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Fully fuzzy initial value problem of Caputo-Fabrizio fractional differential equations

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Abstract

We aim at presenting results including analytical solutions to linear fully fuzzy Caputo-Fabrizio fractional differential equations. In such linear equations, the coefficients are fuzzy numbers and, as a useful approach, the cross product has been considered as a multiplication between the fuzzy data. This approach plays an essential role in simplifying of computation of analytical solutions of linear fully fuzzy problems. The obtained results have been applied for deriving explicit solutions of linear Caputo-Fabrizio differential equations with fuzzy coefficients and of the corresponding initial-value problems. Some of the topics which are needed for the results of this study from the point of view of the cross product of fuzzy numbers have been explained in detail. We illustrate our technique and compare the effect of uncertainty of the coefficients and initial value on the related solutions.

Keywords. Caputo-Fabrizio operator, Cross product, Initial value problem, Linear fuzzy differential equations.
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1. Introduction

Differential equations of fractional order appear more and more frequently in various research areas and engineering applications. Some applications of fractional differential equations are appeared in the field of chemistry, computer networking, control theory, complex medium with electrodynamics, aerodynamics, image processing phenomenon, polymer rheology, signal, etc., [8, 9, 29]. Theory of fractional differential equations is making use of the derivatives of fractional orders such as Riemann-Liouville, Caputo, Gtünwall-Letnikev, etc. Recently in [18], Caputo-Fabrizio fractional derivative has been presented in where the singular kernel has been replaced with a nonsingular kernel. Many studies have been done by several authors on the theory of Caputo-Fabrizio differential equations [13, 21, 22, 26, 31]. The advantages of the large number of fractional derivatives become apparent in different models of mechanical and electrical properties of real materials which provide the best reflection of the behavior of systems.

The idea of arithmetic operations on fuzzy numbers, a spacial kind of fuzzy subsets with extra conditions, are mainly based on the Zadeh's extension principle. It is well known that the calculation based on the Zadeh's product operation depends on the shape of the fuzzy numbers are complicated. Considering the extension principle-based product on the same kind of fuzzy numbers, the shape of the resulting fuzzy number is not preserved. In many situations, these problems are solved by redefining the product operation named as the cross product [15].

As soon as the idea of a function with fuzzy number values was born, it also raised the idea of fuzzy ordinary, partial, and fractional derivatives and consequently fuzzy differential equations [2–4, 7, 10–12, 16, 17, 30]. Many of the fuzzy differential equations that describe real natural phenomena are fuzzy linear differential equations. Such fuzzy linear differential equations can admit crisp or fuzzy coefficients. There are different approaches to interpreting the concept of a solution to linear fuzzy differential equations with crisp and fuzzy coefficients [5, 6, 23–25, 27, 28]. Methods, described in detail in [23–25, 27, 28] for linear fuzzy differential equations with crisp coefficients, do not work in the case of fuzzy coefficients. There are some studies in this fields such that the concept of the product of a fuzzy number and unknown function is the Zadeh's product [19]. However, this approach is complicated in computational

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point of view. On the other hand, there are some numerical methods described in [20], which allow solutions of linear differential equations with fuzzy coefficients but they work effectively only for numerical solutions of homogenous fuzzy differential equations.

The authors in [5, 6] apply a method which is free of these disadvantages and suitable for a wide class of initial value problems for differential equations. The method has been applied for linear fuzzy partial and ordinary differential equations and is based on the cross product of fuzzy numbers.

As we know, linear fuzzy fractional differential equations have not been considered with fuzzy coefficients until now and effective general methods for solving them cannot be found even in the most useful research on fuzzy fractional differential equations. In [1], fuzzy linear Caputo-Fabrizio fractional differential equations with crisp coefficient have been investigated. Method, described in detail in [1] for linear fuzzy Caputo-Fabrizio fractional differential equations with crisp coefficients, does not work in the case of fuzzy coefficients and an effective and easy-to-use approach for solving such equations is needed.

In this contribution, we apply the approach based on the cross product for linear fully fuzzy Caputo-Fabrizio fractional differential equations with fuzzy initial values. We investigate all of analytical solutions considering all of the possible options on the data of the problem. We hope that this method could be a useful method for obtaining solutions of different applied problems appearing in physics, chemistry, electrochemistry, engineering, etc. The main objective of this paper is to complement the contents of the papers mentioned above. We aim to present, in a systematic manner, results including the cross product and explicit solutions of linear fully fuzzy Caputo-Fabrizio differential equations. This paper consists of four sections. Section 2 (Preliminaries) provides some basic definitions and properties from such topics of fuzzy numbers, arithmetic operations on them, the cross product of fuzzy numbers, calculus of fuzzy number-valued functions, and so on. The usages of the cross product in calculus of differential and integral has indeed motivated a major part of section 2. Section 3, Fuzzy CF Fractional Derivative and Integral, includes the definitions and some potentially useful properties of fuzzy Caputo-Fabrizio fractional integral and fractional derivative. Sections 2 and 3 are meant to prepare the reader to understand the various mathematical tools and techniques which are applied in the later sections of this paper. The explicit solutions of the fully fuzzy initial value problem of linear Caputo-Fabrizio differential equation are derived in section 4. At the end of this paper, for the convenience of the readers interested in further investigations on these and other closely-related topics, we include a rather large and up-to-date references.

2. Preliminary

In this section, we extended the cross product of fuzzy numbers in which the core of them consists of one element. So, we present new results and their proof in the context of this section.

Fuzzy numbers are spacial kind of fuzzy sets $u : \mathbb{R} \to [0, 1]$ such that they are normal, fuzzy convex, upper semi-continuous, and compactly supported. The α-level of fuzzy number u is defined by $(u)_\alpha = \{t \in \mathbb{R} : u(t) \geq \alpha, 0 < \alpha \leq 1\}$ i.e. $(u)_\alpha = [u^-_\alpha, u^+_\alpha]$. $(u)_1$ is called the core of the fuzzy number and $(u)_0 = cl\{t \in \mathbb{R} : u(t) \geq 0\}$ is called the support of the fuzzy number. For $0 \leq \alpha \leq 1$, the length of α-level set is denoted by $len(u)_\alpha = u^+_\alpha - u^-_\alpha$. The space of fuzzy numbers is denoted by $\mathbb{R}_\mathcal{F}$. Let $u, v \in \mathbb{R}_\mathcal{F}$, $0 \leq \alpha \leq 1$ and $\lambda \in \mathbb{R}$. The triangular and the space of triangular fuzzy numbers are denoted by $u =< u_l, u_c, u_r >$ for which $u_l \leq u_c \leq u_r$ and $\mathbb{R}_\tau$, respectively.

The following definitions about arithmetic operations on $\mathbb{R}_\mathcal{F}$ are based on Zadeh's extension principle [14].

- Addition:

  $(u + v)_\alpha = \{x + y | x \in (u)_\alpha, y \in (v)_\alpha\} = (u)_\alpha + (v)_\alpha.$

- Scalar product:

  $(\lambda u)_\alpha = \{\lambda x | x \in (u)_\alpha\} = \lambda(u)_\alpha.$

- Difference:

  $(u - v)_\alpha = (u)_\alpha + (-1)(v)_\alpha.$

- Product:

$$(uv)_\alpha = \big[min\{u_\alpha^- v_\alpha^-, u_\alpha^- v_\alpha^+, u_\alpha^+ v_\alpha^-, u_\alpha^+ v_\alpha^+\}, max\{u_\alpha^- v_\alpha^-, u_\alpha^- v_\alpha^+, u_\alpha^+ v_\alpha^-, u_\alpha^+ v_\alpha^+\}\big].$$

The following definitions about the difference of fuzzy numbers are not based on Zadeh's extension principle.

Definition 2.1. [14] Let u and v be two fuzzy numbers. If there exists a fuzzy number z such that $z + v = u$. Then z is called Hukuhara difference of u and v and is denoted by $z = u \ominus v$ and we have

$$(z)_\alpha = [u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+].$$

Definition 2.2. [14] Let $u, v \in \mathbb{R}_\mathcal{F}$, the generalized Hukuhara difference (gH-difference for short) is the fuzzy number w, if it exists, such that

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} \quad (i) \quad u = v + w, \\ or \quad (ii) \;\; v = u + (-1)w. \end{cases}$$

Lemma 2.3. *Let $u, v, w, z \in \mathbb{R}_\mathcal{F}$ and $a \in \mathbb{R}$. Then the following assertions are true*

$(au) \ominus (av) = a(u \ominus v)$, $(u \ominus v) + (w \ominus z) = (u + w) \ominus (v + z)$, $u \ominus (v \ominus w) = (u + w) \ominus v$,
$u + (v \ominus w) = (u + v) \ominus w$, $u \ominus (v + w) = (u \ominus v) \ominus w$,

provided the above H-differences exist.

Proposition 2.4. *Let $u \in \mathbb{R}_\mathcal{F}$ and $a, b \in \mathbb{R}$.*

(1) *If $ab \geq 0$, then $(a + b)u = au + bu$.*
(2) *If $ab \leq 0$, then $(a + b)u = au \ominus_{gH} (-1)bu$.*

Proof. In fact, this proposition is a generalized version of Case 1 presented in [14]. In order to prove Case 2, we let $a, b \in \mathbb{R}$ and $ab \leq 0$. There are two choices for the signs of a and b. Firstly, we assume $a \geq 0$ and $b \leq 0$. Therefore, we have $a + b \geq 0$ or $a + b \leq 0$. If $a + b \leq 0$, then from Case 1 we have

$$bu = (a + b - a)u = (a + b)u + (-a)u.$$

Hence, we have

$$(a + b)u = bu \ominus (-1)au. \tag{2.1}$$

Similarly, if $a + b \geq 0$, then from Case 1 we have

$$(a + b)u = au \ominus (-1)bu. \tag{2.2}$$

From (2.1) and (2.2), we deduce

$$(a + b)u = au \ominus_{gH} (-1)bu.$$

Secondly, we let $a \leq 0$ and $b \geq 0$. Therefore, we have $a + b \geq 0$ or $a + b \leq 0$. If $a + b \leq 0$, then from Case 1 we have

$$au = (a + b - b)u = (a + b)u + (-b)u.$$

Then, we have

$$(a + b)u = au \ominus (-1)bu. \tag{2.3}$$

Similarly, if $a + b \geq 0$, then from Case 1 we have

$$(a + b)u = bu \ominus (-1)au. \tag{2.4}$$

From (2.3) and (2.4), we can deduce

$$(a + b)u = au \ominus_{gH} (-1)bu.$$

$\square$

Definition 2.5. Let $u \in \mathbb{R}_\mathcal{F}$ such that the $Core(u)$ consists of exactly one element i.e. $(u)_1 = \{u_c\}$. u is called non-negative ($\succeq 0$) or non-positive ($\preceq 0$) if we have $u_c \geq 0$ or $u_c \leq 0$. The set of non-negative (or non-positive) fuzzy numbers is denoted by $\mathbf{R}_\mathfrak{F}^+ (\mathbf{R}_\mathfrak{F}^-)$.

The following definition is a spacial case of Definition 5.13 in [14]. In fact in this paper, we have used the fuzzy numbers with singleton core.

Definition 2.6. Let $u, v \in \mathbf{R}^{+}_{\mathfrak{F}}$ and $\alpha \in [0,1]$. The cross product of u and v is denoted by $w = u \odot v$ and defined as follows

$$(w)_\alpha = [u^-_\alpha v_c + u_c v^-_\alpha - u_c v_c, u^+_\alpha v_c + u_c v^+_\alpha - u_c v_c].$$

Definition 2.7. Let $u, v \in \mathbf{R}^{+}_{\mathfrak{F}}(\mathbf{R}^{-}_{\mathfrak{F}})$. We say $u \succeq v$, if $u_c \geq v_c$. Also, we say $u \preceq v$, if $u_c \leq v_c$.

Proposition 2.8. *Let $u, v, w \in \mathbf{R}^{+}_{\mathfrak{F}}(\mathbf{R}^{-}_{\mathfrak{F}})$.*

(1) *If $u \succeq 0$ and $v \preceq 0$, then*
$$u \odot v = -\big(u \odot (-v)\big),$$
(2) *If $u \preceq 0$ and $v \succeq 0$, then*
$$u \odot v = -\big((-u) \odot v\big),$$
(3) *If $u, v \preceq 0$, then*
$$u \odot v = (-u) \odot (-v),$$
(4) *If either $u \succeq v \succeq 0$ or $u \preceq v \preceq 0$, then*
$$(u \ominus v) \odot w = u \odot w \ominus v \odot w$$
(5) *If either $v \succeq u \succeq 0$ or $v \preceq u \preceq 0$, then*
$$(u \ominus v) \odot w = \big(v \odot w + (w - w) \odot (u \ominus v)\big) \ominus v \odot w.$$
(6) *If either $u, v \succeq 0$ or $u, v \preceq 0$, then*
$$(u + v) \odot w = u \odot w + v \odot w,$$
(7) *If $u, u + v \succeq 0$ and $v \preceq 0$ or $u, u + v \preceq 0$ and $v \succeq 0$, we have*
$$(u + v) \odot w = \big(u \odot w + v \odot w\big) \ominus (w - w) \odot v,$$

provided the involving H-differences exist.

Proof. Case 4: If we assume $u \succeq v \succeq 0$, then $u \ominus v \succeq 0$. We assume $w \succeq 0$. Hence, we have

$$\begin{aligned}
\big((u \ominus v) \odot w\big)_\alpha &= [(u_c - v_c)w^-_\alpha + (u^-_\alpha - v^-_\alpha)w_c - (u_c - v_c)w_c, (u_c - v_c)w^+_\alpha + (u^+_\alpha - v^+_\alpha)w_c - (u_c - v_c)w_c] \\
&= [(u_c w^-_\alpha + (u^-_\alpha - u_c)w_c) - (v_c w^-_\alpha + (v^-_\alpha - v_c)w_c), (u_c w^+_\alpha + (u^+_\alpha - u_c)w_c) - (v_c w^+_\alpha + (v^+_\alpha - v_c)w_c)] \\
&= \big(u \odot w\big)_\alpha \ominus \big(v \odot w\big)_\alpha.
\end{aligned}$$

We assume $w \preceq 0$. Then, we get

$$\begin{aligned}
\big((u \ominus v) \odot w\big)_\alpha &= [(u_c - v_c)w^-_\alpha + (u^+_\alpha - v^+_\alpha)w_c - (u_c - v_c)w_c, (u_c - v_c)w^+_\alpha + (u^-_\alpha - v^-_\alpha)w_c - (u_c - v_c)w_c] \\
&= [(u_c w^-_\alpha + (u^+_\alpha - u_c)w_c) - (v_c w^-_\alpha + (v^+_\alpha - v_c)w_c), (u_c w^+_\alpha + (u^-_\alpha - u_c)w_c) - (v_c w^+_\alpha + (v^-_\alpha - v_c)w_c)] \\
&= (u \odot w)_\alpha \ominus (v \odot w)_\alpha.
\end{aligned}$$

The proof of other states are similar.

Case 5: Let $0 \preceq u \preceq v$. Hence, $u \ominus v \preceq 0$. We assume $w \succeq 0$. Thus, we get

$$\begin{aligned}
\big((u \ominus v) \odot w\big)_\alpha &= [(u_c - v_c)(w^+_\alpha - w_c) + (u^-_\alpha - v^-_\alpha)w_c, (u_c - v_c)(w^-_\alpha - w_c) + (u^+_\alpha - v^+_\alpha)w_c] \\
&= [(u_c - v_c)(w^-_\alpha - w_c) + (u^-_\alpha - v^-_\alpha)w_c, (u_c - v_c)(w^+_\alpha - w_c) + (u^+_\alpha - v^+_\alpha)w_c] \\
&+ [len(w)_\alpha(u_c - v_c), -len(w)_\alpha(u_c - v_c)] \\
&= \big(u \odot w + (w - w) \odot (u \ominus v)\big)_\alpha \ominus (v \odot w)_\alpha.
\end{aligned}$$

We suppose $w \preceq 0$. Then, we have

$$\begin{aligned}
\big((u \ominus v) \odot w\big)_\alpha &= [(u_c - v_c)(w^+_\alpha - w_c) + (u^+_\alpha - v^+_\alpha)w_c, (u_c - v_c)(w^-_\alpha - w_c) + (u^-_\alpha - v^-_\alpha)w_c] \\
&= [(u_c - v_c)(w^-_\alpha - w_c) + (u^+_\alpha - v^+_\alpha)w_c, (u_c - v_c)(w^+_\alpha - w_c) + (u^-_\alpha - v^-_\alpha)w_c] \\
&+ [len(w)_\alpha(u_c - v_c), -len(w)_\alpha(u_c - v_c)] \\
&= \big(u \odot w + (w - w) \odot (u \ominus v)\big)_\alpha \ominus (v \odot w)_\alpha.
\end{aligned}$$

The proof of other states are similar. □

Lemma 2.9. *If $u, v, w \in \mathbf{R}^+_{\mathfrak{F}}(\mathbf{R}^-_{\mathfrak{F}})$, then*

(1) $(-u) \odot v = u \odot (-v) = -\,(u \odot v)$.
(2) $u \odot v = v \odot u$.
(3) $(u \odot v) \odot w = u \odot (v \odot w)$.

Proof. The proof immediately follows from Proposition 2.8. □

Definition 2.10. [14] The Hausdorff distance of fuzzy numbers is as $D_\infty : \mathbb{R}_\mathcal{F} \times \mathbb{R}_\mathcal{F} \to \mathbb{R}^+$ defined by

$$D_\infty(u, v) = \sup_{\alpha \in [0,1]} \max\{\mid u^-_\alpha - v^-_\alpha \mid, \mid u^+_\alpha - v^+_\alpha \mid\},$$

where $(u)_\alpha = [u^-_\alpha, u^+_\alpha]$ and $(v)_\alpha = [v^-_\alpha, v^+_\alpha]$. The following properties of the Hausdorff metric are well-known

(1) If $u, v \in \mathbb{R}_\mathcal{F}$ and $a \in \mathbb{R}$, then
$$D_\infty(au, av) = \mid a \mid D_\infty(u, v).$$
(2) If $u, v, w \in \mathbb{R}_\mathcal{F}$, then
$$D_\infty(u + w, v + w) = D_\infty(u, v).$$
(3) If $u, v, w, z \in \mathbb{R}_\mathcal{F}$, then
$$D_\infty(u + v, w + z) \leq D_\infty(u, w) + D_\infty(v, z).$$
(4) If $u, v, w, z \in \mathbb{R}_\mathcal{F}$ and the H-differences $u \ominus v$ and $w \ominus z$ exist, then
$$D_\infty(u \ominus v, w \ominus z) \leq D_\infty(u, w) + D_\infty(v, z).$$

Proposition 2.11. *Let $k, u, v \in \mathbf{R}^+_{\mathfrak{F}}(\mathbf{R}^-_{\mathfrak{F}})$. Then*

$$D_\infty(k \odot u, k \odot v) \leq K D_\infty(u, v),$$

where $K = \Big(\mid k_c \mid + diam(k)\Big)$.

Proof. Since $k, u, v \in \mathbf{R}^+_{\mathfrak{F}}(\mathbf{R}^-_{\mathfrak{F}})$, there are several states for the sign of $k, u,$ and v. We assume $k, u, v \in \mathbf{R}^+_{\mathfrak{F}}$. From Definition 2.10, we have

$$\begin{aligned}
D_\infty(k \odot u, k \odot v) &= \sup_{\alpha \in [0,1]} \max\{\mid (k^-_\alpha - k_c)(u_c - v_c) + k_c(u^-_\alpha - v^-_\alpha) \mid, \mid (k^+_\alpha - k_c)(u_c - v_c) + k_c(u^+_\alpha - v^+_\alpha) \mid\} \\
&\leq \sup_{\alpha \in [0,1]} \max\{\mid (k^-_\alpha - k_c)(u_c - v_c) \mid, \mid (k^+_\alpha - k_c)(u_c - v_c) \mid\} + \sup_{\alpha \in [0,1]} \max\{\mid k_c(u^-_\alpha - v^-_\alpha) \mid, \mid k_c(u^+_\alpha - v^+_\alpha) \mid\} \\
&\leq \mid u_c - v_c \mid \sup_{\alpha \in [0,1]} \max\{\mid k^-_\alpha - k_c \mid, \mid k^+_\alpha - k_c \mid\} + \mid k_c \mid D_\infty(u, v) \\
&\leq \Big(\mid k_c \mid + diam(k)\Big) D_\infty(u, v) = K D_\infty(u, v).
\end{aligned}$$

We assume $k, u \in \mathbf{R}^+_{\mathfrak{F}}$ and $v \in \mathbf{R}^-_{\mathfrak{F}}$. Therefore from Definition 2.10, we have

$$D_\infty(k \odot u, k \odot v) = \sup_{\alpha \in [0,1]} \max\{\mid (k^-_\alpha - k_c)u_c - (k^+_\alpha - k_c)v_c + k_c(u^-_\alpha - v^-_\alpha) \mid, \mid (k^+_\alpha - k_c)u_c - (k^-_\alpha - k_c)v_c + k_c(u^+_\alpha - v^+_\alpha) \mid\} \tag{2.5}$$

$$\begin{aligned}
&\leq \sup_{\alpha \in [0,1]} \max\{\mid k_c(u^-_\alpha - v^-_\alpha) \mid, \mid k_c(u^+_\alpha - v^+_\alpha) \mid\} \\
&+ \sup_{\alpha \in [0,1]} \max\{\mid (k^-_\alpha - k_c)u_c - (k^+_\alpha - k_c)v_c \mid, \mid (k^+_\alpha - k_c)u_c - (k^-_\alpha - k_c)v_c \mid\} \\
&\leq \mid k_c \mid \sup_{\alpha \in [0,1]} \max\{\mid u^-_\alpha - v^-_\alpha \mid, \mid u^+_\alpha - v^+_\alpha \mid\} + diam(k) \mid u_c + v_c \mid \\
&\leq \mid k_c \mid D_\infty(u, v) + diam(k) \mid u_c - v_c \mid \\
&\leq \Big(\mid k_c \mid + diam(k)\Big) D_\infty(u, v) = K D_\infty(u, v).
\end{aligned}$$

The proof of other states are similar. □

Lemma 2.12. *Let* $u, v \in \mathbf{R}^{+}_{\mathfrak{F}}(\mathbf{R}^{-}_{\mathfrak{F}})$*. Then*

$$Core(u \odot v) = Core(u)Core(v).$$

Proof. The proof immediately follows from [5]. □

Remark 2.13. $f : (a, b) \to \mathbb{R}_{\mathcal{F}}$ is called fuzzy function and its $\alpha-$ cuts are as $\big(f(t)\big)_\alpha = [f^-_\alpha(t), f^+_\alpha(t)]$ for all $\alpha \in [0, 1]$.

Definition 2.14. [14] Let the integrable function $f : (a, b) \to \mathbb{R}_{\mathcal{F}}$ be such that $(f(t))_\alpha = [f^-_\alpha(t), f^+_\alpha(t)]$. We define

$$\left(\int_a^b f(s)ds\right)_\alpha = \Big[\int_a^b f^-_\alpha(s)ds, \int_a^b f^+_\alpha(s)ds\Big].$$

Definition 2.15. [14] Let $f : (a, b) \to \mathbb{R}_{\mathcal{F}}$ and $x_0 \in (a, b)$. We say f is strongly generalized differentiable at x_0 if there exists an element $f'(x_0) \in \mathbb{R}_{\mathcal{F}}$ such that one of the following statements is true

(i) For all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $f(x_0) \ominus f(x_0 - h)$ such that

$$\lim_{h \searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0).$$

(ii) For all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h)$, $f(x_0 - h) \ominus f(x_0)$ such that

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0).$$

(iii) For all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $f(x_0 - h) \ominus f(x_0)$ such that

$$\lim_{h \searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0).$$

(iv) For all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h)$, $f(x_0) \ominus f(x_0 - h)$ such that

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0).$$

Lemma 2.16. *[14] Let* $f : (a, b) \to \mathbb{R}_{\mathcal{F}}$ *such that* $f \in C\big((a, b), \mathbb{R}_{\mathcal{F}}\big)$*. Then*

(1) $F(t) = \int_a^t f(s)ds$ *is* (i)*-differentiable and we have* $F'(t) = f(t)$.
(2) $F(t) = \int_t^a f(s)ds$ *is* (ii)*-differentiable and we have* $F'(t) = -f(t)$.

Lemma 2.17. *[16] Let* $f : (a, b) \to \mathbb{R}_{\mathcal{F}}$ *be a fuzzy function, i.e.* $\big(f(t)\big)_\alpha = [f^-_\alpha(t), f^+_\alpha(t)]$ *for* $\alpha \in [0, 1]$.

(1) *If* f *is* (i)*-differentiable, then* $f^-_\alpha(t), f^+_\alpha(t)$ *are differentiable functions on* (a, b) *and*

$$\big(f'(t)\big)_\alpha = [(f^-_\alpha(t))', (f^+_\alpha(t))']. \tag{2.6}$$

(2) *If* f *is* (ii)*-differentiable, then* $f^-_\alpha(t), f^+_\alpha(t)$ *are differentiable functions on* (a, b) *and*

$$\big(f'(t)\big)_\alpha = [(f^+_\alpha(t))', (f^-_\alpha(t))']. \tag{2.7}$$

Definition 2.18. [8] We say that a point $x_0 \in (a, b)$ is a switching point for the differentiability of f, if in any neighborhood V of x_0 there exist points x_1, x_2 such that

- **type(I)** at x_1 (2.6) holds while (2.7) does not hold and at x_2 (2.7) holds and (2.6) does not hold, or
- **type(II)** at x_1 (2.7) holds while (2.6) does not hold and at x_2 (2.6) holds and (2.7) does not hold.

Remark 2.19. Throughout this paper, $f : J \to \mathbb{R}_{\mathcal{F}}$ is called GH-differentiable if it is (i)- or (ii)-differentiable on J and we consider $J = [0, T]$.

Lemma 2.20. *[14] Let* f *be GH-differentiable on* J*. Then*

$$\int_0^T f'(s)ds = f(T) \ominus_{gH} f(0).$$

Lemma 2.21. *[14] Let* $f : J \to \mathbb{R}_{\mathcal{F}}$ *be GH-differentiable on* J*, then we have*

(1) *If* $f(t)$ *is* (i)*-differentiable, then* $diam\big(f(t)\big)$ *is increasing with respect to* t.

TABLE 1. GH-differentiability of $f+g$ and $f\ominus g$.

Case	*Diff* f	*Diff* g	*Diff* $(f+g)$	$(f+g)'$	*Diff* $(f\ominus g)$	$(f\ominus g)'$
1	(i)	(i)	(i)	$f'+g'$	(i)	$f'\ominus g'$
2	(ii)	(ii)	(ii)	$f'+g'$	(ii)	$f'\ominus g'$
3	(i)	(ii)	(i)	$f'\ominus(-1)g'$	(i)	$f'+(-1)g'$
4	(ii)	(i)	(ii)	$f'\ominus(-1)g'$	(ii)	$f'+(-1)g'$

TABLE 2. GH-differentiability of fg.

Case	f	f'	*Diff* g	*Diff* (fg)	$(fg)'$
1	≥ 0	≥ 0	(i)	(i)	$f'g+fg'$
2	≥ 0	≤ 0	(i)	(i)	$fg'\ominus(-1)f'g$
3	≤ 0	≥ 0	(i)	(ii)	$fg'\ominus(-1)f'g$
4	≤ 0	≤ 0	(i)	(ii)	$f'g+fg'$
5	≥ 0	≥ 0	(ii)	(ii)	$fg'\ominus(-1)f'g$
6	≥ 0	≤ 0	(ii)	(ii)	$f'g+fg'$
7	≤ 0	≥ 0	(ii)	(i)	$f'g+fg'$
8	≤ 0	≤ 0	(ii)	(i)	$fg'\ominus(-1)f'g$
9	≥ 0	≥ 0	(ii)	(i)	$f'g\ominus(-1)fg'$

(2) *If $f(t)$ is (ii)-differentiable, then diam$\big(f(t)\big)$ is decreasing with respect to t.*

Lemma 2.22. *[4] Let $f: J\to\mathbb{R}_{\mathcal{F}}$ be such that $\big(f(t)\big)_\alpha=[f_\alpha^-(t),f_\alpha^+(t)]$. Suppose that real functions $f_\alpha^-(t)$, $f_\alpha^+(t)$ are differentiable with respect to t.*

(1) *If the intervals $[(f_\alpha^-(t))',(f_\alpha^+(t))']$ for all $\alpha\in[0,1]$ and $t\in J$, determine valid α cuts of a fuzzy number, then the H-differences $f(t+h)\ominus f(t)$ and $f(t)\ominus f(t-h)$ exist for all $h>0$ sufficiently small.*
(2) *If the intervals $[(f_\alpha^+(t))',(f_\alpha^-(t))']$ for all $\alpha\in[0,1]$ and $t\in J$, determine valid α cuts of a fuzzy number, then the H-differences $f(t)\ominus f(t+h)$ and $f(t-h)\ominus f(t)$ exist for all $h>0$ sufficiently small.*

Lemma 2.23. *[4, 14, 27] Let $f,g: J\to\mathbb{R}_{\mathcal{F}}$ be GH-differentiable on J. Then $f+g, f\ominus g: J\to\mathbb{R}_{\mathcal{F}}$ are GH-differentiable on J provided that the involving H-differences exist and the details of their kind of GH-differentiability are stated in Table 1.*

Lemma 2.24. *[25] Let $f: J\to\mathbb{R}$ and $g: J\to\mathbb{R}_{\mathcal{F}}$ be GH-differentiable on J. Then $fg: J\to\mathbb{R}_{\mathcal{F}}$ is GH-differentiable on J provided that the involving H-differences exist and the details of their kind of GH-differentiability is stated in Table 2.*

Lemma 2.25. *Let $f: J\to\mathbf{R}_{\mathfrak{F}}^+(\mathbf{R}_{\mathfrak{F}}^-)$ be GH-differentiable on J and $k\in\mathbf{R}_{\mathfrak{F}}^+(\mathbf{R}_{\mathfrak{F}}^-)$. Then $k\odot f: J\to\mathbf{R}_{\mathfrak{F}}^+(\mathbf{R}_{\mathfrak{F}}^-)$ is GH-differentiable on J provided that the involving H-differences exist and the details of their kind of GH-differentiability is stated in Table 3.*

Proof. Case 1: Since $k\in\mathbf{R}_{\mathfrak{F}}^+(\mathbf{R}_{\mathfrak{F}}^-)$ and $f\odot f'\succeq 0$, we have four states with the sign of k, f and f'. We suppose $k\in\mathbf{R}_{\mathfrak{F}}^-$ and $f,f'\succeq 0$. Therefore

$$\big(k\odot f(t)\big)_\alpha=[(k_\alpha^- -k_c)f_c(t)+k_cf_\alpha^+(t),(k_\alpha^+ -k_c)f_c(t)+k_cf_\alpha^-(t)].$$

On the other hand, f is (i)-differentiable. Hence, we have

$$(k\odot f'(t))_\alpha=[(k_\alpha^- -k_c)(f_c(t))'+k_c(f_\alpha^+(t))',(k_\alpha^+ -k_c)(f_c(t))'+k_c(f_\alpha^-(t))'].$$

Since $\big((k\odot f(t))_\alpha^-\big)'=(k\odot f'(t))_\alpha^-$ and $\big((k\odot f(t))_\alpha^+\big)'=(k\odot f'(t))_\alpha^+$, we claim that $k\odot f(t)$ is (i)-differentiable. So, from Lemma 2.22, the H-differences $k\odot f(t+h)\ominus k\odot f(t)$ and $k\odot f(t)\ominus k\odot f(t-h)$ exist. Also, from (i)-differentiability

TABLE 3. GH-differentiability of $k \odot f$

Case	$f \odot f'$	*Diff f*	*Diff* $(k \odot f)$	$(k \odot f)'$
1	$\succeq 0$	(i)	(i)	$k \odot f'$
2	$\preceq 0$	(i)	(i)	$k \odot f' \ominus (k-k) \odot f'$
3	$\preceq 0$	(i)	(ii)	$(k-k) \odot f' \ominus (-1)k \odot f'$
4	$\preceq 0$	(ii)	(ii)	$k \odot f'$
5	$\succeq 0$	(ii)	(ii)	$k \odot f' \ominus (k-k) \odot f'$
6	$\succeq 0$	(ii)	(i)	$(k-k) \odot f' \ominus (-1)k \odot f'$

of f the H-differences $f(t+h) \ominus f(t)$ and $f(t) \ominus f(t-h)$ exist. Consequently, from Lemma 2.8 and Proposition 2.11 along with $f, f' \succeq 0$, we have

$$\begin{aligned}
\lim_{h \searrow 0} D_\infty \left(\frac{k \odot f(t+h) \ominus k \odot f(t)}{h}, k \odot f'(t) \right) &= \lim_{h \searrow 0} D_\infty \left(k \odot \frac{f(t+h) \ominus f(t)}{h}, k \odot f'(t) \right) \\
&\leq \big(\mid k_c \mid + diam(k) \big) \lim_{h \searrow 0} D_\infty \left(\frac{f(t+h) \ominus f(t)}{h}, f'(t) \right) \\
&= 0.
\end{aligned} \tag{2.8}$$

Similarly, we can deduce

$$\begin{aligned}
\lim_{h \searrow 0} D_\infty \left(\frac{k \odot f(t) \ominus k \odot f(t-h)}{h}, k \odot f'(t) \right) &= \lim_{h \searrow 0} D_\infty \left(k \odot \frac{f(t) \ominus f(t-h)}{h}, k \odot f'(t) \right) \\
&\leq \big(\mid k_c \mid + diam(k) \big) \lim_{h \searrow 0} D_\infty \left(\frac{f(t) \ominus f(t-h)}{h}, f'(t) \right) \\
&= 0.
\end{aligned} \tag{2.9}$$

Therefore from Definition 2.15, we can conclude $\big(k \odot f\big)' = k \odot f'$. The proof of other states are similar.
Case 2: Since $k \in \mathbf{R}^+_{\mathfrak{F}} (\mathbf{R}^-_{\mathfrak{F}})$ and $f \odot f' \preceq 0$, we have several states with the sign of k, f and f'. We suppose $k \in \mathbf{R}^+_{\mathfrak{F}}$ and $f \preceq 0, f' \succeq 0$. Therefore, we have

$$\big(k \odot f(t)\big)_\alpha = [(k^+_\alpha - k_c) f_c(t) + k_c f^-_\alpha(t), (k^-_\alpha - k_c) f_c(t) + k_c f^+_\alpha(t)].$$

On the other hand, f is (i)-differentiable. Hence, we have

$$(k \odot f'(t))_\alpha = [(k^-_\alpha - k_c)(f_c(t))' + k_c (f^-_\alpha(t))', (k^+_\alpha - k_c)(f_c(t))' + k_c (f^+_\alpha(t))'].$$

Now, we suppose that the H-difference $k \odot f'(t) \ominus (k-k) \odot f'(t)$ for $t \in J$ exists. On the other hand, $\big((k \odot f(t))^-_\alpha\big)' = (k \odot f'(t) \ominus (k-k) \odot f'(t))^-_\alpha$ and $\big((k \odot f(t))^+_\alpha\big)' = (k \odot f'(t) \ominus (k-k) \odot f'(t))^+_\alpha$ which are valid α-level set of a fuzzy number. Therefore, we claim that $k \odot f(t)$ is (i)-differentiable. So, from Lemma 2.22, the H-differences $k \odot f(t+h) \ominus k \odot f(t)$ and $k \odot f(t) \ominus k \odot f(t-h)$ exist. Also, from (i)-differentiability of f the H-differences $f(t+h) \ominus f(t)$ and $f(t) \ominus f(t-h)$ exist. Consequently, from Definition 2.10, Lemma 2.8, and Proposition 2.11 along with $f \succeq 0$ and $f' \preceq 0$, we have

$$\begin{aligned}
&\lim_{h \searrow 0} D_\infty \left(\frac{k \odot f(t+h) \ominus k \odot f(t)}{h}, k \odot f'(t) \ominus (k-k) \odot f'(t) \right) \\
&\qquad = \lim_{h \searrow 0} D_\infty \left(k \odot \frac{f(t+h) \ominus f(t)}{h} \ominus (k-k) \odot \frac{f(t+h) \ominus f(t)}{h}, k \odot f'(t) \ominus (k-k) \odot f'(t) \right) \\
&\qquad \leq \left(\mid k_c \mid + 3\, diam(k) \right) \lim_{h \searrow 0} D_\infty \left(\frac{f(t+h) \ominus f(t)}{h}, f'(t) \right) = 0.
\end{aligned}$$

Table 4. CF-differentiability of $f+g$ and $f\ominus g$

Case	Diff f	Diff g	${}^{CF}D_*^{\beta}(f+g)$		${}^{CF}D_*^{\beta}(f\ominus g)$	
1	(i)	(i)	${}^{CF}D_*^{\beta}f+{}^{CF}D_*^{\beta}g$	(i)-CF	${}^{CF}D_*^{\beta}f\ominus(-1){}^{CF}D_*^{\beta}g$	(i)-CF
2	(ii)	(ii)	${}^{CF}D_*^{\beta}f+{}^{CF}D_*^{\beta}g$	(ii)-CF	${}^{CF}D_*^{\beta}f\ominus(-1){}^{CF}D_*^{\beta}g$	(ii)-CF
3	(i)	(ii)	${}^{CF}D_*^{\beta}f\ominus(-1){}^{CF}D_*^{\beta}g$	(i)-CF	${}^{CF}D_*^{\beta}f+(-1){}^{CF}D_*^{\beta}g$	(i)-CF
4	(ii)	(i)	${}^{CF}D_*^{\beta}f\ominus(-1){}^{CF}D_*^{\beta}g$	(ii)-CF	${}^{CF}D_*^{\beta}f+(-1){}^{CF}D_*^{\beta}g$	(ii)-CF

Similarly, we can deduce

$$\begin{aligned}\lim_{h\searrow 0} D_\infty\Big(\frac{k\odot f(t)\ominus k\odot f(t-h)}{h}, k\odot f'(t)\ominus(k-k)\odot f'(t)\Big)\\ = \lim_{h\searrow 0} D_\infty\Big(k\odot\frac{f(t)\ominus f(t-h)}{h}\ominus(k-k)\odot\frac{f(t)\ominus f(t-h)}{h}, k\odot f'(t)\ominus(k-k)\odot f'(t)\Big)\\ \le (\mid k_c\mid +3\,diam(k))\lim_{h\searrow 0} D_\infty\Big(\frac{f(t)\ominus f(t-h)}{h}, f'(t)\Big)=0.\end{aligned}$$

Therefore from Definition 2.15, we can conclude $(k\odot f)'=k\odot f'\ominus(k-k)\odot f'$. The proof of other states are similar. □

3. Fuzzy CF fractional derivative and integral

In this section, we present some definitions and properties of fuzzy Caputo-Fabrizio fractional derivative and integral which we will use throughout this study. Moreover, we give some theorems about the solution of linear fuzzy fractional differential equation involving Caputo-Fabrizio operator. Throughout this paper, we assume $0<\beta<1$.

Definition 3.1. [1] Let $f: J\to\mathbb{R}_{\mathcal{F}}$ be GH-differentiable such that $f'\in L^1(J,\mathbb{R}_{\mathcal{F}})$. The fuzzy CF derivative of f is defined as

$${}^{CF}D_*^{\beta}f(t)=\frac{1}{1-\beta}\int_0^t \exp\big(\frac{-\beta}{1-\beta}(t-s)\big)f'(s)ds, \quad t\in J.$$

Definition 3.2. Let $f: J\to\mathbb{R}_{\mathcal{F}}$ be GH-differentiable such that $f'\in L^1(J,\mathbb{R}_{\mathcal{F}})$.

(1) If f is (i)-differentiable, then

$$\big({}^{CF}D_*^{\beta}f(t)\big)_\alpha=\big[{}^{CF}D_*^{\beta}f_\alpha^-(t), {}^{CF}D_*^{\beta}f_\alpha^+(t)\big].$$

and f is called (i)-CF differentiable.

(2) If f is (ii)-differentiable, then

$$\big({}^{CF}D_*^{\beta}f(t)\big)_\alpha=\big[{}^{CF}D_*^{\beta}f_\alpha^+(t), {}^{CF}D_*^{\beta}f_\alpha^-(t)\big].$$

and f is called (ii)-CF differentiable.

Remark 3.3. Throughout this study, f is called CF differentiable on J if it is (i)- or (ii)-CF differentiable on J.

Definition 3.4. Let $f: J\to\mathbb{R}_{\mathcal{F}}$ such that $f\in L^1(J,\mathbb{R}_{\mathcal{F}})$. The fuzzy CF integral of f is defined as

$${}^{CF}I_*^{\beta}f(t)=(1-\beta)f(t)+\beta\int_0^t f(s)ds, \quad t\in J.$$

Lemma 3.5. *Let f, g, $f+g$ and $f\ominus g$ be GH-differentiable on J and their derivatives are integrable on J. Then $f+g$ and $f\ominus g$ are CF differentiable on J provided that the involving H-differences exist and the details of their kind of differentiability is as Table 4.*

Proof. Case 1: According to assumptions, f and g are (i)-differentiable. From Lemma 2.23, $f+g$ and $f\ominus g$ are (i)-differentiable and we have $(f+g)'=f'+g'$ and $(f\ominus g)'=f'\ominus g'$ provided the involving H-difference exists. Therefore, from Definition 3.2, $f+g$ and $f\ominus g$ are (i)-CF differentiable. Consequently, applying Definition 3.1, we have

$$\begin{aligned}{}^{CF}D_*^{\beta}\big(f(t)+g(t)\big) &= \frac{1}{1-\beta}\int_0^t \exp\big(-\frac{\beta}{1-\beta}(t-s)\big)\big(f'(s)+g'(s)\big)ds\\ &= \frac{1}{1-\beta}\int_0^t \exp\big(-\frac{\beta}{1-\beta}(t-s)\big)f'(s)ds+\frac{1}{1-\beta}\int_0^t \exp\big(-\frac{\beta}{1-\beta}(t-s)\big)g'(s)ds\\ &= {}^{CF}D_*^{\beta}f(t)+{}^{CF}D_*^{\beta}g(t),\end{aligned}$$

and

$$\begin{aligned}{}^{CF}D_*^{\beta}\big(f(t)\ominus g(t)\big) &= \frac{1}{1-\beta}\int_0^t \exp\big(-\frac{\beta}{1-\beta}(t-s)\big)\big(f'(s)\ominus g'(s)\big)ds\\ &= \frac{1}{1-\beta}\int_0^t \exp\big(-\frac{\beta}{1-\beta}(t-s)\big)f'(s)ds\ominus\frac{1}{1-\beta}\int_0^t \exp\big(-\frac{\beta}{1-\beta}(t-s)\big)g'(s)ds\\ &= {}^{CF}D_*^{\beta}f(t)\ominus{}^{CF}D_*^{\beta}g(t).\end{aligned}$$

Case 3: Let f be (i)-differentiable and g be (ii)-differentiable. Therefore, from Lemma 2.23, $f+g$ and $f\ominus g$ are (i)-differentiable i.e. $(f+g)'=f'\ominus(-1)g'$ and $(f\ominus g)'=f'+(-1)g'$ provided the involving H-difference exists. Therefore, from Definition 3.2, $f+g$ and $f\ominus g$ are (i)-CF differentiable. Consequently, from Definition 3.1, we have

$$\begin{aligned}{}^{CF}D_*^{\beta}\big(f(t)+g(t)\big) &= \frac{1}{1-\beta}\int_0^t \exp\big(-\frac{\beta}{1-\beta}(t-s)\big)\big(f'(s)\ominus(-1)g'(s)\big)ds\\ &= \frac{1}{1-\beta}\int_0^t \exp\big(-\frac{\beta}{1-\beta}(t-s)\big)f'(s)ds\ominus(-1)\frac{1}{1-\beta}\int_0^t \exp\big(-\frac{\beta}{1-\beta}(t-s)\big)g'(s)ds\\ &= {}^{CF}D_*^{\beta}f(t)\ominus(-1){}^{CF}D_*^{\beta}g(t),\end{aligned}$$

and

$$\begin{aligned}{}^{CF}D_*^{\beta}\big(f(t)\ominus g(t)\big) &= \frac{1}{1-\beta}\int_0^t \exp\big(-\frac{\beta}{1-\beta}(t-s)\big)\big(f'(s)+(-1)g'(s)\big)ds\\ &= \frac{1}{1-\beta}\int_0^t \exp\big(-\frac{\beta}{1-\beta}(t-s)\big)f'(s)ds+(-1)\frac{1}{1-\beta}\int_0^t \exp\big(-\frac{\beta}{1-\beta}(t-s)\big)g'(s)ds\\ &= {}^{CF}D_*^{\beta}f(t)+(-1){}^{CF}D_*^{\beta}g(t).\end{aligned}$$

□

Remark 3.6. If $k\in\mathbb{R}_{\mathcal{F}}$, then ${}^{CF}D_*^{\beta}k=0$.

Proof. It follows immediately from $(k)'=0$. □

Lemma 3.7. *Let $k\in\mathbf{R}^+_{\mathfrak{F}}(\mathbf{R}^-_{\mathfrak{F}})$ and $f:J\to\mathbf{R}^+_{\mathfrak{F}}(\mathbf{R}^-_{\mathfrak{F}})$ such that $f\in L^1(J,\mathbb{R}_{\mathcal{F}})$. Then*

$$ {}^{CF}I_*^{\beta}\big(k\odot f(t)\big)=k\odot{}^{CF}I_*^{\beta}f(t).$$

Proof. There are four states for the sign of k and f. We assume $k,f\succeq 0$. From Definition 2.14 and Remark 2.6, we have

$$\begin{aligned}\Big({}^{CF}I_*^{\beta}(k\odot f(t))\Big)_{\alpha} &= \Big[\Big({}^{CF}I_*^{\beta}\big(k\odot f(t)\big)\Big)^-_{\alpha},\Big({}^{CF}I_*^{\beta}\big(k\odot f(t)\big)\Big)^+_{\alpha}\Big]\\ &= \Big[(k^-_{\alpha}-k_c)\Big((1-\beta)f_c(t)+\beta\int_0^t f_c(s)ds\Big)+k_c\Big((1-\beta)f^-_{\alpha}(t)+\beta\int_0^t f^-_{\alpha}(s)ds\Big)\\ &\quad,(k^+_{\alpha}-k_c)\Big((1-\beta)f_c(t)+\beta\int_0^t f_c(s)ds\Big)+k_c\Big((1-\beta)f^+_{\alpha}(t)+\beta\int_0^t f^+_{\alpha}(s)ds\Big)\Big]\\ &= \Big(k\odot{}^{CF}I_*^{\beta}f(t)\Big)_{\alpha}.\end{aligned}$$

In a similar way, we can prove the other states. □

Lemma 3.8. *Consider $f : J \to \mathbb{R}_{\mathcal{F}}$ such that f is GH-differentiable and $f' \in L^1(J, \mathbb{R}_{\mathcal{F}})$.*

(1) *If ${}^{CF}I_*^{\beta} f(t)$ is (i)-differentiable, then*

$$ {}^{CF}D_*^{\beta}\, {}^{CF}I_*^{\beta} f(t) = f(t) \ominus \exp(\frac{-\beta t}{1-\beta}) f(0). $$

(2) *If ${}^{CF}I_*^{\beta} f(t)$ is (ii)-differentiable, then*

$$ {}^{CF}D_*^{\beta}\, {}^{CF}I_*^{\beta} f(t) = (-1)\left(\exp(\frac{-\beta t}{1-\beta}) f(0) \ominus f(t)\right). $$

Proof. Case 1: Since ${}^{CF}I_*^{\beta} f(t)$ is (i)-differentiable and f is GH-differentiable on J, so f is (i) or (ii)-differentiable on J. Firstly, we assume f is (i)-differentiable. Since $0 < \beta < 1$, from Lemma 2.24, $\exp(\frac{\beta t}{1-\beta}) f(t)$ is (i)-differentiable. Therefore, from Case 1 of Lemma 3.5, we have

$$ {}^{CF}D_*^{\beta}\, {}^{CF}I_*^{\beta} f(t) = {}^{CF}D_*^{\beta}\left((1-\beta) f(t) + \beta \int_0^t f(s) ds\right) \tag{3.1} $$

$$ = (1-\beta)\, {}^{CF}D_*^{\beta} f(t) + \beta\, {}^{CF}D_*^{\beta}\big(\int_0^t f(s) ds\big) $$

$$ = \exp(\frac{-\beta t}{1-\beta}) \int_0^t \exp(\frac{\beta s}{1-\beta}) f'(s) ds + \frac{\beta}{1-\beta} \int_0^t \exp\big(-\frac{\beta}{1-\beta}(t-s)\big) f(s) ds. \tag{3.2} $$

For computing $\int_0^t \exp\big(\frac{\beta s}{1-\beta}\big) f'(s) ds$, we apply Case 1 of Lemma 2.24 as follows

$$ \int_0^t \left(\exp\big(\frac{\beta s}{1-\beta}\big) f(s)\right)' ds = \int_0^t \exp\big(\frac{\beta s}{1-\beta}\big) f'(s) ds + \frac{\beta}{1-\beta} \int_0^t \exp\big(\frac{\beta s}{1-\beta}\big) f(s) ds. $$

Therefore, from Lemma 2.20, we have

$$ \exp\big(\frac{\beta t}{1-\beta}\big) f(t) \ominus f(0) = \int_0^t \exp\big(\frac{\beta s}{1-\beta}\big) f'(s) ds + \frac{\beta}{1-\beta} \int_0^t \exp\big(\frac{\beta s}{1-\beta}\big) f(s) ds. $$

Thus, we infer

$$ \int_0^t \exp\big(\frac{\beta s}{1-\beta}\big) f'(s) ds = \left(\exp\big(\frac{\beta t}{1-\beta}\big) f(t) \ominus f(0)\right) \ominus \frac{\beta}{1-\beta} \int_0^t \exp\big(\frac{\beta s}{1-\beta}\big) f(s) ds. $$

Putting the above relation into Eq. (3.1), we have

$$ {}^{CF}D_*^{\beta}\, {}^{CF}I_*^{\beta} f(t) = \left(\big(f(t) \ominus \exp(\frac{-\beta t}{1-\beta}) f(0)\big) \ominus \frac{\beta}{1-\beta} \int_0^t \exp(-\frac{\beta}{1-\beta}(t-s)) f(s) ds\right) + \frac{\beta}{1-\beta} \int_0^t \exp\big(-\frac{\beta}{1-\beta}(t-s)\big) f(s) ds. $$

From (i)-differentiability of $\exp(\frac{\beta t}{1-\beta}) f(t)$ along with Lemma 2.20, the H-difference $f(t) \ominus \exp(\frac{-\beta t}{1-\beta}) f(0)$ for $t \geq 0$ exists. On the other hand, ${}^{CF}D_*^{\beta} f(t) \in \mathbb{R}_{\mathcal{F}}$. Hence, from the results of Lemma 2.3, we deduce

$$ {}^{CF}D_*^{\beta}\, {}^{CF}I_*^{\beta} f(t) = \left(f(t) + \frac{\beta}{1-\beta} \int_0^t \exp\big(-\frac{\beta}{1-\beta}(t-s)\big) f(s) ds\right) \ominus \left(\exp(\frac{-\beta t}{1-\beta}) f(0) + \frac{\beta}{1-\beta} \int_0^t \exp\big(-\frac{\beta}{1-\beta}(t-s)\big) f(s) ds\right) $$

$$ = f(t) \ominus \exp(\frac{-\beta t}{1-\beta}) f(0). $$

Secondly, we assume f is (ii)-differentiable. Since ${}^{CF}I_*^{\beta} f(t)$ is (i)-differentiable, therefore the H-difference $\beta f(t) \ominus (-1)(1-\beta) f'(t)$ for $t \geq 0$ exists. Therefore, $\exp(\frac{\beta t}{1-\beta}) f(t)$ must be (i)-differentiable. In a similar way, from Lemmas 2.25 and 3.5, we have

$$ {}^{CF}D_*^{\beta}\, {}^{CF}I_*^{\beta} f(t) = \beta\, {}^{CF}D_*^{\beta}\big(\int_0^t f(s) ds\big) \ominus (-1)(1-\beta)\, {}^{CF}D_*^{\beta} f(t) $$

$$ = \frac{\beta}{1-\beta} \int_0^t \exp\big(-\frac{\beta}{1-\beta}(t-s)\big) f(s) ds $$

$$ \ominus \left(\frac{\beta}{1-\beta} \int_0^t \exp\big(-\frac{\beta}{1-\beta}(t-s)\big) f(s) ds \ominus \big(f(t) \ominus \exp(\frac{-\beta t}{1-\beta}) f(0)\big)\right). $$

Since $\exp(\frac{\beta t}{1-\beta})f(t)$ is (i)-differentiable, from Lemma 2.20, the H-difference $f(t) \ominus \exp(\frac{-\beta t}{1-\beta})f(0)$ for $t \geq 0$ always exists. On the other hand, ${}^{CF}D_*^{\beta} f(t) \in \mathbb{R}_{\mathcal{F}}$. Therefore, from Lemma 2.3, we have

$$\begin{aligned} {}^{CF}D_*^{\beta}\, {}^{CF}I_*^{\beta} f(t) &= \Big(\frac{\beta}{1-\beta}\int_0^t \exp\big(-\frac{\beta}{1-\beta}(t-s)\big)f(s)ds + \big(f(t)\ominus \exp(\frac{-\beta t}{1-\beta})f(0)\big)\Big) \ominus \frac{\beta}{1-\beta}\int_0^t \exp\big(-\frac{\beta}{1-\beta}(t-s)\big)f(s)ds \\ &= f(t)\ominus \exp(\frac{-\beta t}{1-\beta})f(0). \end{aligned}$$

Case 2: According to the assumptions of Case 2, ${}^{CF}I_*^{\beta} f(t)$ is (ii)-differentiable. To verify this assumption, we must have (ii)-differentiability for f and the existence of the H-difference $(1-\beta)f'(t) \ominus (-1)\beta f(t)$. From Lemma 2.24, we can conclude $\exp(\frac{\beta t}{1-\beta})f(t)$ is (ii)-differentiable. From Case 4 of Lemma 3.5 and Definition 3.1, we can conclude

$$\begin{aligned} {}^{CF}D_*^{\beta}\, {}^{CF}I_*^{\beta} f(t) &= {}^{CF}D_*^{\beta}\Big((1-\beta)f(t) + \beta\int_0^t f(s)ds\Big) \\ &= (1-\beta)\, {}^{CF}D_*^{\beta} f(t) \ominus (-\beta)\, {}^{CF}D_*^{\beta}\big(\int_0^t f(s)ds\big) \\ &= \exp(\frac{-\beta t}{1-\beta})\int_0^t \exp(\frac{\beta s}{1-\beta})f'(s)ds \ominus \frac{-\beta}{1-\beta}\int_0^t \exp\big(-\frac{\beta}{1-\beta}(t-s)\big)f(s)ds. \end{aligned} \tag{3.3}$$

For computing $\int_0^t \exp\big(\frac{\beta s}{1-\beta}\big)f'(s)ds$, we apply Case 3 of Lemma 2.24 as follows

$$\int_0^t \Big(\exp\big(\frac{\beta s}{1-\beta}\big)f(s)\Big)' ds = \int_0^t \exp(\frac{\beta s}{1-\beta})f'(s)ds \ominus \frac{-\beta}{1-\beta}\int_0^t \exp\big(\frac{\beta s}{1-\beta}\big)f(s)ds.$$

Therefore, from Lemma 2.20, we have

$$(-1)\Big(f(0)\ominus \exp(\frac{\beta t}{1-\beta})f(t)\Big) = \int_0^t \exp\big(\frac{\beta s}{1-\beta}\big)f'(s)ds \ominus \frac{-\beta}{1-\beta}\int_0^t \exp\big(\frac{\beta s}{1-\beta}\big)f(s)ds.$$

Thus, we infer

$$\int_0^t \exp\big(\frac{\beta s}{1-\beta}\big)f'(s)ds = (-1)\Big(f(0)\ominus \exp(\frac{\beta t}{1-\beta})f(t)\Big) + \frac{-\beta}{1-\beta}\int_0^t \exp\big(\frac{\beta s}{1-\beta}\big)f(s)ds.$$

Putting the above relation into Eq. (3.3), we have

$$\begin{aligned} {}^{CF}D_*^{\beta}\, {}^{CF}I_*^{\beta} f(t) =& (-1)\Big(\big(\exp(\frac{-\beta t}{1-\beta})f(0)\ominus f(t)\big) + \frac{\beta}{1-\beta}\int_0^t \exp\big(-\frac{\beta}{1-\beta}(t-s)\big)f(s)ds\Big) \\ &\ominus \frac{-\beta}{1-\beta}\int_0^t \exp\big(-\frac{\beta}{1-\beta}(t-s)\big)f(s)ds. \end{aligned}$$

Consequently, form Lemma 2.20, the H-difference $\exp(\frac{-\beta t}{1-\beta})f(0)\ominus f(t)$ for $t \geq 0$ exists. Hence we deduce

$${}^{CF}D_*^{\beta}\, {}^{CF}I_*^{\beta} f(t) = (-1)\Big(\exp(\frac{-\beta t}{1-\beta})f(0)\ominus f(t)\Big).$$

□

4. Linear fuzzy CF fractional differential equations with fuzzy coefficients and fuzzy initial value

In this section, we investigate the following fuzzy initial value problem of linear fuzzy CF fractional differential equations with fuzzy coefficients

$$\begin{aligned} &{}^{CF}D_*^{\beta} y(t) = a \odot y(t) + f(t), \qquad t \in J, \\ &y(0) = y_0, \end{aligned} \tag{4.1}$$

where $a, y_0 \in \mathbb{R}_{\mathcal{F}}$ and $f : J \to \mathbb{R}_{\mathcal{F}}$ is generalized differentiable such that $f' \in L^1(J, \mathbb{R}_{\mathcal{F}})$.
This section is divided into two subsections. The first subsection is devoted to non-homogenous linear fuzzy CF

fractional differential equation with crisp force function and the second to non-homogenous linear fuzzy CF fractional differential equation with fuzzy force function.

Definition 4.1. We say that $y \in C(J, \mathbb{R}_{\mathcal{F}})$ is (i)-solution of Problem (4.1), if y is (i)-CF differentiable and satisfies in Problem (4.1). Also, $y \in C(J, \mathbb{R}_{\mathcal{F}})$ is (ii)-solution of Problem (4.1), if y is (ii)-CF differentiable and satisfies in Problem (4.1).

4.1. **Non-homogenous linear fuzzy CF fractional differential equation with uncertainty in speed and initial value.**

We consider the initial valve problem

$$\begin{aligned} &{}^{CF}D_*^{\beta} y(t) = a \odot y(t) + f(t), \qquad t \in J \\ &y(0) = y_0, \end{aligned} \tag{4.2}$$

where $a, y_0 \in \mathbf{R}_{\mathfrak{F}}^{+}(\mathbf{R}_{\mathfrak{F}}^{-})$ and $f : J \to \mathbb{R}$ is differentiable such that $f' \in L^1(J, \mathbb{R})$. Throughout the paper, we consider $p = (1-\beta)a_c$, $q = \beta a_c$, and $\lambda = \frac{q}{1-p}$.

According to the Problem (4.2) and the fact that ${}^{CF}D_*^{\beta} y(0) = 0$, we deduce $a \odot y(0) + f(0) = 0$. Therefore, we solve Problem (4.2) under one of the following conditions.

(1) $f(0) = 0$, $a \in \mathbf{R}_{\mathfrak{F}}$, $a_c \neq \{0\}$, and $y_0 = 0 \in \mathbf{R}$.
(2) $f(0) = 0$, $a = 0 \in \mathbf{R}$, and $y_0 \in \mathbf{R}_{\mathfrak{F}}$.
(3) $f(0) = 0$, $a, y_0 \in \mathbf{R}_{\mathfrak{F}}$, and $a_c = y_c^0 = \{0\}$.

For the sake of simplicity, we suppose the following conditions.

Definition 4.2. We say that the function $h(t)$ satisfies conditions $(A_1) - (A_5)$, if

(A_1) $\beta h(t) + (1-\beta)h'(t) \geq 0, \forall t \in (0, T)$.
(A_2) $\beta h(t) + (1-\beta)h'(t) \leq 0, \forall t \in (0, T)$.
(A_3) $\int_0^t \big(\beta h(s) + (1-\beta)h'(s)\big)ds + y_c^0 \geq 0, \forall t \in (0, T)$.
(A_4) $\int_0^t \big(\beta h(s) + (1-\beta)h'(s)\big)ds + y_c^0 \leq 0, \forall t \in (0, T)$.
(A_5)

$$\begin{aligned} &\left(\int_0^t \big(1-\beta+\frac{\beta}{1-p}(t-s)\big)\big(\beta h(s) + (1-\beta)h'(s)\big)e^{-\lambda s}ds + \beta t y_c^0\right)\bigg((1-\beta)\big(\beta h(t) + (1-\beta)h'(t)\big) \\ &+ \frac{\beta}{1-p}e^{\lambda t}\int_0^t \big(\lambda(t-s)+p+1\big)\big(\beta h(s) + (1-\beta)h'(s)\big)e^{-\lambda s}ds\big) + \beta(1+\lambda t)e^{\lambda t}y_c^0\bigg) \geq 0, \forall t \in (0, T). \end{aligned}$$

Theorem 4.3. *Let $f : J \to \mathbb{R}$ be differentiable such that $f' \in L^1(J, \mathbb{R})$. Moreover, let $y_0, a \in \mathbf{R}_{\mathfrak{F}}^{+}(\mathbf{R}_{\mathfrak{F}}^{-})$. Consider*

$$\begin{aligned} y_{1N}(t) &= \frac{e^{\lambda t}}{1-p}\int_0^t \big(\beta f(s) + (1-\beta)f'(s)\big)e^{-\lambda s}ds + e^{\lambda t}y_0 + (a - a_c) \odot g(t), \\ y_{2N}(t) &= \frac{e^{\lambda t}}{1-p}\int_0^t \big(\beta f(s) + (1-\beta)f'(s)\big)e^{-\lambda s}ds + \Big(e^{\lambda t}y_0 \ominus (-1)(a - a_c) \odot g(t)\Big), \end{aligned}$$

where

$$g(t) = \frac{e^{\lambda t}}{(1-p)^2}\left(\beta t y_0 + \int_0^t \big(1-\beta+\frac{\beta}{1-p}(t-s)\big)\big(\beta f(s) + (1-\beta)f'(s)\big)e^{-\lambda s}ds\right).$$

Then, the kind of solution to Problem (4.2) along with required conditions in the details are presented in Tables 5, 6, and 7.

Proof. Here, we just prove Case 1 of Table 5. The other cases of Tables 5, 6, and 7 can be investigated in a similar manner. Since $p = 0$, so $\lambda = \frac{q}{1-p} = 0$. Let $f, f' \geq 0$ and $y_0 \succeq 0$.

$$(y_{1N}(t))_1 = \int_0^t \big(\beta f(s) + (1-\beta)f'(s)\big)ds + y_c^0 \geq 0.$$

Table 5. The obtained results of Theorem 4.3 with $p = 0$ for $t \in (0, T)$ in details.

Case	*condition on f*	*solution*	*kind of Diff for solution*
1	$f, f' \geq 0,\ y_c^0 \geq 0$	y_{1N}	(i)
2	$f, f' \leq 0,\ y_c^0 \leq 0$	y_{1N}	(i)
3	$f, f' \leq 0,\ y_c^0 \geq 0,\ (A_3),\ (A_5)$	y_{1N}	(i)
4	$f, f' \geq 0,\ y_c^0 \leq 0,\ (A_4),\ (A_5)$	y_{1N}	(i)
5	$ff' \leq 0,\ y_c^0 \geq 0,\ (A_1)$	y_{1N}	(i)
6	$ff' \leq 0,\ y_c^0 \leq 0,\ (A_2)$	y_{1N}	(i)
7	$ff' \leq 0,\ ,\ y_c^0 \geq 0,\ (A_3),\ (A_5)$	y_{1N}	(i)
8	$ff' \leq 0,\ ,\ y_c^0 \leq 0,\ (A_4),\ (A_5)$	y_{1N}	(i)
9	$f, f' \geq 0,\ y_c^0 \geq 0$	y_{2N}	(ii)
10	$f, f' \leq 0,\ y_c^0 \leq 0$	y_{2N}	(ii)
11	$f, f' \leq 0,\ y_c^0 \geq 0,\ (A_3),\ (A_5)$	y_{2N}	(ii)
12	$f, f' \geq 0,\ y_c^0 \leq 0,\ (A_4),\ (A_5)$	y_{2N}	(ii)
13	$ff' \leq 0,\ y_c^0 \geq 0,\ (A_1)$	y_{2N}	(ii)
14	$ff' \leq 0,\ y_c^0 \leq 0,\ (A_2)$	y_{2N}	(ii)
15	$ff' \leq 0,\ ,\ y_c^0 \geq 0,\ (A_3),\ (A_5)$	y_{2N}	(ii)
16	$ff' \leq 0,\ ,\ y_c^0 \leq 0,\ (A_4),\ (A_5)$	y_{2N}	(ii)

Table 6. The obtained results of Theorem 4.3 with $0 < p < 1$ for $t \in (0, T)$ in details.

Case	*condition on f*	*solution*	*kind of Diff for solution*
1	$ff' \geq 0$	y_{1N}	(i)
2	$ff' \leq 0,\ (A_1)$	y_{1N}	(i)
3	$ff' \leq 0,\ (A_2)$	y_{1N}	(i)

Table 7. The obtained results of Theorem 4.3 with $p > 1$ for $t \in (0, T)$ in details.

Case	*condition on f*	*solution*	*kind of Diff for solution*
1	$ff' \geq 0,\ (A_5)$	y_{1N}	(i)
2	$ff' \leq 0,\ (A_2),\ (A_5)$	y_{1N}	(i)
3	$ff' \leq 0,\ (A_1),\ (A_5)$	y_{1N}	(i)

Consequently, $y_{1N}(t) \succeq 0$. From Lemma 2.3, Proposition 2.8, and condition $a \odot y_0 + f(0) = 0$, we can deduce

$$a \odot y_{1N}(t) + f(t) = \Big(\int_0^t (\beta f(s) + (1-\beta) f'(s)) e^{-\lambda s} ds\Big) a + f(t). \tag{4.3}$$

From Lemmas 2.23 and 2.24, $g(t)$ is (i)-differentiable. It is easy to check that $Core\big(g(t) \odot g'(t)\big) = \big(g(t)\big)_1 . \big(g'(t)\big)_1 \geq 0$. Then, from Lemma 2.25, $a \odot g(t)$ is (i)-differentiable. Consequently, y_{1N} is (i)-differentiable and we have

$$y'_{1N}(t) = \beta f(t) + (1-\beta) f'(t) + a \odot g'(t).$$

TABLE 8. The required conditions for Theorem 4.4.

Case	*condition on* f_c
1	$f_c, f_c' \geq 0,\ y_c^0 \geq 0$
2	$f_c, f_c' \leq 0,\ y_c^0 \leq 0$
3	$f_c, f_c' \leq 0,\ y_c^0 \geq 0,\ (A_3),\ (A_5)$
4	$f_c, f_c' \geq 0,\ y_c^0 \leq 0,\ (A_4),\ (A_5)$
5	$f_c f_c' \leq 0,\ y_c^0 \geq 0,\ (A_1)$
6	$f_c f_c' \leq 0,\ y_c^0 \leq 0,\ (A_2)$
7	$f_c f_c' \leq 0,\ ,\ y_c^0 \geq 0,\ (A_3),\ (A_5)$
8	$f_c f_c' \leq 0,\ ,\ y_c^0 \leq 0,\ (A_4),\ (A_5)$

From Definition 3.1, Lemma 2.3, and condition $a \odot y_0 + f(0) = 0$, we have

$$\begin{aligned} {}^{CF}D_*^{\beta} y_{1N}(t) &= \frac{1}{1-\beta}\int_0^t \exp\big(\frac{-\beta}{1-\beta}(t-s)\big)y_{1N}'(s)ds \\ &= f(t) - e^{\frac{-\beta t}{1-\beta}} f(0) + \left(\int_0^t \big(\beta f(s) + (1-\beta)f'(s)\big)ds + (1 - e^{\frac{-\beta t}{1-\beta}})y_0\right) \odot a \\ &= f(t) + \left(\int_0^t \big(\beta f(s) + (1-\beta)f'(s)\big)ds\right) a. \end{aligned} \tag{4.4}$$

From Eq. (4.3) and (4.4), we can deduce

$${}^{CF}D_*^{\beta} y_{1N}(t) = a \odot y_{1N}(t) + f(t).$$

Therefore, y_{1N} is (i)-solution of Problem (4.2).

□

4.2. **Non-homogenous linear fuzzy CF fractional differential equation with uncertainty in speed, source and initial value.**

We consider the initial valve problem

$$\begin{aligned} {}^{CF}D_*^{\beta} y(t) &= a \odot y(t) + f(t), \qquad t \in J \\ y(0) &= y_0, \end{aligned} \tag{4.5}$$

where $a, y_0 \in \mathbb{R}_{\mathcal{F}}$ and $f : J \to \mathbb{R}_{\mathcal{F}}$ is generalized differentiable such that $f' \in L^1(J, \mathbb{R}_{\mathcal{F}})$.
For simplicity, we present some notations which will be used in the next theorem.

(I) For $t \in J$ the following H-difference exists.

$$\beta y_0 \ominus (-1)\left((1-\beta)\big(\beta f(t) + (1-\beta)f'(t)\big) + \beta\int_0^t \big(\beta f(s) + (1-\beta)f'(s)\big)ds\right).$$

Theorem 4.4. *Let $f : J \to \mathbf{R}_{\mathfrak{F}}^{+}(\mathbf{R}_{\mathfrak{F}}^{-})$ be (i)-differentiable such that $f' \in L^1(J, \mathbf{R}_{\mathfrak{F}})$ and $p = 0$. Moreover, let $a, y_0 \in \mathbf{R}_{\mathfrak{F}}^{+}(\mathbf{R}_{\mathfrak{F}}^{-})$. Consider*

$$y_{3N}(t) = e^{\lambda t} y_0 \ominus (-1)\left((a - a_c) \odot \bar{g}(t) + \frac{e^{\lambda t}}{1-p}\int_0^t \big(\beta f(s) + (1-\beta)f'(s)\big)e^{-\lambda s}ds\right),$$

where

$$\bar{g}(t) = \frac{e^{\lambda t}}{(1-p)^2}\left(\beta t y_0 \ \ominus (-1)\int_0^t \big(1 - \beta + \frac{\beta}{1-p}(t-s)\big)\big(\beta f(s) + (1-\beta)f'(s)\big)e^{-\lambda s}ds\right),$$

provided the above H-differences exist.

(1) *Let f be (i)-differentiable. If one of the conditions presented in Table 8 satisfies, then y_{1N} is (i)-solution of Problem (4.5) for $t \in (0, T)$.*

TABLE 9. The required conditions for Theorem 4.4.

Case	*condition on* f_c
1	$f_c, f_c' \geq 0,\ y_c^0 \geq 0$
2	$f_c, f_c' \leq 0,\ y_c^0 \leq 0$
3	$f_c, f_c' \leq 0,\ y_c^0 \geq 0,\ (A_3),\ (A_5)$
4	$f_c, f_c' \geq 0,\ y_c^0 \leq 0,\ (A_4),\ (A_5)$
5	$f_c f_c' \leq 0,\ y_c^0 \geq 0,\ (A_1)$
6	$f_c f_c' \leq 0,\ y_c^0 \leq 0,\ (A_2)$
7	$f_c f_c' \leq 0,\ ,\ y_c^0 \geq 0,\ (A_3),\ (A_5)$
8	$f_c f_c' \leq 0,\ ,\ y_c^0 \leq 0,\ (A_4),\ (A_5)$

(2) *Let f be (i)-differentiable. If one of the conditions presented in Table 9 along with the Condition (I) satisfy, then y_{3N} is (ii)-solution of Problem (4.5) for $t \in (0, T)$.*

Notice that y_{1N} has been presented in Theorem 4.3 with this difference that here the function f appeared in y_{1N} is a fuzzy function.

Proof. (1) Here, we just prove theorem under the condition in Row 1 of Table 8. The other cases can be investigated in an analogous manner. Let f be (i)-differentiable, $f, f' \succeq 0$, and $y_0 \succeq 0$. Since $p = 0$, so $\lambda = \frac{q}{1-p} = 0$. Thus, we have

$$(y_{1N}(t))_1 = y_c^0 + \int_0^t \big(\beta f_c(s) + (1-\beta) f_c'(s)\big) ds \geq 0.$$

Therefore, $y_{1N}(t) \succeq 0$. Also, by the results of Lemma 2.3, Proposition 2.8, and condition $a \odot y_0 + f(0) = 0$ we can deduce

$$a \odot y_{1N}(t) + f(t) = \Big(\int_0^t (\beta f(s) + (1-\beta) f'(s)) ds\Big) \odot a + f(t). \tag{4.6}$$

From Lemmas 2.16 and 2.24, ty_0 and

$$\int_0^t \big(\beta f(s) + (1-\beta) f'(s)\big) ds,$$

$$\int_0^t \big(1 - \beta + \beta(t-s)\big)\big(\beta f(s) + (1-\beta) f'(s)\big) ds,$$

are (i)-differentiable. Therefore, from Lemma 2.23, $g(t)$ is (i)-differentiable. It is easy to check that $Core\big(g(t) \odot g'(t)\big) = \big(g(t)\big)_1 . \big(g'(t)\big)_1 \geq 0$. Then from Lemma 2.25, $a \odot g(t)$ is (i)-differentiable. Consequently, y_{1N} is (i)-differentiable and we have

$$y_{1N}'(t) = \int_0^t (\beta f(s) + (1-\beta) f'(s)) ds + \beta f(t) + (1-\beta) f'(t) + a \odot g'(t).$$

Utilizing Definition 3.1 and Lemma 2.3, and condition $a \odot y_0 + f(0) = 0$, we have

$$\begin{aligned} {}^{CF}D_*^{\beta} y_{1N}(t) &= \frac{1}{1-\beta} \int_0^t \exp\big(\frac{-\beta}{1-\beta}(t-s)\big) y_{1N}'(s) ds \\ &= \Big(\int_0^t (\beta f(s) + (1-\beta) f'(s)) ds + (1 - e^{\frac{-\beta t}{1-\beta}}) y_0\Big) \odot a + (f(t) \ominus e^{\frac{-\beta t}{1-\beta}} f(0)) \\ &= \Big(\int_0^t (\beta f(s) + (1-\beta) f'(s)) ds\Big) \odot a + f(t). \end{aligned} \tag{4.7}$$

It follows from Eq. (4.6) and (4.7) that y_{1N} satisfies Eq. (4.5) and we have

$${}^{CF}D_*^{\beta} y_{1N}(t) = a \odot y_{1N}(t) + f(t).$$

TABLE 10. The required conditions for Theorem 4.5.

Row	*condition on* $f_c(t)$
1	$f_c f_c' \geq 0$
2	$f_c f_c' \leq 0,\ (A_1)$
3	$f_c f_c' \leq 0,\ (A_2)$

Consequently, using the above facts, y_{1N} is (i)-solution of Problem (4.5) for $t \in (0,T)$.
(2) Here, we just prove Case 1 under the condition in Row 1 of Table 9. The other cases can be investigated in an analogous manner. Let f be (i)-differentiable, $f, f' \succeq 0$, and $y_0 \succeq 0$. Since $p = 0$, so $\lambda = \frac{q}{1-p} = 0$. Hence, we have

$$(y_{3N}(t))_1 = y_c^0 + \int_0^t \big(\beta f_c(s) + (1-\beta) f_c'(s)\big) ds \geq 0.$$

Therefore, $y_{3N}(t) \succeq 0$. By the results of Lemma 2.3, Proposition 2.8, and condition $a \odot y_0 + f(0) = 0$, we can deduce

$$a \odot y_{3N}(t) + f(t) = \Big(\int_0^t (\beta f(s) + (1-\beta) f'(s)) ds \Big) \odot a + f(t). \tag{4.8}$$

Let $\bar{g}(t)$ be (i)-differentiable. It is easy to check that $Core\big(\bar{g}(t) \odot \bar{g}'(t)\big) = \big(\bar{g}(t)\big)_1 . \big(\bar{g}'(t)\big)_1 \geq 0$. Consequently, from Lemma 2.25, $a \odot \bar{g}(t)$ is (i)-differentiable. It follows from Condition (I) that y_{3N} is (ii)-differentiable and we have

$$y_{3N}'(t) = y_0 + a \odot \bar{g}'(t) + \beta f(t) + (1-\beta) f'(t),$$

From Definition 3.1, Lemma 2.3, and condition $a \odot y_0 + f(0) = 0$, we have

$$\begin{aligned}
{}^{CF}D_*^{\beta} y_{3N}(t) &= \frac{1}{1-\beta} \int_0^t \exp\big(\frac{-\beta}{1-\beta}(t-s)\big) y_{3N}'(s) ds \\
&= \Big(\int_0^t (\beta f(s) + (1-\beta) f'(s)) ds + (1 - e^{\frac{-\beta t}{1-\beta}}) y_0 \Big) \odot a + \big(f(t) \ominus e^{\frac{-\beta t}{1-\beta}} f(0)\big) \\
&= \Big(\int_0^t (\beta f(s) + (1-\beta) f'(s)) e^{-\lambda s} ds \Big) \odot a + f(t).
\end{aligned} \tag{4.9}$$

From Eq. (4.8) and (4.9), we have

$${}^{CF}D_*^{\beta} y_{3N}(t) = a \odot y_{3N}(t) + f(t).$$

Consequently, y_{3N} is (ii)-solution of Problem (4.5). □

Theorem 4.5. *Let $f : J \to \mathbf{R}_{\mathfrak{F}}^{+}(\mathbf{R}_{\mathfrak{F}}^{-})$ be (i)-differentiable such that $f' \in L^1(J, \mathbf{R}_{\mathfrak{F}})$ and let $a \in \mathbf{R}_{\mathfrak{F}}^{+}$ and $0 < p < 1$. If one of conditions presented in Table 10 satisfies, then y_{1N} is (i)-solution of Problem (4.5) for $t \in (0,T)$.*

Proof. Here, we just prove theorem under the condition in Row 1 of Table 10. The other cases can be investigated in an analogous manner. Let f be (i)-differentiable, $f, f' \succeq 0$. Since $0 < p < 1$, hence from condition $a \odot y_0 + f(0) = 0$, we have $y_0 = 0$ and

$$(y_{1N}(t))_1 = \frac{e^{\lambda t}}{1-p} \int_0^t \big(\beta f_c(s) + (1-\beta) f_c'(s)\big) e^{-\lambda s} ds \geq 0.$$

Therefore, $y_{1N}(t) \succeq 0$. Also, by the results of Lemma 2.3 and Proposition 2.8, and condition $a \odot y_0 + f(0) = 0$, we can deduce

$$\begin{aligned}
a \odot y_{1N}(t) + f(t) &= \Big(\frac{e^{\lambda t}}{1-p} \int_0^t \big(\beta f(s) + (1-\beta) f'(s)\big) e^{-\lambda s} ds \Big) \odot a + f(t) \\
&+ \frac{e^{\lambda t}}{(1-p)^2} \Big(\int_0^t \big(p + \frac{q}{1-p}(t-s)\big) \big(\beta f(s) + (1-\beta) f'(s)\big) e^{-\lambda s} ds \Big) \odot (a - a_c).
\end{aligned} \tag{4.10}$$

Since $0 < p < 1$ and $a \in \mathbf{R}_{\mathfrak{F}}^{+}$, so $\lambda = \frac{q}{1-p} > 0$. Hence, from Lemma 2.24

$$\frac{e^{\lambda t}}{1-p} \int_0^t \big(\beta f(s) + (1-\beta) f'(s)\big) e^{-\lambda s} ds,$$

and

$$e^{\lambda t}\int_0^t \big(1-\beta+\frac{\beta}{1-p}(t-s)\big)\big(\beta f(s)+(1-\beta)f'(s)\big)e^{-\lambda s}ds,$$

are (i)-differentiable. Therefore, from Lemma 2.23, $g(t)$ is (i)-differentiable. It is easy to check that $Core\big(g(t)\odot g'(t)\big)=\big(g(t)\big)_1.\big(g'(t)\big)_1\geq 0$. Then from Lemma 2.25, $(a-a_c)\odot g(t)$ is (i)-differentiable. Consequently, y_{1N} is (i)-differentiable and we have

$$y'_{1N}(t)=\frac{1}{1-p}\Big(\lambda e^{\lambda t}\int_0^t \big(\beta f(s)+(1-\beta)f'(s)\big)e^{-\lambda s}ds+\beta f(t)+(1-\beta)f'(t)\Big)+(a-a_c)\odot g'(t).$$

Utilizing Definition 3.1 and Lemma 2.3, we have

$$\begin{aligned}
{}^{CF}D_*^{\beta}y_{1N}(t)&=\frac{1}{1-\beta}\int_0^t \exp\big(\frac{-\beta}{1-\beta}(t-s)\big)y'_{1N}(s)ds\\
&=\frac{a_c}{1-p}e^{\lambda t}\int_0^t \big(\beta f(s)+(1-\beta)f'(s)\big)e^{-\lambda s}ds+\big(f(t)\ominus e^{\frac{-\beta t}{1-\beta}}f(0)\big)\\
&+\Big(\frac{e^{\lambda t}}{(1-p)^2}\Big(\int_0^t \big(1+\frac{q}{1-p}(t-s)\big)\big(\beta f(s)+(1-\beta)f'(s)\big)e^{-\lambda s}ds\Big)\Big)\odot(a-a_c)\\
&=\Big(\frac{e^{\lambda t}}{1-p}\int_0^t \big(\beta f(s)+(1-\beta)f'(s)\big)e^{-\lambda s}ds\Big)\odot a+\big(f(t)\ominus e^{\frac{-\beta t}{1-\beta}}f(0)\big)\\
&+\frac{e^{\lambda t}}{(1-p)^2}\Big(\int_0^t \big(p+\frac{q}{1-p}(t-s)\big)\big(\beta f(s)+(1-\beta)f'(s)\big)e^{-\lambda s}ds\Big)\odot(a-a_c)\\
&=\Big(\frac{e^{\lambda t}}{1-p}\int_0^t \big(\beta f(s)+(1-\beta)f'(s)\big)e^{-\lambda s}ds\Big)\odot a+f(t)\\
&+\frac{e^{\lambda t}}{(1-p)^2}\Big(\int_0^t \big(p+\frac{q}{1-p}(t-s)\big)\big(\beta f(s)+(1-\beta)f'(s)\big)e^{-\lambda s}ds\Big)\odot(a-a_c).
\end{aligned}\tag{4.11}$$

It follows from Eq. (4.10) and (4.11) that y_{1N} satisfies Eq. (4.5) and we have

$${}^{CF}D_*^{\beta}y_{1N}(t)=a\odot y_{1N}(t)+f(t).$$

Consequently, using the above facts, y_{1N} is (i)-solution of Problem (4.5) for $t\in(0,T)$. □

For the convenience of the readers, we need some notations which are used in the following theorem.

(N_1) For $t\in(0,T)$, $g(t)$ is (i)-differentiable and either the H-differences

$$\lambda e^{\lambda t}\int_0^t (\beta f(s)+(1-\beta)f'(s))e^{-\lambda s}ds\ominus(-1)\big(\beta f(t)+(1-\beta)f'(t)\big),$$

or

$$(a-a_c)\odot g'(t)\ominus\frac{1}{p-1}\Big(\big(\beta f(t)+(1-\beta)f'(t)\big)\ominus(-\lambda)e^{\lambda t}\int_0^t \big(\beta f(s)+(1-\beta)f'(s)\big)e^{-\lambda s}ds\Big),$$

exist.

(N_2) For $t\in(0,T)$, $g(t)$ is (ii)-differentiable and the H-differences

$$(a-a)\odot g'(t)\ominus a\odot g'(t),$$

and

$$\lambda e^{\lambda t}\int_0^t (\beta f(s)+(1-\beta)f'(s))e^{-\lambda s}ds\ominus(-1)\big(\beta f(t)+(1-\beta)f'(t)\big),$$

exist.

Theorem 4.6. *Let $f:J\to\mathbf{R}^+_{\mathfrak{F}}(\mathbf{R}^-_{\mathfrak{F}})$ be (i)-differentiable such that $f'\in L^1(J,\mathbf{R}_{\mathfrak{F}})$ and let $a\in\mathbf{R}^+_{\mathfrak{F}}$ and $p>1$. Consider*

$$y_{4N}(t)=\Big(e^{\lambda t}y_0+(a-a_c)\odot g(t)\Big)\ominus\frac{e^{\lambda t}}{p-1}\int_0^t \big(\beta f(s)+(1-\beta)f'(s)\big)e^{-\lambda s}ds,$$

provided the above H-difference exists. If one of conditions presented in Table 11 along with the condition either (N_1) or (N_2) satisfy, then y_{4N} is (i)-solution of Problem (4.5) for $t\in(0,T)$.

TABLE 11. The required conditions for Theorem 4.6.

Row	*condition on* $f_c(t)$
1	$f_c f_c' \geq 0$, (A_5)
2	$f_c f_c' \leq 0$, (A_2), (A_5)
3	$f_c f_c' \leq 0$, (A_1), (A_5)

Proof. Here, we just prove theorem under the conditions in Row 1 of Table 11. The other cases can be investigated in an analogous manner. Let f be (i)-differentiable, $f, f' \succeq 0$. Since $p > 1$, hence from condition $a \odot y_0 + f(0) = 0$, we have $y_0 = 0$

$$(y_{4N}(t))_1 = \frac{e^{\lambda t}}{1-p}\int_0^t \big(\beta f_c(s) + (1-\beta) f_c'(s)\big)e^{-\lambda s} ds \leq 0.$$

Therefore, $y_{4N}(t) \succeq 0$. By the results of Lemma 2.3 and condition $a \odot y_0 + f(0) = 0$, we can deduce

$$\begin{aligned} a \odot y_{4N}(t) + f(t) = \Bigg(& \frac{e^{\lambda t}}{(1-p)^2}\Big(\int_0^t \big(p + \frac{q}{1-p}(t-s)\big)\big(\beta f(s) + (1-\beta) f'(s)\big)e^{-\lambda s} ds\Big) \odot (a - a_c) \\ & \ominus \Big(\frac{e^{\lambda t}}{1-p}\int_0^t \big(\beta f(s) + (1-\beta) f'(s)\big)e^{-\lambda s} ds\Big) \odot a\Bigg) + f(t). \end{aligned} \tag{4.12}$$

Let $g(t)$ be (i)-differentiable. Since $f_c(t)$ satisfies Condition (A_5), we have $Core\big(g(t) \odot g'(t)\big) = \big(g(t)\big)_1.\big(g'(t)\big)_1 \geq 0$. Consequently, from Lemma 2.25, $(a - a_c) \odot g(t)$ is (i)-differentiable. It follows from Condition (N_1) that y_{2N} is (i)-differentiable and we have

$$y_{4N}'(t) = (a - a_c) \odot g'(t) \ominus \frac{1}{p-1}\Big(\big(\beta f(t) + (1-\beta) f'(t)\big) \ominus (-\lambda) e^{\lambda t}\int_0^t \big(\beta f(s) + (1-\beta) f'(s)\big)e^{-\lambda s} ds\Big),$$

or

$$y_{4N}'(t) = (a - a_c) \odot g'(t) + (-1)\Big(\lambda e^{\lambda t}\int_0^t (\beta f(s) + (1-\beta) f'(s))e^{-\lambda s} ds \ominus (-1)\big(\beta f(t) + (1-\beta) f'(t)\big)\Big).$$

From Definition 3.1 and Lemma 2.3, we have

$$\begin{aligned} {}^{CF}D_*^{\beta} y_{4N}(t) &= \frac{1}{1-\beta}\int_0^t \exp\big(\frac{-\beta}{1-\beta}(t-s)\big) y_{4N}'(s) ds \\ &= \Bigg(\Big(\frac{e^{\lambda t}}{(1-p)^2}\int_0^t \big(1 + \frac{q}{1-p}(t-s)\big)\big(\beta f(s) + (1-\beta) f'(s)\big)e^{-\lambda s} ds\Big) \odot (a - a_c) \\ &\quad + \big(f(t) \ominus e^{\frac{-\beta t}{1-\beta}} f(0)\big)\Bigg) \ominus \frac{a_c}{1-p} e^{\lambda t}\int_0^t \big(\beta f(s) + (1-\beta) f'(s)\big)e^{-\lambda s} ds \\ &= \Bigg(\frac{e^{\lambda t}}{(1-p)^2}\Big(\int_0^t \big(p + \frac{q}{1-p}(t-s)\big)\big(\beta f(s) + (1-\beta) f'(s)\big)e^{-\lambda s} ds\Big) \odot (a - a_c) + f(t)\Bigg) \\ &\quad \ominus \Big(\frac{e^{\lambda t}}{1-p}\int_0^t \big(\beta f(s) + (1-\beta) f'(s)\big)e^{-\lambda s} ds\Big) \odot a \\ &= \Bigg(\frac{e^{\lambda t}}{(1-p)^2}\Big(\int_0^t \big(p + \frac{q}{1-p}(t-s)\big)\big(\beta f(s) + (1-\beta) f'(s)\big)e^{-\lambda s} ds\Big) \odot (a - a_c) \\ &\quad \ominus \Big(\frac{e^{\lambda t}}{1-p}\int_0^t \big(\beta f(s) + (1-\beta) f'(s)\big)e^{-\lambda s} ds\Big) \odot a\Bigg) + f(t). \end{aligned} \tag{4.13}$$

From Eq. (4.12) and (4.13), we have

$${}^{CF}D_*^{\beta} y_{4N}(t) = a \odot y_{4N}(t) + f(t).$$

Consequently, y_{4N} is (i)-solution of Problem (4.5). □

Example 4.7. Consider the following initial value problem

$$
\begin{aligned}
&{}^{CF}D_*^{\frac{1}{2}}y(t) =< \frac{-1}{2}, 0, 1 > \odot y(t)+ < 1-e^t, \frac{1-e^t}{2}, e^t-1 >, \quad 0 \le t \le \frac{3}{2} \\
&y(0) =< -7, 0, 8 > .
\end{aligned}
\tag{4.14}
$$

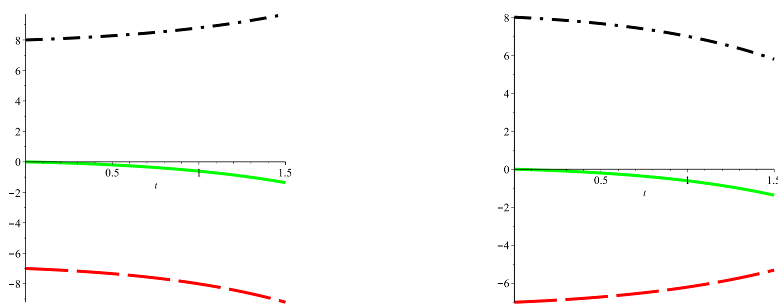
Since $f =< 1-e^t, \frac{1-e^t}{2}, e^t-1 >$ is (i)-differentiable, $p=0$, $f, f' \preceq 0$, and $y_0 \preceq 0$ for $t \in (0, \frac{3}{2})$. Therefore, from Theorem 4.4 under the conditions in Row 2 of Table 8, (i)-solution of Problem (4.14) is as

$$y_{1N} =< y_l(t), y_c(t), y_r(t) >=< -\frac{25}{4} + \frac{5t}{16} - \frac{t^2}{32} - \frac{3e^t}{4}, \frac{1}{2} + \frac{t}{4} - \frac{e^t}{2}, \frac{15}{2} - \frac{t}{8} + \frac{t^2}{16} + \frac{e^2}{2} > .$$

Also, from Theorem 4.4 under the conditions in Row 2 of Table 9, (ii)-solution of Problem (4.14) is as

$$y_{3N} =< y_l(t), y_c(t), y_r(t) >=< -\frac{15}{2} - \frac{t}{8} + \frac{t^2}{16} + \frac{e^t}{2}, \frac{1}{2} + \frac{t}{4} - \frac{e^t}{2}, \frac{35}{4} + \frac{5t}{16} - \frac{t^2}{32} - \frac{3e^t}{4} > .$$

Figure 3 illustrates the level sets of the (i) and (ii)-solutions of Problem (4.14) obtained under the conditions of Theorem 4.4.



(a) (b)

FIGURE 1. (a) The level sets of the (i)-solution of Problem (4.14). (b) The level sets of the (ii)-solution of Problem (4.14).

Example 4.8. Consider the following initial value problem

$$
\begin{aligned}
&{}^{CF}D_*^{\frac{1}{4}}y(t) =< -2, 1, 5 > \odot y(t)+ < -2t\sin(t), t, 2t\sin(t) >, \quad 0 \le t \le 1 \\
&y(0) = 0.
\end{aligned}
\tag{4.15}
$$

$f =< -2t\sin(t), t, 2t\sin(t) >$ is (i)-differentiable, $p = \frac{3}{4}$, $f, f' \succeq 0$ for $t \in (0,1)$. Therefore, from Theorem 4.5 under the conditions in Row 1 of Table 10, (i)-solution of Problem (4.15) is as

$$
\begin{aligned}
&y_{1N} =< y_l(t), y_c(t), y_r(t) > \\
&=< -3\Big((t - \frac{4\cos(t)+2\sin(t)}{3} + 8)e^{-t} + 16t - \frac{20}{3}\Big)e^t, 4e^t(1 - \frac{te^{-t}}{4} - e^{-t}), 4\Big((t - \cos(t) + \frac{\sin(t)}{2} + 8)e^{-t} + 16t - 7\Big)e^t > .
\end{aligned}
$$

Figure 2 illustrates the level sets of (i)-solutions of Problem (4.15) obtained under the conditions of Theorem 4.5.

Example 4.9. Consider the following initial value problem

$$
\begin{aligned}
&{}^{CF}D_*^{\frac{1}{2}}y(t) =< -7, 6, 10 > \odot y(t)+ < -t, t, 2t >, \quad 0 \le t \le 2 \\
&y(0) = 0.
\end{aligned}
\tag{4.16}
$$

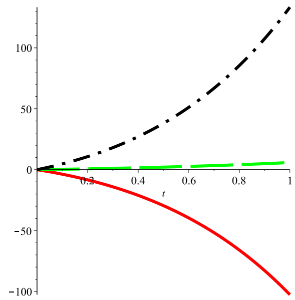


FIGURE 2. Solid line shows $y_l(t)$, long dash line shows $y_c(t)$, dash dot line shows $y_r(t)$ for Example 4.8.

$f =< -t, t, 2t >$ is (i)-differentiable, $p = 3$, $f, f' \succeq 0$ for $t \in (0, 2)$. Therefore, from Theorem 4.6 under the conditions in Row 1 of Table 10, (i)-solution of Problem (4.16) is as

$$\begin{aligned}
&y_{1N} =< y_l(t), y_c(t), y_r(t) > \\
&= \frac{e^{\frac{-3t}{2}}}{18} < \frac{-1}{24}\left(e^{\frac{3t}{2}}(300t + 152) + 39t - 152\right), -\left(e^{\frac{3t}{2}}(3t + 1) - 1\right), \frac{1}{6}\left(e^{\frac{3t}{2}}(30t + 14) + 3t - 14\right) > .
\end{aligned}$$

Figure 3 illustrates the level sets of (i)-solutions of Problem (4.16) obtained under the conditions of Theorem 4.6.

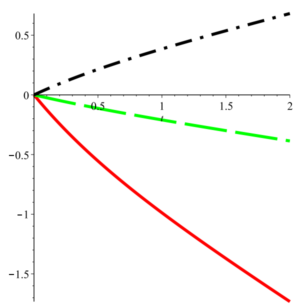


FIGURE 3. Solid line shows $y_l(t)$, long dash line shows $y_c(t)$, dash dot line shows $y_r(t)$ for Example 4.9.

Example 4.10. Consider the following initial value problem

$$\begin{aligned}
&{}^{CF}D_*^{\frac{2}{3}} y(t) = a \odot y(t) + f(t), \qquad 0 \le t \le 1 \qquad (4.17)\\
&y(0) =< -3, 0, 1 > .
\end{aligned}$$

(1) If $f = t + t^2$ and $a = 0$, then we have $p = 0$, $f, f' \ge 0$, and $y_c^0 = 0$ for $t \in (0, 1)$. Therefore, from Case 1 of Table 5, (i)-solution of Problem (4.17) is

$$y_{1N}(t) =< y_l(t), y_c(t), y_r(t) >= \frac{2}{9}t^3 + \frac{2}{3}t^2 + \frac{1}{3}t + < -3, 0, 1 > .$$

Since the H-difference appeared in y_{2N} does not exist, the (ii)-solution of Problem (4.17) does not exist.

(2) If $f =< -e^t, t + t^2, e^t + 2t >$ and $a = 0$, then f is (i)-differentiable, $p = 0$, $f, f' \succeq 0$, and $y_c^0 = 0$ for $t \in (0, 1)$. Therefore, from Theorem 4.4 under the conditions in Row 1 of Table 8, the (i)-solution of Problem (4.17) is as

$$y_{1N}(t) =< y_l(t), y_c(t), y_r(t) >=< -2 - e^t, \frac{2}{9}t^3 + \frac{2}{3}t^2 + \frac{1}{3}t, e^t + \frac{2}{9}t^3 + \frac{1}{3}t^2 > .$$

Since the H-difference appeared in y_{3N} does not exist, (ii)-solution of Problem (4.17) does not exist.

(3) If $f = t + t^2$ and $a =< -7, 0, 10 >$, then we have $p = 0$, $f, f' \geq 0$, and $y_c^0 = 0$ for $t \in (0,1)$. Therefore, from Case 1 of Table 5, (i)- solution of Problem (4.17) is

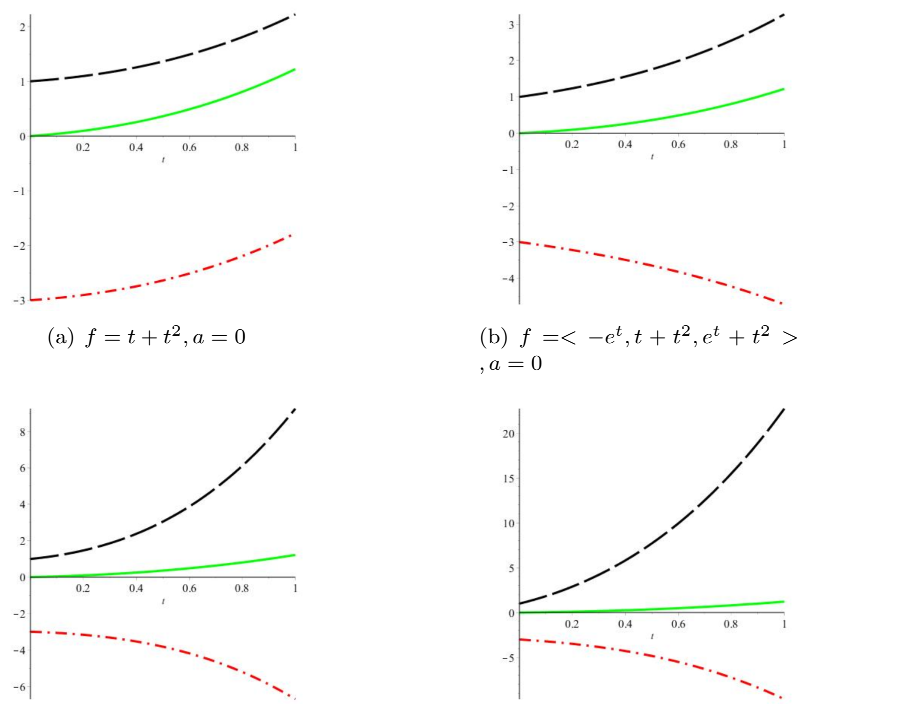
$$y_{1N}(t) =< y_l(t), y_c(t), y_r(t) >$$
$$=< -\frac{7}{27}t^4 - \frac{4}{3}t^3 - \frac{5}{3}t^2 - \frac{4}{9}t - 3, \frac{2}{9}t^3 + \frac{2}{3}t^2 + \frac{1}{3}t, \frac{10}{27}t^4 + \frac{22}{9}t^3 + 4t^2 + \frac{13}{9}t + 1 > .$$

Since the H-difference appeared in y_{2N} does not exist, the (ii)-solution of Problem (4.17) does not exist.

(4) If $f =< -e^t, t + t^2, e^t + 2t >$ and $a =< -7, 0, 10 >$, then f is (i)-differentiable, $p = 0$, $f, f' \succeq 0$, and $y_c^0 = 0$ for $t \in (0,1)$. Therefore, from Theorem 4.4 under the conditions in Row 1 of Table 8, (i)-solution of Problem (4.17) is

$$y_{1N}(t) =< y_l(t), y_c(t), y_r(t) >$$
$$=< -\frac{7}{27}t^4 - \frac{14}{9}t^3 - \frac{7}{3}t^2 - e^t - \frac{7}{9}t - 2, \frac{2}{9}t^3 + \frac{2}{3}t^2 + \frac{1}{3}t, -10 + 11e^t + \frac{46}{9}t^2 - \frac{34}{9}t + \frac{40}{27}t^3 > .$$

Since the H-difference appeared in y_{3N} does not exist, the (ii)-solution of Problem (4.17) does not exist.



(a) $f = t + t^2, a = 0$

(b) $f =< -e^t, t + t^2, e^t + t^2 >, a = 0$

(c) $f = t + t^2, a =< -7, 0, 10 >$

(d) $f =< -e^t, t + t^2, e^t + t^2 >, a =< -7, 0, 10 >$

FIGURE 4. Dash dot line shows $y_l(t)$, solid line shows $y_c(t)$, long dash line shows $y_r(t)$.

Figure 4 illustrates the solution of Problem (4.17) for various values of "a" and "f". In fact, the core of four solutions of Problem (4.17) in Cases 1-4 are the same. We compare the uncertainty of $y_{1N}(t)$ in four cases. The comparison between Figures 4(b) and 4(d) or 4(a) and 4(c) verifies that when "a" is a fuzzy number, diam$(y_{1N}(t))$ dramatically increases. However, comparing Figures 4(a) and 4(b) or 4(c) and 4(d) show that $diam\big(y_{1N}(t)\big)$ increases but it is less than previous state. It means that the effect of parameter "a" on uncertainty of the solution is more than "f".

5. Conclusion

In this paper, we presented analytical solutions for linear fully fuzzy Caputo-Fabrizio fractional differential equations with fuzzy coefficients. The cross product of fuzzy number was considered as a product operator between the fuzzy numbers. We investigated the explicit solutions of initial-value problems of linear Caputo-Fabrizio differential equations with fuzzy coefficients. We have explained some of the topics needed for the results of this paper related to the cross product of fuzzy numbers in details.

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