = I_{61},\\
&\mathcal{D}_1 I_{51} = I_{61} + \alpha\, I_{10}, && \mathcal{D}_2 I_{51} = I_{52} + \alpha\, I_{01},\\
&\mathcal{D}_1 I_{42} = I_{52} + 2\alpha\, I_{51} I_{10}, && \mathcal{D}_2 I_{42} = I_{43} + 2\alpha\, I_{51} I_{01},\\
&\mathcal{D}_1 I_{33} = I_{43} + 3\alpha\, I_{42} I_{10}, && \mathcal{D}_2 I_{33} = I_{34} + 3\alpha\, I_{42} I_{01},\\
&\mathcal{D}_1 I_{24} = I_{34} + 4\alpha\, I_{33} I_{10}, && \mathcal{D}_2 I_{24} = I_{25} + 4\alpha\, I_{33} I_{01},\\
&\mathcal{D}_1 I_{15} = I_{25} + 5\alpha\, I_{24} I_{10}, && \mathcal{D}_2 I_{15} = I_{16} + 5\alpha\, I_{24} I_{01},\\
&\mathcal{D}_1 I_{06} = I_{16} + 6\alpha\, I_{15} I_{10}, && \mathcal{D}_2 I_{06} = I_{07} + 6\alpha\, I_{15} I_{01}.
\end{aligned} \tag{3.17}$$

Theorem 3.1. *The set $\{I_{10}, I_{01}\}$ generates the entire differential invariant algebra of the equation (1.3).*

Proof. The recurrence formula (3.16) and (3.17) implies all differential invariants up to the seventh-order are generated by functions of I_{10} and I_{10} and their derivatives. By continuing the differentiation, we find that all differential invariants are generated by the first-order functions of I_{10} and I_{10} and their derivatives. □

Finally, the invariant differential operators $\mathcal{D}_1$ and $\mathcal{D}_2$ satisfy the commutator relation

$$[\mathcal{D}_1, \mathcal{D}_2] = Y_1 \mathcal{D}_1 + Y_2 \mathcal{D}_2, \tag{3.18}$$

where coefficients named the *commutator invariants*. As a result of general recurrence formulae for invariant horizontal differential one-forms, [12, 34], we have

$$Y_1 = \sum_{\kappa=1}^{r} \left[R_2^\kappa \iota(D_x \xi_\kappa^1) - R_1^\kappa \iota(D_t \xi_\kappa^1) \right], \qquad Y_2 = \sum_{\kappa=1}^{r} \left[R_2^\kappa \iota(D_x \xi_\kappa^2) - R_1^\kappa \iota(D_t \xi_\kappa^2) \right],$$

in which ξ_κ^i is the coefficients of ∂_{x^i}, in the infinitesimal generator X_κ.

Substituting formula (3.15) for the Maurer-Cartan invariants yields

$$Y_1 = \alpha I_{10}, \quad Y_2 = 0. \tag{3.19}$$

Thus, from (3.18) and (3.19), we have

$$[\mathcal{D}_1, \mathcal{D}_2] = \alpha I_{10} \mathcal{D}_1, \tag{3.20}$$

As discussed in section 2.1. The basic syzygy is in form of

$$\mathcal{D}_2 I_{10} - \mathcal{D}_1 I_{01} - \alpha\, I_{10}^2 = 0, \tag{3.21}$$

which is a consequence of the recurrence formulae.

The generating differential invariants $\{I_{10}, I_{01}\}$, the recurrence formulas (3.16) and (3.17), along with the commutation relations (3.20), provide a complete specification of the structure of the differential invariant algebra of equation (1.3).

4. Discussions

In this paper, we considered a special case of the KdV equation (1.1), with the specific right-hand side $f(u, u_x, u_{xx}) = \alpha\, u\, u_x$. We used classical Lie point transformations (3.1), which do not preserve the differential structure of equations containing arbitrary functions. Fortunately, equivalence transformations preserve the differential structure of equations containing arbitrary functions, e.g. [7, 23, 43]. Obtaining equivalence transformations for equation (1.1) and applying our method to the equation is desirable for further research.

5. Conclusions

In this paper, using the moving frames method, we located a finite generating set of differential invariants for the Lie symmetry group of a fifth-order KdV type, known as Kawahara KdV equation, and we obtained the recurrence relations and syzygies among the generating differential invariants. We proved that the differential invariant algebra of the equation can be generated by two first-order differential invariants. In our approach, we also obtained the Maurer-Cartan invariants. Our results can be used for the construction of sets of PDEs, which possess the same symmetry properties, and it is important for not only mathematics but also physical interpretation.

References

[1] A. Akbulut, M. Kaplan, and M. K. A. Kaabar, *New conservation laws and exact solutions of the special case of the fifth-order KdV equation*, J. Ocean Eng. Sci., *7*(4) (2022), 377–382.
[2] N. H. Aljahdaly, et al. *The analysis of the fractional-order system of third-order KdV equation within different operators*, Alex. Eng. J., *61*(12) (2022), 11825–11834.
[3] Ö. Arnaldsson, *Involutive moving frames*, Differential Geom. Appl., *69* (2020), 101603.
[4] Ö. Arnaldsson, *Involutive moving frames II; The Lie-Tresse theorem*, Differential Geom. Appl. *79* (2021), 101802.
[5] M. Bazghandi, *Lie Symmetries and Similarity Solutions of Phi-Four Equation*, Indian J. Math. *61*(2) (2019), 187–197.
[6] A. Bihlo, J. Jackaman, and F. Valiquette, *Invariant variational schemes for ordinary differential equations*, Stud. Appl. Math., *148*(3) (2022), 991–1020.
[7] Y. A. Chirkunov, *Generalized equivalence transformations and group classification of systems of differential equations*, J. Appl. Mech. Tech. Phys. *53* (2012), 147–155.
[8] E. Cartan, *Les problémes déquivalence, in Oeuvres completes, Part II*, Vol. 2, Gauthiers-Villars, Paris, (1953), 1311–1334.
[9] W. Chen, J. Li, C. Miao, and J. Wu, *Low regularity solutions of two fifth-order KdV type equations*, J. Anal. Math., *107* (2009), 221–238.
[10] J. Diehl, R. Preiß, M. Ruddy, and N. Tapia, *The Moving-Frame Method for the Iterated-Integrals Signature: Orthogonal Invariants*, Found. Comput. Math., (2022), 1–61.
[11] M. Fels and P. J. Olver, *Moving coframes. I. A practical algorithm*, Acta Appl. Math., *51* (1998), 161–213.
[12] M. Fels and P. J. Olver, *Moving coframes. II. Regularization and theoretical foundations*, Acta Appl. Math., *55* (1999), 127–208.
[13] P. A. Griffiths, *On Cartan's method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry*, Duke Math. J., *41* (1974), 775–814.
[14] Gh. Haghighatdoost, M. Bazghandi, and F. Pashaie, *Differential Invariants of coupled Hirota-Satsuma KDV Equations*, Kragujev. J. Math., *49*(5) (2025), 793–805.
[15] G. H. Halphen, *Sur les invariant différentiels, in : Oeuvres*, Vol. 2, Gauthiers-Villars, Paris, (1913), 197–253.
[16] Y. Hu, F. Zhang, and X. Xin, *Lie symmetry analysis, optimal system and exact solutions of variable-coefficients Sakovich equation*, J. Geom. Phys., *184* (2023), 104712.
[17] P. E. Hydon, *Symmetry Methods for Differential Equations*, Cambridge University Press, Cambridge, UK, 2000.
[18] D. Kaya and S. M. El-Sayed, *On a generalized fifth order KdV equations*, Phys. Lett. A, *310*(1) (2003), 44–51.
[19] T. Kawaharara, *Oscillatory solitary waves in dispersive media*, J. Phys. Soc. Japan., *33*(1) (1972), 260–264.
[20] F. Klein, *A comparative review of recent researches in geometry*, Bull. Am. Math., *2*(10) (1893), 215–249.

[21] I. A. Kogan, *Inductive Approach to Cartan's Moving Frame Method with Applications to Classical Invariant Theory*, 2000, University of Minnesota, Phd dissertation.

[22] B. Kruglikov and V. Lychagin, *Global Lie–Tresse theorem*, Sel. Math., *22* (2016), 1357–1411.

[23] R. Li, X. Yong, Y. Chen, and Y. Huang. *Equivalence transformations and differential invariants of a generalized cubic–quintic nonlinear Schrödinger equation with variable coefficients*, Nonlinear Dyn., *102* (2020), 339–348.

[24] S. Lie, and G. Scheffers, *Vorlesungen über continuierliche Gruppen mit geometrischen und anderen Anwendungen*, B.G. Teubner, Leipzig, 1893.

[25] H. Liu, J. Li, and L. Liu, *Lie symmetry analysis, optimal systems and exact solutions to the fifth-order KdV types of equations*, J. Math. Anal. Appl., *368*(2) (2010), 551–558.

[26] Q. Meng and C. Zhang, *A third-order KdV solution for internal solitary waves and its application in the numerical wave tank*, J. Ocean Eng. Sci., *1*(2) (2016), 93–108.

[27] F. Oliveri, *Lie symmetries of differential equations: classical results and recent contributions*, Symmetry, *2*(2) (2010), 658–706.

[28] P. J. Olver, *Equivalence, invariants and symmetry*, Cambridge University Press, 1995.

[29] P. J. Olver, *Applications of Lie groups to differential equations*, Vol. 107. Springer Science & Business Media, 2000.

[30] P. J. Olver and J. Pohjanpelto, *Maurer-Cartan forms and the structure of Lie pseudo-groups*, Selecta Math., *11* (2005), 99–126.

[31] P. J. Olver and J. Pohjanpelto, *Moving frames for Lie pseudo-groups*, Canadian J. Math., *60* (2008), 1336–1386.

[32] P. J. Olver and F. Valiquette, *Recursive moving frames for Lie pseudo-groups*, Results Math., *73*(2) (2018), 1–64.

[33] P. J. Olver, *Generating differential invariants*, J. Math. Anal. Appl., *333* (2007), 450–471.

[34] P. J. Olver *Differential invariants of surfaces*, Differential Geom. Appl., *27*(2) (2009), 230–239.

[35] P. J. Olver and J. Pohjanpelto. *Differential invariant algebras of Lie pseudo-groups*, Adv. Math., *222*(5) (2009), 1746–1792.

[36] G. G. Polat and P. J. Olver, *Joint differential invariants of binary and ternary forms*, Port. Math., *76*(2) (2020), 169–204.

[37] M. Sabzevari, *Convergent Normal Form for Five Dimensional Totally Nondegenerate CR Manifolds in $\mathbb{C}^4$*, J. Geom. Anal., *31*(8) (2021), 7900–7946.

[38] A. Tresse, *Sur les invariants différentiels des groupes continus de transformations*, Acta Math., *18* (1894), 1–88.

[39] A. M. Wazwaz, *The extended tanh method for new solitons solutions for many forms of the fifth-order KdV equations*, Appl. Math. Comput., *184*(2) (2007), 1002-1014.

[40] G. Wang, X. Liu, and Y. Zhang, *Lie symmetry analysis to the time fractional generalized fifth-order KdV equation*, Commun. Nonlinear Sci. Numer. Simul., *18*(9) (2013), 2321–2326.

[41] G. Wang, T. Z. Xu, and X. Q. Liu, *New explicit solutions of the fifth-order KdV equation with variable coefficients*, Bull. Malays. Math. Sci. Soc, *37*(3) (2014), 769–778.

[42] X. Yan, J. Liu, J. Yang, and X. Xin, *Lie symmetry analysis, optimal system and exact solutions for variable-coefficients (2+ 1)-dimensional dissipative long-wave system* J. Math. Anal. Appl., *518*(1) (2023), 126671.

[43] X. Yong, X. Yang, L. Wu, and J. Gao, *Equivalence transformations of a fifth-order partial differential equation with variable-coefficients*, Appl. Math. Lett., *123* (2022), 107564.