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Fitted mesh numerical scheme for singularly perturbed delay reaction diffusion problem with integral boundary condition

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Abstract

This article presents a numerical treatment of the singularly perturbed delay reaction diffusion problem with an integral boundary condition. In the considered problem, a small parameter ε, is multiplied on the higher order derivative term. The presence of this parameter causes the existence of boundary layers in the solution. The solution also exhibits an interior layer because of the large spatial delay. Simpson's $\frac{1}{3}$ rule is applied to approximate the integral boundary condition given on the right end plane. A standard finite difference scheme on piecewise uniform Shishkin mesh is proposed to discretize the problem in the spatial direction, and the Crank-Nicolson method is used in the temporal direction. The developed numerical scheme is parameter uniformly convergent, with nearly two orders of convergence in space and two orders of convergence in time. Two numerical examples are considered to validate the theoretical results.

Keywords. Singularly perturbed problems, Fitted mesh scheme, Integral boundary condition.
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1. Introduction

Many biological, chemical, and physical systems that are characterized by both spatial and temporal variables are commonly modeled by differential equations [18, 23]. Differential equations with integral boundary conditions constitute a very interesting and important class of problems [1]. Such differential equations are said to be singularly perturbed delay differential equations with integral boundary conditions if they include at least one delay term, involve unknown functions occurring with various arguments, and also have the highest derivative term multiplied by a small parameter. Delay differential equations play an important role in the mathematical modeling of various practical phenomena and are also widely applied within fields like bio-sciences, control theory, economics, material science, medicine, robotics, micro-scale heat transfer, HIV infection models, hydrodynamics of liquid helium, oceanography, population dynamics, cellular systems, meteorology, physical science, and blood flow models, etc. [19] and the reference cited there.

The existence of the singular perturbation parameter ε on the highest-order derivative term causes oscillations in the computed solution when using the classical numerical method on a uniform mesh [25]. When ε is very small, these methods require an unacceptably large number of mesh points to avoid the oscillations. This is impractical due to computer memory limitations and rounding off errors. Fitted numerical schemes are used by the authors to overcome the limitations of traditional numerical methods. Recently, numerical solutions of singularly perturbed time delay PDEs with appropriate conditions have gotten great attention, and different authors in [2–4, 11, 12] have developed uniformly convergent numerical schemes. Currently, singularly perturbed delay ordinary differential equations with integral boundary conditions are discussed, and various numerical methods have been incorporated in [5–8, 14, 20, 21].

Most recently, few authors have discussed a numerical solution for solving singularly perturbed delay partial differential equations (SPDPDEs) with integral boundary conditions. The authors in [22] proposed a standard finite

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difference method on a piecewise uniform Shishkin mesh for spatial direction and the backward Euler method for time direction for solving SPDPDEs with integral boundary condition. In [9], researchers used the fitted operator finite difference method on a uniform mesh for spatial discretization and the backward Euler method for the resulting system of initial value problems in the temporal direction by using the procedures of a method of lines. The authors in [10] proposed an exponential fitted finite difference method on uniform mesh for solving SPDPDEs with integral boundary conditions. Hailu and Duressa [13] developed the cubic spline method on a piecewise uniform mesh for spatial direction and the Implicit Euler method for temporal direction to formulate a parameter uniform numerical scheme. The main objective of this paper is to formulate an accurate and uniformly convergent numerical scheme for solving singularly perturbed delay partial differential equations with integral boundary conditions and to establish the stability and uniform convergence analysis for the scheme. We construct a fitted finite difference method on a piecewise uniform Shishkin mesh in spatial direction and the Crank-Nicholson method in time direction.

The contents of the paper are arranged in the following manner. In section 2, the formulation of the problem is given. In section 3, the properties of continuous problem are discussed. The bounds of a solution and its derivative, and the decomposition of the solution are also discussed. In section 4, the formulation of a numerical scheme is studied. The error estimate is given in section 5. Numerical results and discussion are also discussed in section 6. Lastly, the conclusion of the paper is given in section 7.

Notation: In this paper, N_x, N_t denotes the number of mesh intervals in space and time direction respectively. C is the notation for a positive constant independent of ε, N_x and N_t. The norm $||.||$ denotes the maximum norm defined as $||u(x,t)|| = \sup |u(x,t)|$, for $(x,t) \in \mathcal{D}$.

2. Problem Formulation

We considered singularly perturbed parabolic delay partial differential equation with integral boundary conditions of the form

$$\begin{cases} \mathcal{L}u(x,t) = \left(-\varepsilon \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t} + a(x,t)\right) u(x,t) + b(x,t)u(x-1,t) = f(x,t), (x,t) \in \mathcal{D}, \\ u(x,t) = \phi_l(x,t),\ \phi_l(x,t) \in \Gamma_l = \{(x,t); -1 \le x \le 0 \text{ and } 0 \le t \le T\}, \\ \mathcal{K}u(x,t) = u(2,t) - \varepsilon \int_0^2 g(x)u(x,t)dx = \phi_r(x,t),\ \phi_r(x,t) \in \Gamma_r = \{(2,t); 0 \le t \le T\}, \\ u(x,t) = \phi_b(x,t),\ \phi_b(x,t) \in \Gamma_b = \{(x,0)\}, \end{cases} \tag{2.1}$$

where $(x,t) \in \mathcal{D} = (0,2) \times (0,T],\ \bar{\mathcal{D}} = [0,2] \times [0,T],\ \mathcal{D}_1 = (0,1) \times [0,T],\ \mathcal{D}_2 = (1,2) \times [0,T],\ \mathcal{D}^* = \mathcal{D}_1 \cup \mathcal{D}_2$ and ε is a small positive parameter $(0 < \varepsilon \ll 1)$. Assume that $a(x,t) \ge \alpha > 0,\ b(x,t) \le \beta < 0,\ \alpha + \beta \ge 2\gamma > 0,\ f(x,t),\ \phi_l,\ \phi_r,\ \phi_b$ are sufficiently smooth and $g(x)$ is monotonically non negative function and satisfy $\int_0^2 g(x)dx < 1$. The above problem (2.1) is equivalent to

$$\mathcal{L}u(x,t) = F(x,t), \tag{2.2}$$

with boundary conditions

$$\begin{cases} u(x,t) = \phi_l(x,t),\ \phi_l(x,t) \in \Gamma_l = \{(x,t); -1 \le x \le 0 \text{ and } 0 \le t \le T\}, \\ u(1^-,t) = u(1^+),\ \frac{\partial u}{\partial x}(1^-,t) = \frac{\partial u}{\partial x}(1^+,t), \\ \mathcal{K}u(x,t) = u(2,t) - \varepsilon \int_0^2 g(x)u(x,t)dx = \phi_r(x,t), \phi_r(x,t) \in \Gamma_r = \{(2,t); 0 \le t \le T\}, \\ u(x,t) = \phi_b(x,t),\ \phi_b(x,t) \in \Gamma_b = \{(x,0)\}, \end{cases} \tag{2.3}$$

where

$$\mathcal{L}u(x,t) = \begin{cases} \mathcal{L}_1 u(x,t) = \left(-\varepsilon \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t} + a(x,t)\right) u(x,t),\ (x,t) \in \mathcal{D}_1, \\ \mathcal{L}_2 u(x,t) = \left(-\varepsilon \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t} + a(x,t)\right) u(x,t) + b(x,t)u(x-1,t), \\ (x,t) \in \mathcal{D}_2, \end{cases}$$

$$F(x,t) = \begin{cases} f(x,t) - b(x,t)\phi_l(x-1,t),\ (x,t) \in \mathcal{D}_1, \\ f(x,t),\ (x,t) \in \mathcal{D}_2. \end{cases}$$

The existence and uniqueness of the solution of (2.1)-(2.3) can be established under the assumption that the data are Holder continuous and by imposing proper compatibility conditions at the corners points $(0,0),(2,0),(-1,0)$ and $(1,0)$ [15]. The problem necessarily satisfy the following compatibility conditions:

$$\phi_b(0,0)=\phi_l(0,0), \phi_b(2,0)=\phi_r(2,0),$$

and

$$-\varepsilon\frac{\partial^2\phi_b(0,0)}{\partial x^2}+a(0,0)\phi_b(0,0)+\frac{\partial\phi_l(0,0)}{\partial t}+b(0,0)\phi_l(-1,0)=f(0,0),$$
$$-\varepsilon\frac{\partial^2\phi_b(2,0)}{\partial x^2}+a(2,0)\phi_b(2,0)+\frac{\partial\phi_r(2,0)}{\partial t}+b(2,0)\phi_r(1,0)=f(2,0).$$

3. Properties of continuous problem

Here we present some properties of the continuous problem (2.1) which ensure the existence and uniqueness of the exact solution. A replication of this property in the semi-discrete form will be utilized to analyze the numerical method which we display in the next section.

Lemma 3.1. *(Continuous Maximum Principle) [22] If $\psi(x,t)\in C^{(0,0)}(\bar{\mathcal{D}})\cap C^{(1,0)}(\mathcal{D})\cap C^{(2,1)}(\mathcal{D}_1\cup\mathcal{D}_2)$ such that $\psi(0,t)\geq 0, \psi(x,0)\geq 0, \mathcal{K}\psi(2,t)\geq 0, \mathcal{L}_1\psi(x,t)\geq 0, \forall(x,t)\in\mathcal{D}_1, \mathcal{L}_2\psi(x,t)\geq 0, \forall(x,t)\in\mathcal{D}_2$ and $[\psi_x](1,t)=\psi_x(1^+,t)-\psi_x(1^-,t)\leq 0$, then $\psi(x,t)\geq 0$, for all $(x,t)\in\bar{\mathcal{D}}$.*

Proof. Using the basic idea used in the proof of lemma (1) of [22] and the test function $S(x,t)$ given by,

$$S(x,t)=\begin{cases}\frac{1}{8}+\frac{x}{2}, & (x,t)\in\mathcal{D}_1,\\ \frac{3}{8}+\frac{x}{4}, & (x,t)\in\mathcal{D}_2.\end{cases} \tag{3.1}$$

the above lemma can be proved. □

An immediate result of the continuous maximum principle is the following stability result.

Lemma 3.2. *(Stability Result) The solution $u(x,t)$ for the problems (2.1) satisfies the bound*

$$||u||_{\bar{\mathcal{D}}}=C\max\left\{||u||_{\Gamma_l},||u||_{\Gamma_b},||\mathcal{K}u||_{\Gamma_r},||\mathcal{L}u||_{\mathcal{D}^*}\right\}.$$

Proof. We prove by using maximum principle Lemma 3.1 and by constructing the barrier functions $\Theta^{\pm}(x,t)=CMS(x,t)\pm u(x,t),(x,t)\in\bar{\mathcal{D}}$, where $M=\max\left\{||u||_{\Gamma_l},||u||_{\Gamma_b},||u||_{\Gamma_r},||\mathcal{L}u||_{\mathcal{D}^*}\right\}$ and $S(x,t)$ is the test function in (3.1). □

The existence of the solution is unique if the sufficient conditions satisfy the following theorem.

Lemma 3.3. *If the coefficient satisfies $a(x,t),b(x,t),f(x,t)\in C^{(\alpha_1,\alpha_1/2)}(\bar{\mathcal{D}})$ and boundary conditions satisfies $\phi_l\in C^{1+\alpha_1/2}([0,T]),\phi_b\in C^{(2+\alpha_1,1+\alpha_1/2)}(\Gamma_b),\phi_r\in C^{1+\alpha_1/2([0,T])},\alpha_1\in(0,1)$ and assume that the compatibility conditions are satisfied. Then, the problem (2.1) have a unique solution is u which is satisfy $u\in C^{(2+\alpha_1,1+\alpha_1/2)}(\bar{\mathcal{D}})$.*

Proof. See [15]. □

3.1. **Bounds on the solution and its derivatives.** To estimate an error for the fitted mesh finite difference method under the assumption that the solution of (2.1) is more regular than the one guaranteed by the result in Lemma (3.4). To get this greater regularity, stronger compatibility conditions are imposed at the corners.

Lemma 3.4. *If the coefficient satisfies $a(x,t),b(x,t),f(x,t)\in C^{(2+\alpha_1,1+\alpha_1/2)}(\bar{\mathcal{D}})$ and boundary conditions satisfies $\phi_l\in C^{2+\alpha_1/2}([0,T]),\phi_b\in C^{(4+\alpha_1,2+\alpha_1/2)}(\Gamma_b),\phi_r\in C^{2+\alpha_1/2([0,T])}$, where $\alpha_1\in(0,1)$. Then, the problem (2.1) have a unique solution is u which satisfy $u\in C^{(4+\alpha_1,2+\alpha_1/2)}(\bar{\mathcal{D}})$. And also the derivatives of solution u are bounded, $\forall i,j\in\mathbf{Z}\geq 0$ such that $0\leq i+2j\leq 4$,*

$$\left|\left|\frac{\partial^{i+j}u}{\partial r^i\partial t^j}\right|\right|\leq C\varepsilon^{\frac{-i}{2}}.$$

Proof. For the proof, interested reader can refer [22]. □

3.2. **Decomposition of the solution.** The bounds on the derivatives of the solution given in Lemma (3.4) were derived from classical results. They are not adequate for the proof of the ε -uniform error estimate. Stronger bounds on these derivatives are now obtained by a method originally given in [24]. The key step is to decompose the solution u into smooth and singular components. The solution u of (2.1) -(2.3) is decomposed into v-smooth and w- singular components.

Lemma 3.5. *If the coefficient satisfy* $a(x,t), b(x,t), f(x,t) \in C^{(4+\alpha_1, 2+\alpha_1/2)}(\bar{\mathcal{D}})$, *and the boundary conditions satisfies* $\phi_l \in C^{(3+\alpha_1/2)}([0,T])$, $\phi_b \in C^{(6+\alpha_1, 3+\alpha_1/2)}(\Gamma_b)$, $\phi_r \in C^{(3+\alpha_1/2)}([0,T])$, *where* $\alpha_1 \in (0,1)$. *Then we have*

$$\left|\left|\frac{\partial^{i+j} v}{\partial x^i \partial t^j}\right|\right|_{\bar{\mathcal{D}}} \leq C\left(1+\varepsilon^{1-i/2}\right), \tag{3.2}$$

$$\left|\frac{\partial^{i+j} w_l}{\partial x^i \partial t^j}\right| \leq \begin{cases} C\varepsilon^{\frac{-i}{2}} e^{\frac{x}{\sqrt{\varepsilon}}}, & (x,t) \in \mathcal{D}_1, \\ C\varepsilon^{\frac{-i}{2}} e^{\frac{-(x-1)}{\sqrt{\varepsilon}}}, & (x,t) \in \mathcal{D}_2, \end{cases} \tag{3.3}$$

$$\left|\frac{\partial^{i+j} w_r}{\partial x^i \partial t^j}\right| \leq \begin{cases} C\varepsilon^{\frac{-i}{2}} e^{\frac{-(1-x)}{\sqrt{\varepsilon}}}, & (x,t) \in \mathcal{D}_1, \\ C\varepsilon^{\frac{-i}{2}} e^{\frac{-(2-x)}{\sqrt{\varepsilon}}}, & (x,t) \in \mathcal{D}_2, \end{cases} \tag{3.4}$$

where C *is constant independent parameter of* ε, $(x,t) \in \bar{\mathcal{D}}$, $i, j \geq 0, 0 \leq i+2j \leq 4$.

Proof. For the proof, one can see [22]. □

4. Formulation of numerical scheme

4.1. **The time semi-discretization.** Let N_t be a positive integer. The uniform mesh Ω^{N_t} which is used in the time semi-discretization is defined as

$$\Omega^{N_t} = \left\{ t_j = j\Delta t, \quad j = 0, 1, \cdots, N_t, \quad \Delta t = \frac{T}{N_t} \right\}.$$

Using the Crank-Nicolson scheme on Ω^{N_t}, the discretized problem in the temporal direction associated with the continuous problem (2.1) is given by

$$\mathcal{L}U^{j+1}(x) = g(x, t_{j+1}), \tag{4.1}$$

where the operator $\mathcal{L}$ and function $g(x, t_{j+1})$ are defined as

$$\begin{aligned} \mathcal{L}U^{j+1}(x) &= -\frac{\varepsilon}{2}U^{j+1}_{xx} + \frac{r(x)}{2}U^{j+1}(x) + \frac{b(x)}{2}U^{j+1}(x-1), \\ g(x, t_{j+1}) &= \frac{\varepsilon}{2}U^{j}_{xx} + \frac{s(x)}{2}U^{j}(x) - \frac{b(x)}{2}U^{j}(x-1) + \frac{1}{2}\left[f(x)^{j+1} + f(x)^{j}\right], \\ &\text{and } r(x) = \frac{2}{\Delta t} + a(x), \ s(x) = \frac{2}{\Delta t} - a(x). \end{aligned}$$

Hence, the problem (2.1)-(2.3) can be written as the following

$$\mathcal{L}_k U^{j+1}(x) = g_k(x, t_{j+1}), \ k = 1, 2, \ x \in \mathcal{D}, \ j = 0, 1, \cdots, N_t - 1, \tag{4.2}$$

$$U^{j+1}(x) = \phi_l^{j+1}(x), \ \phi_l^{j+1}(x) \in \Gamma_l, \tag{4.3}$$

$$\mathcal{K}U^{j+1}(x) = U^{j+1}(2) - \varepsilon \int_0^2 g(x) U^{j+1}(x) dx = \phi_r^{j+1}(x), \ \phi_r^{j+1}(x) \in \Gamma_r, \tag{4.4}$$

$$U^{j+1}(x) = \phi_b^{j+1}(x), \ \phi_b^{j+1}(x) \in \Gamma_b, \tag{4.5}$$

where

$$\begin{aligned}
\mathcal{L}_1 U^{j+1}(x) &= -\frac{\varepsilon}{2} U^{j+1}_{xx} + \frac{r(x)}{2} U^{j+1}(x), \ x \in \mathcal{D}_1 = (0,1),\\
\mathcal{L}_2 U^{j+1}(x) &= -\frac{\varepsilon}{2} U^{j+1}_{xx} + \frac{r(x)}{2} U^{j+1}(x) + \frac{b(x)}{2} U^{j+1}(x-1),\\
g_1(x,t_{j+1}) &= \frac{\varepsilon}{2} U^{j}_{xx} + \frac{s(x)}{2} U^{j}(x) - \frac{b(x)}{2} U^{j}(x-1) + \frac{1}{2}\left[f(x)^{j+1} + f(x)^{j}\right] - \frac{b(x)}{2}\phi_l^{j+1}(x-1),\\
g_2(x,t_{j+1}) &= \frac{\varepsilon}{2} U^{j}_{xx} + \frac{s(x)}{2} U^{j}(x) - \frac{b(x)}{2} U^{j}(x-1) + \frac{1}{2}\left[f(x)^{j+1} + f(x)^{j}\right],
\end{aligned}$$

The discrete operator $\mathcal{L}$ satisfies the following lemma which can be proved by dividing the domain $\mathcal{D}$ into two parts $\mathcal{D}_1$ and $\mathcal{D}_2$.

Lemma 4.1. *For $j = 0,1,2,\cdots N_t - 1$, assume that $\psi^{j+1}(x)$ be any function in $\mathcal{D}$ such that $\psi^{j+1}(0) \geq 0$ and $\mathcal{K}\psi^{j+1}(2) \geq 0$, then $\mathcal{L}\psi^{j+1} \geq 0, \forall x \in \mathcal{D}$, implies $\psi^{j+1}(x) \geq 0$.*

Proof. For contrary assume that the function $[\psi^{j+1} + \lambda_1 S^{j+1}](x)$ attains a minimum value at $x = x_0$. Where S is defined in (3.1) and $\lambda_1 = \max\{-\frac{\psi(x,t)}{S(x,t)} : (x,t) \in \bar{\mathcal{D}}\}$. Without loss of generality we have $[\psi^{j+1} + \lambda_1 S^{j+1}](x_0) = 0$ and $[\psi^{j+1} + \lambda_1 S^{j+1}](x) \geq 0$. Suppose the lemma does not hold true, then $\lambda_1 > 0$.
Case (i): $x_0 \in \mathcal{D}_1$, $\mathcal{L}_1[\psi^{j+1} + \lambda_1 S^{j+1}](x_0) = -\frac{\varepsilon}{2}[\psi^{j+1} + \lambda_1 S^{j+1}]_{xx}(x_0) + \frac{r(x)}{2}[\psi^{j+1} + \lambda_1 S^{j+1}](x_0) \leq 0$.
Case (ii): $x_0 \in \mathcal{D}_2$, $\mathcal{L}_2[\psi^{j+1} + \lambda_1 S^{j+1}](x_0) = -\frac{\varepsilon}{2}[\psi^{j+1} + \lambda_1 S^{j+1}]_{xx}(x_0) + \frac{r(x)}{2}[\psi^{j+1} + \lambda_1 S^{j+1}](x_0) + \frac{b(x)}{2}[\psi^{j+1} + \lambda_1 S^{j+1}](x_0 - 1) \leq 0$.
Case (iii): $x_0 = 2$,

$$\mathcal{K}[\psi^{j+1} + \lambda_1 S^{j+1}](2) = [\psi^{j+1} + \lambda_1 S^{j+1}](2) - \varepsilon \int_0^2 g(x)[\psi^{j+1} + \lambda_1 S^{j+1}](x)dx \leq 0.$$

Observed that in all the cases we arrived at a contradiction. Therefore $\lambda_1 > 0$ is not possible. This implies that $\psi^{j+1}(x) \geq 0$. □

An application of the above lemma is the following uniform stability estimate.

Lemma 4.2. *The solution $U^{j+1}(x)$ of (2.1) satisfies the bound*

$$||U^{j+1}(x)||_{\bar{\mathcal{D}}} \leq C \max\left\{ ||U^{j+1}(0)||_{\Gamma_l}, ||\mathcal{K}U^{j+1}(2)||_{\Gamma_r}, \frac{\Delta t}{\gamma \Delta t + 1} ||g||_{\bar{\mathcal{D}}} \right\}.$$

Proof. The proof can be completed by considering $\mathcal{D}_1$ and $\mathcal{D}_2$ separately. Consider a two comparison functions as follows $\Pi^{\pm}(x,t_{j+1}) = C\max\left\{||U^{j+1}(0)||_{\Gamma_l}, ||\mathcal{K}U^{j+1}(2)||_{\Gamma_r}, \frac{\Delta t}{\gamma\Delta+1}||g||_{\bar{\mathcal{D}}}\right\} S(x,t_{j+1}) \pm U^{j+1}(x)$ where $S(x,t_{j+1})$ is a test function given in (3.1). Hence, the result follows by maximum principle Lemma 4.1. □

Lemma 4.3. *The time derivatives of the solution u of (2.1) are bounded i.e.,*

$$\left| \frac{\partial^j u(x,t)}{\partial t^j} \right| \leq C, \ j = 0,1,2.$$

Proof. See [14] □

Lemma 4.4. *The local truncation error associated with temporal direction satisfies*

$$\| e^{j+1} \| \leq \ C(\Delta t)^3,$$

where C is a constant independent of ε and j .

Proof. See [25]. □

Theorem 4.5. *The global error estimate $E^j = u(x,t_j) - U(x,t_{j+1})$ in the temporal direction satisfies*

$$||E^j|| \leq C(\Delta t)^2 = CN_t^{-2}, \ \forall j \leq T/\Delta t. \tag{4.6}$$

Proof. Using the local error up to the $(j+1)^{th}$ time step given in the above lemma, we obtain the following global error at the $(j+1)^{th}$ time step as

$$\begin{aligned}||E^j|| &\leq \left|\left|\sum_{i=1}^{j+1} E^j\right|\right| = ||E_1|| + ||E_2|| + ||E_3|| + ||E_4|| + \cdots + ||E_{j+1}|| \\ &\leq C((j+1)\Delta t)(\Delta t)^2, \text{ since } \ (j+1)\Delta t = T \\ &= CT(\Delta t)^2 \leq C(\Delta t)^2 \\ &= CN_t^{-2},\end{aligned}$$

where C is a constant independent of ε and Δt. □

The solution $U^{j+1}(x)$ of the problem (4.1) can be written in its decomposition form as

$$U^{j+1}(x) = V^{j+1}(x) + W^{j+1}(x),$$

where $V^{j+1}(x)$ and $W^{j+1}(x)$ are regular and singular component respectively, and $V^{j+1}(x)$ is the solution of the differential equation in $(0,1)$,

$$\begin{cases} -\frac{\varepsilon}{2}\frac{d^2V^{j+1}(x)}{dx^2} + r(x)V^{j+1}(x) = g_1^{j+1}(x) - \frac{b(x)}{2}\phi_l^{j+1}(x), \ x \in (0,1), \\ V^{j+1}(0) = V_0^{j+1}(0), \\ V^{j+1}(1) = 2r(x)^{-1}(1)\left(g_1^{j+1}(1) - \frac{b(1)}{2}\phi_l^{j+1}(0)\right), \end{cases}$$

and also it is a solution of differential equation in $(1,2)$,

$$\begin{cases} -\frac{\varepsilon}{2}\frac{d^2V^{j+1}(x)}{dx^2} + r(x)V^{j+1}(x) + \frac{b(x)}{2}V^{j+1}(x-1) = g_2^{j+1}(x), \\ V^{j+1}(1) = 2r(x)^{-1}(1)\left(g_2^{j+1}(1) - \frac{b(1)}{2}V_0^{j+1}(0)\right), \\ \mathcal{K}V^{j+1}(2) = \mathcal{K}V_0^{j+1}(2), \end{cases}$$

where $V_0^{j+1}(x)$ is the solution of the reduced problem. In the other case $W^{j+1}(x)$ is the solution of

$$\begin{cases} -\frac{\varepsilon}{2}\frac{d^2W^{j+1}(x)}{dx^2} + r(x)W^{j+1}(x) = 0, \ x \in (0,1), \\ -\frac{\varepsilon}{2}\frac{d^2W^{j+1}(x)}{dx^2} + r(x)W^{j+1}(x) + \frac{b(x)}{2}W^{j+1}(x-1), \ x \in (1,2), \\ W^{j+1} = U^{j+1}(x) - V_0^{j+1}(x), \\ \mathcal{K}W^{j+1}(2) = \mathcal{K}U^{j+1}(2) - \mathcal{K}V_0^{j+1}(2). \end{cases}$$

Lemma 4.6. *The derivatives of $V^{j+1}(x)$ and $W^{j+1}(x)$ satisfy the following estimates for $0 \leq k \leq 4$.*

$$\left|\frac{d^kV^{j+1}(x)}{dx^k}\right| \leq C\begin{cases} 1 + \varepsilon^{-\frac{(k-2)}{2}}d_1(x,\alpha), \ x \in (0,1), \ k = 0,1,2,3,4, \\ 1 + \varepsilon^{-\frac{(k-2)}{2}}d_2(x,\alpha), \ x \in (1,2), \ k = 0,1,2,3,4, \end{cases}$$

$$\left|\frac{d^kW^{j+1}(x)}{dx^k}\right| \leq C\begin{cases} \varepsilon^{-\frac{k}{2}}d_1(x,\alpha), \ x \in (0,1), \ k = 0,1,2,3,4, \\ \varepsilon^{-\frac{k}{2}}d_2(x,\alpha), \ x \in (1,2), \ k = 0,1,2,3,4, \end{cases}$$

where $d_1(x,\alpha) = e^{-x\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}} + e^{-(1-x)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}}$ and $d_2(x,\alpha) = e^{-(x-1)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}} + e^{-(2-x)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}}$.

Proof. See [16]. □

4.2. **The spatial discretization.** The piecewise-uniform mesh having $N_x(\geq 8)$ mesh elements on $[0,2]$ is generated by dividing the first half interval $[0,1]$ into three subintervals as $\Omega_1 = \Omega_l \cup \Omega_c \cup \Omega_r$ where $\Omega_1 = [0,1]$, $\Omega_l = [0,\mu]$, $\Omega_c = (\mu, 1-\mu]$, and $\Omega_r = (1-\mu, 1]$. To obtain a piecewise-uniform mesh, we place $\frac{N_x}{4}$ mesh elements in Ω_c and $\frac{N_x}{8}$ mesh elements in each of the subintervals Ω_l and Ω_r. Hence, the piecewise-uniform mesh is given by

$$x_i = \begin{cases} 0, \ i = 0, \\ x_{i-1} + h_i, \ i = 1,2,3,\cdots, \frac{N_x}{2}, \end{cases}$$

where h_i's are given by

$$h_i = \begin{cases} \frac{8\mu}{N_x}, \ i = 1,2,3,\cdots, \frac{N_x}{8}, \\ \frac{4(1-2\mu)}{N_x}, \ i = \frac{N_x}{8}+1,\cdots, \frac{3N_x}{8}, \\ \frac{8\mu}{N_x}, \ i = \frac{3N_x}{8}+1,\cdots, \frac{N_x}{2}. \end{cases}$$

Similarly $\Omega_2 = [1,2]$ divided into three subintervals. $\Omega_2 = \Omega_l \cup \Omega_c \cup \Omega_r$ where $\Omega_l = (1, 1+\mu]$, $\Omega_c = (1+\mu, 2-\mu]$, and $\Omega_r = (2-\mu, 2]$. The nodal points are then given by $x_i = x_{i-1} + h_i, \ i = \frac{N_x}{2}+1,\cdots, N_x$, where h_i's are given by

$$h_i = \begin{cases} \frac{8\mu}{N_x}, \ i = \frac{N_x}{8},\cdots, \frac{5N_x}{8}, \\ \frac{4(1-2\mu)}{N_x}, \ i = \frac{5N_x}{8}+1,\cdots, \frac{7N_x}{8}, \\ \frac{8\mu}{N_x}, \ i = \frac{7N_x}{8}+1,\cdots, N_x, \end{cases}$$

where the transition parameter μ separates the non-uniform mesh into uniform meshes and is given by $\mu = \min\left\{\frac{1}{4}, \ 2\sqrt{\varepsilon}\ln N_x\right\}$, where N_x denotes the number of mesh elements in the x-direction. The following difference formula used to discretize the problem. $D_x^- U^{j+1}(x) = \frac{U_i^{j+1} - U_{i-1}^{j+1}}{h_i}$, $D_x^+ U^{j+1}(x) = \frac{U_{i+1}^{j+1} - U_i^{j}}{h_{i+1}}$, and $D_x^+ D_x^- U^{j+1}(x) = 2\dfrac{(D_x^+ - D_x^-)}{h_i + h_{i+1}} U_i^{j+1}$, where $h_i = x_i - x_{i-1}, \ h_{i+1} = x_{i+1} - x_i$.

Now, we apply finite difference method for semi-discrete problem (4.1), i.e, we replace the second order derivative by central difference scheme.

$$\mathcal{L} U^{j+1}(x_i) = G^{j+1}(x_i),$$

where the discrete operator $\mathcal{L}^{N_x}$ and $G^{j+1}(x_i)$ is defined as

$$\mathcal{L} U(x_i) = \begin{cases} -\frac{\varepsilon}{2} D_x^+ D_x^- U_i + \frac{r_i}{2} U_i, \ i = 0,1,2,\cdots, \frac{N_x}{2}, \\ -\frac{\varepsilon}{2} D_x^+ D_x^- U_i + \frac{r_i}{2} U_i + \frac{b_i}{2} U_{i-\frac{N_x}{2}}, \ i = \frac{N_x}{2}+1, \frac{N_x}{2}+2,\cdots, N_x, \end{cases} \tag{4.7}$$

$$G^{j+1}(x_i) = \begin{cases} g_i - \frac{b_i}{2} \phi_{l_{i-\frac{N_x}{2}}}, \ i = 0,1,2,\cdots, \frac{N_x}{2}, \\ g_i, \ i = \frac{N_x}{2}+1, \frac{N_x}{2}+2,\cdots, N_x, \end{cases} \tag{4.8}$$

with boundary conditions

$$\begin{cases} U_i = \phi_{l_i}, \ i = -\frac{N_x}{2}, -\frac{N_x}{2}, \cdots, 0, \\ D_x^- U_{\frac{N_x}{2}} = D_x^+ U_{\frac{N_x}{2}}, \\ \mathcal{K}^{N_x} U_{N_x} = U_{N_x} - \varepsilon \sum_{i=1}^{N_x} \dfrac{g_{i-1} U_{i-1}^{j+1} + 4 g_i U_i^{j+1} + g_{i+1} U_{i+1}^{j+1}}{3} h_i = \phi_{r_{N_x}}, \\ U_i = \phi_b, \ i = 0,1,2,3,\cdots, N_x, \end{cases} \tag{4.9}$$

where $U_i = U(x_i), r_i = r(x_i), g_i = g(x_i), b_i = b(x_i)$. Here, for $i = N_x$, we use Simpson's $\frac{1}{3}$ rule to approximate the integral $\int_0^2 g(x) u(x,t) dx$.

Lemma 4.7. *(Semi-discrete Maximum Principle): Assume that* $\sum_{i=1}^{N} \dfrac{g_{i-1}+4g_i+g_{i+1}}{3} h = \rho < 1$ *and* $\mathcal{Z}^{j+1}$ *be any mesh function satisfying* $\mathcal{Z}_0^{j+1} \geq 0$, $\mathcal{Z}_i^{j+1} \geq 0$, $\mathcal{K}^{N_x}\mathcal{Z}_{N_x}^{j+1} \geq 0$, $\mathcal{L}_1\mathcal{Z}_i^{j+1} \geq 0, \forall i \in D_1^{N_x}$, $\mathcal{L}_2\mathcal{Z}_i^{j+1} \geq 0, \forall i \in D_2^{N_x}$, *and* $[D_x]\,\mathcal{Z}_{\frac{N_x}{2}}^{j+1} = D_x^+\mathcal{Z}_{\frac{N_x}{2}}^{j+1} - D_x^-\mathcal{Z}_{\frac{N_x}{2}}^{j+1} \leq 0$, *then* $\mathcal{Z}_i^{j+1} \geq 0$, *for all* $i = 0, 1, 2, \cdots, N_x$.

Proof. Define a test function $S^{j+1}(x_i) = \begin{cases} \frac{1}{8} + \frac{x_i}{2}, & x_i \in \mathcal{D}_1^{N_x}, \\ \frac{3}{8} + \frac{x_i}{4}, & x_i \in \mathcal{D}_2^{N_x}. \end{cases}$

Note that $S^{j+1}(x_i) \geq 0 \ \forall x_i \in \bar{\mathcal{D}}^{N_x}$, $\mathcal{L}^{N_x}S^{j+1}(x_i) > 0, \forall x_i \in \mathcal{D}_1^{N_x} \cup \mathcal{D}_2^{N_x}$, $S^{j+1}(x_0) > 0, \mathcal{K}^{N_x}S^{j+1}(x_{N_x}) > 0$ and $[D_x]\,S^{j+1}(x_{\frac{N_x}{2}}) < 0$. Let

$$\lambda = \max\left\{ \frac{-\mathcal{Z}^{j+1}(x_i)}{S^{j+1}(x_i)};\ i = 1, 2, 3, \cdots, N_x \right\}.$$

Then there exist x_i^* such that $\mathcal{Z}^{j+1}(x_i^*) + \lambda S^{j+1}(x_i^*) = 0$ and $\mathcal{Z}^{j+1}(x_i) + \lambda S^{j+1}(x_i) \geq 0$, for all $i = 1, 2, 3, \cdots, N_x, j = 1, 2, 3, \cdots, N_t - 1$. Hence the function $\mathcal{Z}^{j+1}(x_i) + \lambda S^{j+1}(x_i)$ attains a minimum value at x_i^*. Assume that the lemma does not hold true, then $\lambda > 0$.

case (i) $x_i^* = x_0$, $0 < (\mathcal{Z}^{j+1} + \lambda S^{j+1})(x_0) = 0$.

case (ii) $x_i^* = x_i,\ i = 1, 2, 3, \cdots . \frac{N_x}{2}$,

$$0 < \mathcal{L}_1(\mathcal{Z}^{j+1} + \lambda S^{j+1})(x_i^*) = \left(-\frac{\varepsilon}{2}D_x^+D_x^- + \frac{r_i}{2}\right)\left(\mathcal{Z}^{j+1} + \lambda S^{j+1}\right)(x_i) \leq 0.$$

case (iii) $x_i^* = x_{\frac{N_x}{2}}$, $0 \leq [D_x]\,(\mathcal{Z}^{j+1} + \lambda S^{j+1})(x_{\frac{N_x}{2}}) < 0$.

case (iv) $x_i^* = x_i,\ i = \frac{N_x}{2} + 1, \frac{N_x}{2} + 2, \cdots, N_x$,

$$\begin{aligned} 0 < \mathcal{L}_2(\mathcal{Z}^{j+1} + \lambda S^{j+1})(x_i^*) &= \left(-\frac{\varepsilon}{2}D_x^+D_x^- + \frac{r_i}{2}\right)\left(\mathcal{Z}^{j+1} + \lambda S^{j+1}\right)(x_i) + b\left(\mathcal{Z}^{j+1} + \lambda S^{j+1}\right)(x_i - \frac{N_x}{2}) \\ &\leq 0. \end{aligned}$$

case (v) $x_i^* = x_{N_x}$,

$$\begin{aligned} &0 < \mathcal{K}\left(\mathcal{Z}^{j+1} + \lambda S^{j+1}\right)(x_{N_x}) = \left(\mathcal{Z}^{j+1} + \lambda S^{j+1}\right)(x_{N_x}) \\ &\quad - \frac{\varepsilon}{3}\sum_{i=1}^{N_x}\left[g_{i-1}\left(\mathcal{Z}^{j+1} + \lambda S^{j+1}\right)(x_{i-1}) + 4g_i\left(\mathcal{Z}^{j+1} + \lambda S^{j+1}\right)(x_i)\right] h_i \\ &\quad - \frac{\varepsilon}{3}\sum_{i=1}^{N_x} g_{i+1}\left(\mathcal{Z}^{j+1} + \lambda S^{j+1}\right)(x_{i+1}) h_i \leq 0. \end{aligned}$$

Observed that in all the cases we arrived at a contradiction. Therefore $\lambda > 0$ is not possible. This implies that $\mathcal{Z}^{j+1}(x_i) \geq 0$.

□

An application of the above lemma is the following uniform stability estimate.

Lemma 4.8. *Let* $U^{j+1}(x_i),\ i = 1, 2,$ *be any mesh functions.Then,*

$$\left|\left|U_i^{j+1}\right|\right| \leq C\max\left\{\left|\left|U_0^{j+1}\right|\right|,\ \left|\left|\mathcal{K}U_{N_x}^{j+1}\right|\right|,\ \max_{1\leq i\leq N_x}\left|\left|\mathcal{L}^{N_x}U_i^{j+1}\right|\right|\right\}.$$

Proof. By considering the two barrier functions

$$\varpi_i^{\pm} = C\max\left\{\left|\left|U_0^{j+1}\right|\right|,\ \left|\left|\mathcal{K}U_{N_x}^{j+1}\right|\right|,\ \max_{1\leq i\leq N_x}\left|\left|\mathcal{L}^{N_x}U_i^{j+1}\right|\right|\right\} S_i^{j+1} \pm U_i^{j+1},$$

the result is obtained by applying the discrete maximum principle. □

5. Error estimate

In this section, the parameter-uniform error will be estimated by decomposing the solution U_i^{j+1} into the smooth and singular components as $U_i^{j+1} = V_i^{j+1} + W_i^{j+1}$, where V_i^{j+1} is the solution of

$$\begin{cases} \mathcal{L}V_i^{j+1} = g_i^{j+1}, \; i = 1, 2, \cdots, N_x, j = 0, 1, \cdots, N_t - 1, \\ V_0^{j+1} = {\phi_l}_0^{j+1}, \\ \mathcal{K}^{N_x} V_{N_x}^{j+1} = \mathcal{K}V_0^{j+1}(x_{N_x}), \\ V_i^{j+1} = {\phi_b}_i^{j+1}, \end{cases} \tag{5.1}$$

and therefore W_i^{j+1} must satisfy

$$\begin{cases} \mathcal{L}^{N_x} W_i^{j+1} = 0, \; i = 1, 2, \cdots, N_x, j = 0, 1, \cdots, N_t - 1, \\ W_i^{j+1} = U_i^{j+1} - v_i^{j+1}, \\ \mathcal{K}^{N_x} W_{N_x}^{j+1} = \mathcal{K}^{N_x} U_{N_x}^{j+1} - \mathcal{K}^{N_x} V_{N_x}^{j+1}. \end{cases} \tag{5.2}$$

Theorem 5.1. *Let $U^{j+1}(x)$ and U_i^{j+1} are the solutions of the problem (4.2)-(4.5) and (4.7)-(4.9) respectively, and assume that the coefficients $r(x), b(x), g(x) \in C^{4+\alpha_1}(\bar{\mathcal{D}})$, and the boundary conditions satisfy $\phi_l^{j+1} \in C^{3+\alpha_1/2}(0, T)$, $\phi_b^{j+1} \in C^{6+\alpha_1}(\Gamma_b)$, $\phi_r^{j+1}(0, T)$, where $\alpha_1 \in (0, 1)$, then we have*

$$\sup_{0<\varepsilon\ll 1} \left|\left| U^{j+1}(x) - U_i^{j+1} \right|\right| \leq C N_x^{-2} \ln^2 N_x.$$

Proof. The error can be written in the form $U_i^{j+1} - U^{j+1}(x_i) = \left(V_i^{j+1} - V^{j+1}(x_i) \right) + \left(W_i^{j+1} - W^{j+1}(x_i) \right)$. We prove the error estimates of smooth and singular components separately. First, we derive the error estimate for the smooth component using the following classical argument.

At the point $x_i = x_{N_x}$,

$$\begin{aligned} \mathcal{K}^{N_x} \left(V_i^{j+1} - V^{j+1}(x_i) \right) =& \mathcal{K}^{N_x} V_i^{j+1} - \mathcal{K}^{N_x} V^{j+1}(x_i) = \phi_r - \mathcal{K}^{N_x} V_i^{j+1} \\ =& \mathcal{K}V^{j+1}(x_i) - \mathcal{K}^{N_x} V^{j+1}(x_i) \\ =& V^{j+1}(x_i) - \varepsilon \int_{x_0}^{x_{N_x}} g(x) V^{j+1}(x) dx - V^{j+1}(x_i) \\ & + \varepsilon \sum_{i=1}^{N_x} \frac{g_{i-1} V_{i-1}^{j+1} + 4 g_i V_i^{j+1} + g_{i+1} V_{i+1}^{j+1}}{3} h_i \\ =& \varepsilon \frac{g_0 V_0^{j+1} + 4 g_1 V_1^{j+1} + g_2 V_2^{j+1}}{3} h_1 + \cdots \\ & + \varepsilon \frac{g_{N_x - 1} V_{N_x - 1}^{j+1} + 4 g_{N_x} V_{N_x}^{j+1} + g_{N_x + 1} V_{N_x + 1}^{j+1}}{3} h_{N_x} \\ & - \varepsilon \int_{x_0}^{x_1} g(x) V^{j+1}(x) dx - \cdots - \varepsilon \int_{x_{N_x}}^{x_{N_x + 1}} g(x) V^{j+1}(x) dx \\ =& - \varepsilon \frac{h_1^4}{90} g^{iv}(\xi_1) \frac{\partial^4 V^{j+1}}{\partial x^4}(\xi_1) - \cdots - \frac{h_1^4}{90} g^{iv}(\xi_{N_x + 1}) \frac{\partial^4 V^{j+1}}{\partial x^4}(\xi_{N_x + 1}), \end{aligned}$$

$$
\begin{aligned}
\left|\mathcal{K}^{N_x}\left(V_i^{j+1}-V^{j+1}(x_i)\right)\right| &= \left|C\varepsilon\left(h_1^4\frac{\partial^4 V^{j+1}}{\partial x^4}(\xi_1)+\cdots+h_{N_x+1}^4\frac{\partial^4 V^{j+1}}{\partial x^4}(\xi_{N_x+1})\right)\right| \\
&\le C\varepsilon\left(h_1^4\frac{\partial^4 V^{j+1}}{\partial x^4}(\xi_1)+\cdots+h_{N_x+1}^4\frac{\partial^4 V^{j+1}}{\partial x^4}(\xi_{N_x+1})\right) \\
&\le C\left(h_1^4 e^{-\xi_1\sqrt{\frac{\alpha}{\varepsilon}}}+\cdots+h_{N_x}^4 e^{-\xi_{N_x}\sqrt{\frac{\alpha}{\varepsilon}}}\right) \\
&\le C\left(h_1^4+h_2^4+\cdots+h_{N_x+1}^4\right) \\
&\le CN_x^{-2},
\end{aligned}
$$

where $x_{i-1}\le \xi_i \le x_i,\ 1\le i\le N_x$, and C is chosen to be arbitrary positive constant. From the difference and discrete equations, we have

$$
\mathcal{L}_1\left(V_i^{j+1}-V^{j+1}(x_i)\right)=g_1^{j+1}-\mathcal{L}_1V^{j+1}(x_i)=\left(\mathcal{L}_1-\mathcal{L}_1^{N_x}\right)V^{j+1}(x_i).
$$

Then, it implies that $\mathcal{L}_1^{N_x}\left(V_i^{j+1}-V^{j+1}(x_i)\right)=-\frac{\varepsilon}{2}\left(\frac{\partial^2}{\partial x^2}-D_x^+D_x^-\right)V^{j+1}(x_i)$. It follows from classical estimates [17], at each point $x_i\in\mathcal{D}_1^{N_x}$,

$$
\begin{aligned}
\mathcal{L}_1^{N_x}\left(V_i^{j+1}-V^{j+1}(x_i)\right) &\le \begin{cases}\frac{\varepsilon}{6}\left(x_{i+1}-x_{i-1}\right)\left\|\frac{\partial^3 V^{j+1}(x)}{\partial x^3}\right\|, & \text{if } x_i=\mu \text{ or } x_i=1-\mu,\\ \frac{\varepsilon}{12}\left(x_i-x_{i-1}\right)^2\left\|\frac{\partial^4 V^{j+1}(x)}{\partial x^4}\right\|, & \text{otherwise,}\end{cases} \\
&\le C\begin{cases}\sqrt{\varepsilon}\ N_x^{-1}, & \text{if } x_i=\mu \text{ or } x_i=1-\mu,\\ N_r^{-2}, & \text{otherwise.}\end{cases}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathcal{L}_2^{N_x}\left(V_i^{j+1}-V^{j+1}(x_i)\right) &\le \begin{cases}\frac{\varepsilon}{6}\left(x_{i+1}-x_{i-1}\right)\left\|\frac{\partial^3 V^{j+1}(x)}{\partial x^3}\right\|, & \text{if } x_i=1+\mu \text{ or } x_i=2-\mu,\\ \frac{\varepsilon}{12}\left(x_i-x_{i-1}\right)^2\left\|\frac{\partial^4 V^{j+1}(x)}{\partial x^4}\right\|, & \text{otherwise,}\end{cases} \\
&\le C\begin{cases}\sqrt{\varepsilon}\ N_x^{-1}, & \text{if } x_i=1+\mu \text{ or } x_i=2-\mu,\\ N_x^{-2}, & \text{otherwise.}\end{cases}
\end{aligned}
$$

Now, introduce the barrier functions

$$
\Phi^{j+1}(x_i)=C\begin{cases}\frac{\mu}{\varepsilon}\theta_1(x_i)N_x^{-2}+N_x^{-2},\\ \frac{(1+\mu)}{\varepsilon}\theta_2(x_i)+N_x^{-2},\end{cases}
$$

where θ_1 and θ_2 are the piecewise linear polynomial

$$
\theta_1=\begin{cases}\frac{x}{\mu}, & \text{for } 0\le x\le \mu,\\ 1, & \text{for } \mu\le x\le 1-\mu,\\ \frac{1-x}{\mu}, & \text{for } 1-\mu\le x\le 1,\end{cases}
$$

and

$$
\theta_2=\begin{cases}\frac{x}{1+\mu}, & \text{for } 1\le x\le 1+\mu,\\ 1, & \text{for } 1+\mu\le x\le 2-\mu,\\ \frac{2-x}{1+\mu}, & \text{for } 2-\mu\le x\le 2.\end{cases}
$$

Then, for all $x_i, i=1,2,3,\cdots,N_x$, $0\le\Phi^{j+1}(x_i)\le CN_x^{-2}\ln N_x$ and also

$$
\mathcal{L}_1^{N_x}\Phi^{j+1}(x_i)=\begin{cases}C\sqrt{\varepsilon}N_x^{-1}+N_x^{-2} \text{ if } x_i=\mu \text{ or } x_i=1-\mu,\\ CN_x^{-2}, \ \text{otherwise.}\end{cases}
$$

Where the observations that $\frac{\mu}{\sqrt{\varepsilon}} \leq 2\ln N_x$ and

$$\mathcal{L}_1^{N_x}\theta(x_i) = \begin{cases} \frac{\varepsilon N_x}{\mu} + r(x_i), \text{ if } x_i = \mu \text{ or } x_i = 1-\mu, \\ a(x_i)\theta(x_i), \text{ otherwise,} \end{cases}$$

Similarly,

$$\mathcal{L}_2^{N_x}\Phi^{j+1}(x_i) = \begin{cases} C\sqrt{\varepsilon}N_x^{-1} + N_x^{-2} \text{ if } x_i = 1+\mu \text{ or } x_i = 2-\mu, \\ CN_x^{-2}, \text{ otherwise.} \end{cases}$$

Where the observations that $\frac{1+\mu}{\sqrt{\varepsilon}} \leq 2\ln N_x$ and

$$\mathcal{L}_2^{N_x}\theta(x_i) = \begin{cases} \frac{\varepsilon N_x}{1+\mu} + r(x_i), \text{ if } x_i = 1+\mu \text{ or } x_i = 2-\mu, \\ r(x_i)\theta(x_i) + b(x_i)\theta(x_i - \frac{N_x}{2}), \text{ otherwise.} \end{cases}$$

and for all $x_i \in \Gamma^{N_x}$, then $\phi^{j+1}(x_i) \geq 0$. Observe that $\mathcal{K}^{N_x}\theta^{j+1}(x_i) \geq 0$.

Define a barrier functions $\Xi^{\pm}(x_i) = \Phi(x_i) \pm \left(V_i^{j+1} - V^{j+1}(x_i)\right)$, it follows that at $\forall x_i, i = 1, 2, \cdots, N_x$

$$\mathcal{L}_1^{N_x}\Xi^{\pm}(x_i) \geq 0 \text{ and } \mathcal{L}_2^{N_x}\Xi^{\pm}(x_i) \geq 0.$$

Then, from the semi-discrete maximum principle, $\Xi^{\pm}(x_i) \geq 0, \forall i = 1, 2, \cdots, N_x$.
Then, we have

$$\left|V_i^{j+1} - V^{j+1}(x_i)\right| \leq CN_x^{-2}\ln N_x. \tag{5.3}$$

To estimate the singular component of the error, we decompose $W^{j+1}(x)$ into $W_l^{j+1}(x)$ and $W_r^{j+1}(x)$.

$$\begin{cases} \mathcal{L}^{N_x}W_l^{j+1}(x_i) = 0, \; i = 1, 2, \cdots, N_x, j = 0, 1, \cdots, N_t - 1, \\ W_l^{j+1}(x_i) = \phi_l^{j+1}(x_i) - v_0^{j+1}(x_i), \; x_i \in \Gamma_l^{N_x}, \\ W_l^{j+1}(x_i) = 0, \; x_i \in \Gamma_r^{N_x}, \\ W_l^{j+1}(x_i) = 0, , \; x_i \in \Gamma_b^{N_x}, \end{cases} \tag{5.4}$$

$$\begin{cases} \mathcal{L}^{N_x}W_l^{j+1}(x_i) = 0, \; i = \frac{N_x}{2} + 1, \cdots, N_x, \\ W_l^{j+1}(x_i) = A, \; x_i \in \Gamma_l^{N_x}, \\ \mathcal{K}^{N_x}W_r^{j+1}(x_i) = 0, \; x_i \in \Gamma_r^{N_x}, \\ W_l^{j+1}(x_i) = 0, \; x_i \in \Gamma_b^{N_x}, \end{cases}$$

and

$$\begin{cases} \mathcal{L}^{N_x}W_r^{j+1}(x_i) = 0, \; i = 1, \cdots, N_x, \\ W_l^{j+1}(x_i) = 0, \; x_i \in \Gamma_l^{N_x}, \\ \mathcal{K}^{N_x}W_r^{j+1}(x_i) = A, \; x_i \in \Gamma_r^{N_x}, \\ W_l^{j+1}(x_i) = 0, \; x_i \in \Gamma_b^{N_x}, \end{cases}$$

$$\begin{cases} \mathcal{L}^{N_x}W_l^{j+1}(x_i) = 0, \; i = \frac{N_x}{2} + 1, \cdots, N_x, \\ W_l^{j+1}(x_i) = 0, \; x_i \in \Gamma_l^{N_x}, \\ \mathcal{K}^{N_x}W_r^{j+1}(x_i) = \mathcal{K}^{N_x}W^{j+1}(x_i), \; x_i \in \Gamma_r^{N_x}, \\ W_l^{j+1}(x_i) = 0, \; x_i \in \Gamma_b^{N_x}. \end{cases}$$

The singular component error is equivalent to

$$W_i^{j+1} - W^{j+1}(x_i) = \left({W_l}_i^{j+1} - {W_l}^{j+1}(x_i)\right) + \left({W_r}_i^{j+1} - {W_r}^{j+1}(x_i)\right),$$

$$\mathcal{L}^{N_x}\left(W_i^{j+1}-W^{j+1}(x_i)\right)\leq -\frac{\varepsilon}{2}\left(\frac{\partial^2}{\partial x^2}-D^+D^-\right)W^{j+1}(x_i).$$

It follows from classical estimates [17], at each point $x_i,\ i=1,2,\cdots,N_x$,

$$\mathcal{L}^{N_x}\left(W_i^{j+1}-W^{j+1}(x_i)\right)\leq C\begin{cases}(N_x^{-1}\ln N_x)^2, & \text{if } i=1,2,\cdots,\frac{N_x}{2},\\ (N_x^{-1}\ln N_x)^2, & \text{if } i=\frac{N_x}{2}+1,\cdots,N_x.\end{cases}$$

First, the estimate for ${W_l}_i^{j+1}-{W_l}^{j+1}(x_i)$ is given. The argument depends on whether $\mu=\frac{1}{4}$ or $\mu=2\sqrt{\varepsilon}\ln N_x$.

Case-I : $\mu=\frac{1}{4}$

In this case the mesh is uniform and $\mu=2\sqrt{\varepsilon}\ln N_x\geq\frac{1}{4}$. It is clear that $x_i-x_{i-1}=N_x^{-1}$ and $\varepsilon^{-\frac{1}{2}}\leq C\ln N_x$. By [17], we have

$$\begin{aligned}\mathcal{K}^{N_x}\left({W_l}_i^{j+1}-{W_l}^{j+1}(x_i)\right)&=\mathcal{K}^{N_x}{W_l}_i^{j+1}-\mathcal{K}^{N_x}{W_l}^{j+1}(x_i)\\ &=\phi_r-\mathcal{K}^{N_x}{W_l}^{j+1}(x_i)\\ &=\mathcal{K}{W_l}^{j+1}(x_i)-\mathcal{K}^{N_x}{W_l}^{j+1}(x_i),\\ \left|\mathcal{K}^{N_x}\left({W_l}_i^{j+1}-{W_l}^{j+1}(x_i)\right)\right|&\leq C\varepsilon\left(h_1^4\frac{\partial^4}{\partial^4}{W_l}^{j+1}(\xi_i)+\cdots+h_{N_x}^4\frac{\partial^4}{\partial^4}{W_l}^{j+1}(\xi_{N_x})\right)\\ &\leq C\varepsilon^{-1}\left(h_1^4+\cdots+h_{N_x}^4\right)\\ &\leq CN_x^{-2}\ln^2 N_x,\end{aligned}$$

where $x_{i-1}\leq\xi_i\leq x_i$. By applying semi-discrete uniform stability, Lemma 4.8, we obtain

$$\left|{W_l}_i^{j+1}-{W_l}^{j+1}(x_i)\right|\leq CN_x^{-2}\ln^2 N_x.$$

Case-II $\mu<\frac{1}{4}$. Since the mesh is piecewise uniform with the subinterval $[\mu,1-\mu]$, mesh elements are $4(1-2\mu)/N_x$ and the rest of the intervals $[0,\mu]$ and $[1-\mu,1]$ with $8\mu/N$ mesh elements. By [17], we have

$$\mathcal{K}^{N_x}\left({W_l}_i^{j+1}-{W_l}^{j+1}(x_i)\right)\leq CN_x^{-2}\ln^2 N_x,$$

and

$$\begin{aligned}\left|\mathcal{K}^{N_x}\left({W_l}_i^{j+1}-{W_l}^{j+1}(x_i)\right)\right|&\leq C\varepsilon\left(h_1^4\frac{\partial^4}{\partial^4}{W_l}^{j+1}(\xi_i)+\cdots+h_{N_x}^4\frac{\partial^4}{\partial^4}{W_l}^{j+1}(\xi_{N_x})\right)\\ &\leq C\varepsilon^{-1}\left(h_1^4+\cdots+h_{N_x}^4\right)\\ &\leq CN_x^{-2}\ln^2 N_x,\end{aligned}$$

where $x_{i-1}\leq\xi_i\leq x_i$. By applying semi-discrete uniform stability, Lemma 4.8, we obtain

$$\left|{W_l}_i^{j+1}-{W_l}^{j+1}(x_i)\right|\leq C(N_x^{-2}\ln^2 N_x).\tag{5.5}$$

Similar arguments are used to establish the error estimate for W_r. By combining equation (5.3) and (5.5), we have

$$\left|U_i^{j+1}-U^{j+1}(x_i)\right|\leq CN_x^{-2}\ln^2 N_x.\tag{5.6}$$

□

We summarizes the results of this work by considering the semidiscrete error estimate obtained in (4.6) and (5.5) then we conclude by the following theorem.

Theorem 5.2. *The error estimate for the solution of continuous and fully discrete problem is given by*

$$\sup_{0<\varepsilon\le 1}\left|\left|u(x,t)-U_i^{j+1}\right|\right| \le C\left(N_x^{-2}\ln^2 N_x + N_t^{-2}\right),$$

where $u(x,t)$ and U_i^{j+1} are the solutions of the problem (2.1)-(2.3) and (4.2)-(4.5) respectively.

Proof. The proof follows from Theorems 4.5 and 5.1. □

6. Numerical Results and Discussions

Below we present two examples to validate the main result of this study. We compute the maximum point wise error and the rate of convergence. These results are displayed in tables for different values of $N, \Delta t$ and ε. Since the exact solution of the problem is not known to estimate the error, we use the method of double mesh principle as follows. The maximum absolute error of the developed numerical scheme is computed as $E_\varepsilon^{N,\Delta t} = \max_{(x_i,t_j)\in\bar{\mathcal{D}}}\left|U^{N,\Delta t} - U^{2N,\Delta t/2}\right|$. We determine the uniform error $E^{N,\Delta t} = \max_\varepsilon E_\varepsilon^{N,\Delta t}$ and rate of convergence $P^{N,\Delta t} = \log 2(E^{N,\Delta t}) - \log 2(E^{2N,\Delta t/2})$.

Example 6.1. [22]

$$\begin{cases} -\varepsilon\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} + 5u(x,t) - u(x-1,t) = e^{-x}, \ (x,t)\in(0,2)\times(0,2], \\ u(x,t) = 0, \ \forall (x,t)\in\Gamma_L, \\ \mathcal{K}u(2,t) = u(2,t) - \varepsilon\int_0^2 \frac{x}{3}u(x,t)dx = 0, \ \forall\ (x,t)\in\Gamma_r, \\ u(x,t) = 0, \ \forall\ (x,t)\in\Gamma_b. \end{cases} \tag{6.1}$$

Example 6.2. [22]

$$\begin{cases} -\varepsilon\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} + 5u(x,t) - xu(x-1,t) = 1, \ (x,t)\in(0,2)\times(0,2], \\ u(x,t) = 0, \ \forall (x,t)\in\Gamma_L, \\ \mathcal{K}u(2,t) = u(2,t) - \varepsilon\int_0^2 \frac{1}{6}u(x,t)dx = 0, \ \forall\ (x,t)\in\Gamma_r, \\ u(x,t) = \sin(\pi x), \ \forall\ (x,t)\in\Gamma_b. \end{cases} \tag{6.2}$$

Table 1. Maximum absolute errors, uniform error and rate of convergence for Example 6.1.

ε $\downarrow$	N=32 Δt =0.1/4	64 0.1/8	128 0.1/16	256 0.1/32	512 0.1/64
10^{0}	4.0072e-03	2.0250e-03	1.0177e-03	5.1010e-04	2.5536e-04
10^{-1}	1.1374e-03	6.9735e-04	3.5906e-04	1.8526e-04	9.2506e-05
10^{-2}	1.0686e-03	3.4677e-04	1.7196e-04	1.1169e-04	8.1812e-05
10^{-3}	4.5524e-03	2.3661e-03	7.8475e-04	2.1043e-04	5.3542e-05
10^{-4}	4.5511e-03	2.3657e-03	9.3776e-04	3.2531e-04	1.0796e-04
10^{-5}	4.5508e-03	2.3657e-03	9.3775e-04	3.2531e-04	1.0796e-04
10^{-6}	4.5507e-03	2.3656e-03	9.3775e-04	3.2531e-04	1.0796e-04
10^{-7}	4.5507e-03	2.3656e-03	9.3775e-04	3.2531e-04	1.0796e-04
10^{-8}	4.5507e-03	2.3656e-03	9.3775e-04	3.2531e-04	1.0796e-04
10^{-9}	4.5507e-03	2.3656e-03	9.3775e-04	3.2531e-04	1.0796e-04
$E^{N,\Delta t}$	4.5524e-03	2.3661e-03	9.3776e-04	3.2531e-04	1.0796e-04
$P^{N,\Delta t}$	9.4388e-01	1.3349e+00	1.5274e+00	1.5913e+00	-

Figures 1(a) and 1(b) indicate the numerical solution profiles of Examples 6.1 and 6.2 respectively. We observed that for a small values of ε, the solution of the problem exhibit a boundary layer at $x = 0$ and $x = 2$ and interior layer occurred at $x = 1$, because of the delay term in spatial direction. We can also observed that the width of the layers decreases as the parameter ε decreases. Tables 1 and 2 indicates ε-uniform maximum pointwise error $E^{N,\Delta t}$ and the rate of convergence $P^{N,\Delta t}$ for both Examples 6.1 and 6.2. Figures 2(a) and 2(b) show the Log-Log plot

Table 2. Maximum absolute errors, uniform error and rate of convergence for Example 6.2.

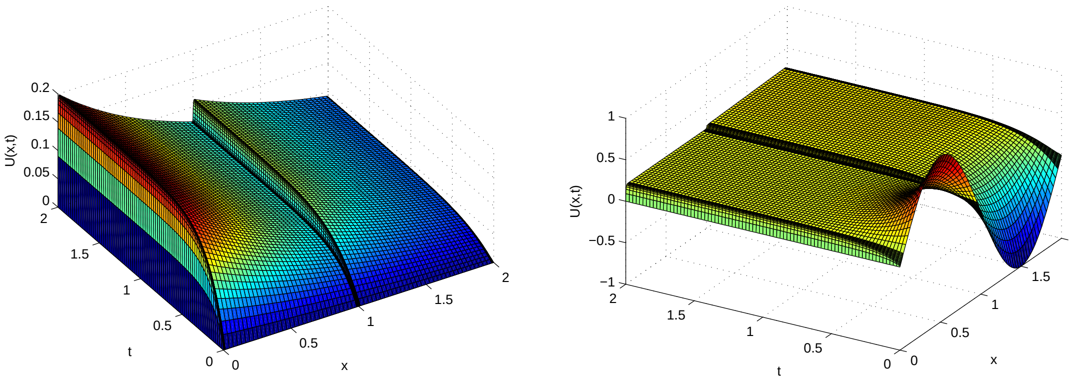
ε $\downarrow$	$N = 32$ $\Delta t = 0.1/4$	64 0.1/8	128 0.1/16	256 0.1/32	512 0.1/64
10^{0}	7.0411e-03	3.6766e-03	1.8801e-03	9.5088e-04	4.7819e-04
10^{-1}	5.2949e-03	2.8424e-03	1.4907e-03	7.6522e-04	3.8789e-04
10^{-2}	1.8802e-03	1.9654e-03	1.2756e-03	7.1103e-04	3.7340e-04
10^{-3}	5.0995e-03	2.3698e-03	7.8500e-04	4.4835e-04	3.1910e-04
10^{-4}	5.1011e-03	2.3669e-03	9.3786e-04	3.2534e-04	2.6596e-04
10^{-5}	5.1010e-03	2.3660e-03	9.3778e-04	3.2531e-04	2.6602e-04
10^{-6}	5.1009e-03	2.3658e-03	9.3776e-04	3.2531e-04	2.6604e-04
10^{-7}	5.1009e-03	2.3657e-03	9.3775e-04	3.2531e-04	2.6604e-04
10^{-8}	5.1009e-03	2.3657e-03	9.3775e-04	3.2531e-04	2.6604e-04
10^{-9}	5.1009e-03	2.3657e-03	9.3775e-04	3.2531e-04	2.6604e-04
$E^{N,\Delta t}$	5.2949e-03	2.8424e-03	1.4907e-03	7.6522e-04	2.6604e-04
$P^{N,\Delta t}$	8.9749e-01	9.3112e-01	9.6204e-01	9.8023e-01	-

Table 3. Comparison of maximum absolute error and rate of convergence for Example 6.1.

	$N = 32$ $\Delta t = 0.1/4$	64 0.1/8	128 0.1/16	256 0.1/32	512 0.1/64
Present Method					
$E^{N,\Delta t}$	4.5524e-03	2.3661e-03	9.3776e-04	3.2531e-04	1.0796e-04
$P^{N,\Delta t}$	0.94388	1.3349	1.5274	1.5913	-
Results in [22]					
$E^{N,\Delta t}$	1.2534e-02	6.9738e-03	3.6873e-03	1.8972e-03	9.6241e-04
$P^{N,\Delta t}$	0.84584	0.91937	0.95873	0.97912	
Results in [9]					
$E^{N,\Delta t}$	2.0026e-03	1.0723e-03	5.5505e-04	2.8241e-04	
$P^{N,\Delta t}$	0.90117	0.95002	0.97483	0.98734	-
Results in [10]					
$E^{N,\Delta t}$	6.7641e-05	1.7419e-05	4.4217e-06	1.1140e-06	2.7958e-07
$P^{N,\Delta t}$	1.9572	1.9780	1.9889	1.9944	-
Results in [13]					
$E^{N,\Delta t}$	1.2974e-02	8.7628e-03	4.6313e-03	2.0155e-03	7.6160e-04
$P^{N,\Delta t}$	0.56616e	0.91997	1.2003	1.4040	-

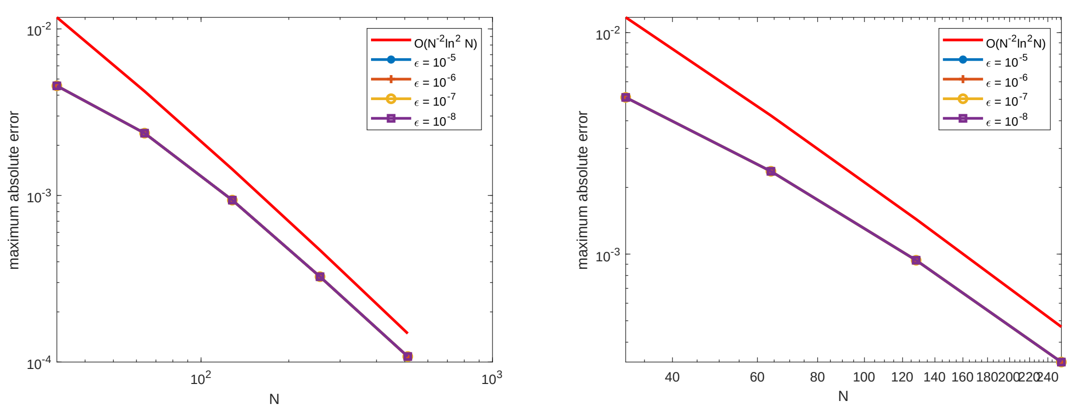
Table 4. Comparison of maximum absolute error and rate of convergence for Example 6.2.

	$N = 32$ $\Delta t = 0.1/4$	64 0.1/8	128 0.1/16	256 0.1/32	512 0.1/64
Present Method					
$E^{N,\Delta t}$	5.2949e-03	2.8424e-03	1.4907e-03	7.6522e-04	2.6604e-04
$P^{N,\Delta t}$	0.89749	0.93112	0.96204	0.98023	-
Results in [22]					
	1.4776e-01	9.7571e-02	5.7092e-02	3.1057e-02	
$P^{N,\Delta t}$	0.59873	0.77316	0.87837	0.93697	
Results in [9]					
$E^{N,\Delta t}$	1.5980e-02	8.3421e-03	4.2681e-03	2.1581e-03	
$P^{N,\Delta t}$	0.93778	0.96682	0.98383	0.99180	
Results in [10]					
$E^{N,\Delta t}$	6.3220e-04	1.5808e-04	3.9500e-05	9.8764e-06	2.4690e-06
$P^{N,\Delta t}$	1.9997	2.0007	1.9998	2.0001	-
Results in [13]					
$E^{N,\Delta t}$	3.1547e-02	1.7445e-02	9.2639e-03	4.7759e-03	2.4253e-03
$P^{N,\Delta t}$	0.85469	0.91312	0.95585	0.97761	-



(a) (b)

FIGURE 1. (a) Graph of numerical solution for Example 6.1, (b) Graph of numerical solution for Example 6.2.



(a) (b)

FIGURE 2. (a) The Log-Log plot of the maximum absolute error for Example 6.1, (b) The Log-Log plot of the maximum absolute error for Example 6.2.

of the maximum absolute error verses N for singular perturbation parameter ranging from $\varepsilon = 10^{-5}$ to 10^{-8} for Example 6.1 and 6.2 respectively. In these figures the graphs are parallel and overlapped as ε goes small, this indicate that the proposed scheme converges independent of the values of perturbation parameter. Tables 3 and 4, shows the comparison of maximum absolute error and rate of convergence for the proposed scheme with the methods existed in the literature. From those tables, we observed that the developed numerical scheme is a more accurate parameter uniformly convergent than the methods presented in [9, 13, 22]. The numerical approach described in [10] yields a higher order but has limitations in resolving layers.

7. Conclusion

A class of singularly perturbed reaction diffusion problem of partial delay differential equations with non-local boundary conditions was considered. We developed a finite difference scheme on a piecewise uniform Shishkin mesh for spatial discretization and the Crank-Nicolson method to discretize the time derivative. The properties of continuous problem were studied and satisfy the continuous stability estimate. To show the boundedness of the solution and its derivatives we decomposed the solution into smooth and regular components. Since the problem has a strong boundary layer at the right end layer, we used the Simpsons rule to treat the integral boundary conditions. The stability estimate and parameter uniform convergence analysis for the numerical scheme are also studied. We also showed that the order of convergence and the error is of order $\mathcal{O}\left(N_x^{-2}\ln^2 N_x + N_t^{-2}\right)$. Our numerical results reflect the theoretical estimates.

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