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Numerical methods for m-polar fuzzy initial value problems

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Abstract

Some problems in science and technology are modeled using ambiguous, imprecise, or lacking contextual data. In the modeling of some real-world problems, differential equations often involve multi-agent, multi-index, multi-objective, multi-attribute, multi-polar information or uncertainty, rather than single bits. These types of differentials are not well represented by fuzzy differential equations or bipolar fuzzy differential equations. Therefore, m-pole fuzzy set theory can be applied to differential equations to deal with problems with multi-polar information. In this paper, we study differential equations in m extremely fuzzy environments. We introduce the concept gH-derivative of m-polar fuzzy-valued function. By considering different types of differentiability, we propose some properties of the gH-differentiability of m-polar fuzzy-valued functions. We consider the m-polar fuzzy Taylor expansion. Using Taylor expansion, Euler's method and modified Euler's method for solving the m-polar fuzzy initial value problem are proposed. We discuss the convergence analysis of these methods. Some numerical examples are described to see the convergence and stability of the proposed method. We compare these methods by computing the global truncation error. From the numerical results, it can be seen that the modified Euler method converges to the exact solution faster than the Euler method.

Keywords. Generalized Hukuhara derivative, Taylor expansion, Initial value problem, Convergence analysis.
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1. Introduction

Differential equations are widely used in modeling physics and engineering problems, such as classical mechanics, thermodynamics, general relativity, and electrodynamics. Typically, the initial conditions are considered to be precisely defined in the model. Uncertain errors in observational, measurement, or experimental data may render the data unclear, incomplete, or inaccurate. To mitigate this uncertainty or inaccuracy, fuzzy differential equations (FDEs) can be used. Zadeh [44] introduced the concept of fuzzy sets to deal with uncertainty due to fuzziness or imprecision rather than randomness. Chang and Zadeh [18] introduced fuzzy numbers, which are special types of fuzzy sets that satisfy certain conditions. They also introduced fuzzy derivatives. Following the approach of [18], the extension principle has been used to define the fuzzy derivative [20]. There are many fuzzy DEs for which analytical solutions do not exist or are difficult to process analytically. For such differential equations, numerical programs are usually used. Abbasbandy and Allahviranloo [2] presented Taylor's method for solving ambiguous DE. Effati and Pakdaman [21] used a novel approach to solving neural network based fuzzy IVP. They replaced the DE equation with a DE system and split the solution into two parts, one part satisfying the initial conditions and the other part satisfying a feed-forward neural network with controllable parameters. Gisilov et al. [25] proposed a linear transformation based method to solve higher order linear differential equations with ambiguous initial values. They proposed a solution in the form of a fuzzy function that satisfies the initial value problem. Pederson and Sambandham [36] solved fuzzy DEs using the Runge-Kutta method. A method based on Euler and an improved Euler method introduced by Tapaswini

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and Chakraverty [41]. Fard and Gal [24] studied an iterative method to resolve a system of first order linear fuzzy DEs having fuzzy coefficients Parandin [35] applied RK-method to obtain numerical solution of n-th order fuzzy DEs. Tapaswini and Chakraverty [40] discussed the homotopy perturbation method and a method based on the Euler method for fuzzy DEs. Jayakumar et al. [30] investigated and applied fifth order RK-method. Palligkinis et al. [34] utilized RK-method to solve more general fuzzy DEs and discussed the convergence analysis for s-stage RK-method. Ghazanfari et al. [26] applied a fourth order R-Kutta like method has used for numerical solution of fuzzy DEs. Ivaz et al. [29] developed algorithms to approximate the solution of first order fuzzy DEs and hybrid fuzzy DEs. Rabiei et al. [27] applied improved RK-nystrom method for second order fuzzy DEs. Characterization theorem has been used by Pederson and Sambandham [37] to obtain the numerical solution of fuzzy DEs. Allahviranloo et al. [6] presented Adam-moulton, Adam-Bashforth, and a predictor-corrector method in which three step Adam-Bashforth is used as a predictor and the two-step Adam-Moulton method is used as a corrector. The stability and convergence conditions of proposed methods have been discussed. Bade and Gal[12] put forward the concept of strongly generalized differentiability and applied it to fuzzy DEs. Under this concept, the solution of fuzzy DE has decreasing length of its support. Cano and Flores [16] introduced the notion of generalized H-differentiability.

Sometimes our information has two poles, one satisfactory, and the other unsatisfactory. Coexistence, harmony and balance of both parties can be seen as a necessary condition for the intellectual and material health of human beings and the strength and success of social institutions. Euler's method for bipolar fuzzy initial value problems under generalized differentiability is discussed in [38].

There are several problems with multipolar information. Analysts assume a multipolar world. Therefore, it is not surprisingly, multipolar data and information play a vital role in different technological fields. Multipolar neurons in the brain collect numerous inputs from other neurons in neurobiology. Multipolar techniques can be used to manage large-scale applications of information technology. Real-world models often contain multiple attributes, multiple indicators, multiple objects, and multiple information. Chen et al. [19] proposed the concept of m-polar fuzzy set models by extending bipolar fuzzy set models. Akram et al. [4] considered m polar fuzzy linear systems. Akram [3] presented the concept and application of m-polar fuzzy graphs.

The rest of the paper is structured as follows: In section 2, we give some definition and basic results. Some properties of gH-differentiability are given in section 3. In section 4, we demonstrate Taylor's theorem for m-polar fuzzy-valued functions. In section 5, we describe the m-polar fuzzy initial value problem and some related theorems. We present Euler and Modified Euler method for m-polar fuzzy IVPs in section 6. The consistency, stability, and convergence of these methods have been discussed in section 7. In section 8, a few numerical examples have been presented. Lastly, we give some concluding remarks in section 9.

2. Preliminaries

Chen *et al.* [19] considered the notion of $m-$polar fuzzy sets. The grade of membership of $m-$polar fuzzy sets ranges over the interval $[0,1]^m$, and it represents m different properties of an object.

Definition 2.1. [4] The parametric form of m-polar fuzzy number (mPFN) $\mathcal{U}$ is a m-tuple $\prec [\underline{\mathcal{U}}^1(\delta_1),\overline{\mathcal{U}}^1(\delta_1)]$, $[\underline{\mathcal{U}}^2(\delta_2),\overline{\mathcal{U}}^2(\delta_2)],\cdots,[\underline{\mathcal{U}}^m(\delta_m),\overline{\mathcal{U}}^m(\delta_m)] \succ$ of the functions $\underline{\mathcal{U}}^i(\delta_i),\overline{\mathcal{U}}^i(\delta_i)$; $0 \le \delta_i \le 1$, $i = 1,2,\cdots,m$, which satisfy the following conditions:

(i) $\underline{\mathcal{U}}^i(\delta_i)$ is a non-decreasing, bounded, right continuous at 0 and left continuous function on the interval (0, 1].
(ii) $\overline{\mathcal{U}}^i(\delta_i)$ is a non-decreasing, bounded, right continuous at 0 and left continuous function on the interval (0, 1].
(iii) $\underline{\mathcal{U}}^i(\delta_i) \le \overline{\mathcal{U}}^i(\delta_i)$, $\forall\ i = 1,2,\cdots,m$.

Note: All over the paper, $i = 1,2,\cdots,m$.

Definition 2.2. [4] For any $\mathcal{U} = \prec [\underline{u}^i(\delta_i),\overline{\mathcal{U}}^i(\delta_i)] \succ$, $\mathcal{V} = \prec [\underline{\mathcal{V}}^i(\delta_i),\overline{\mathcal{V}}^i(\delta_i)] \succ$ and $c \in \mathbb{R}$; $0 \le \delta_i \le 1$, the addition and multiplication are laid out as;

(i) $(\underline{\mathcal{U}^i \oplus \mathcal{V}^i})(\delta_i) = \underline{\mathcal{U}}^i(\delta_i) + \underline{\mathcal{V}}^i(\delta_i)$, $(\overline{\mathcal{U}^i \oplus \mathcal{V}^i})(\delta_i) = \overline{\mathcal{U}}^i(\delta_i) + \overline{\mathcal{V}}^i(\delta_i)$,
(ii) $\underline{\mathcal{U}^i \odot \mathcal{V}^i}(\delta_i) = \min\{\underline{\mathcal{U}}^i(\delta_i)\ \underline{\mathcal{V}}^i(\delta_i),\underline{\mathcal{U}}^i(\delta_i)\ \overline{\mathcal{V}}^i(\delta_i),\overline{\mathcal{U}}^i(\delta_i)\ \underline{\mathcal{V}}^i(\delta_i),\overline{\mathcal{U}}^i(\delta_i)\ \overline{\mathcal{V}}^i(\delta_i)\}$,

(iii) $\overline{\mathcal{U}^i \odot \mathcal{V}^i}(\delta_i) = \max\{\underline{\mathcal{U}}^i(\delta_i)\ \underline{\mathcal{V}}^i(\delta_i), \underline{\mathcal{U}}^i(\delta_i)\ \overline{\mathcal{V}}^i(\delta_i), \overline{\mathcal{U}}^i(\delta_i)\ \underline{\mathcal{V}}^i(\delta_i), \overline{\mathcal{U}}^i(\delta_i)\ \overline{\mathcal{V}}^i(\delta_i)\}$,

(iv) For $c \geq 0$, $(\underline{c \odot \mathcal{U}^i})(\delta_i) = c(\underline{\mathcal{U}}^i)(\delta_i)$, $(\overline{c \odot \mathcal{U}}^i)(\delta_i) = c(\overline{\mathcal{U}}^i)(\delta_i)$,

(v) For $c < 0$, $(\underline{c \odot \mathcal{U}^i})(\delta_i) = c(\overline{\mathcal{U}}^i)(\delta_i)$, $(\overline{c \odot \mathcal{U}}^i)(\delta_i) = c(\underline{\mathcal{U}}^i)(\delta_i)$.

Note: $\mathfrak{X}$ denotes the space of mPFNs.

Definition 2.3. The generalized H-difference of two m-polar fuzzy numbers $\mathcal{U}, \mathcal{V} \in \mathfrak{X}$ is laid out as

$$\mathcal{U} \ominus_{gH} \mathcal{V} = \mathcal{W} \Leftrightarrow \begin{cases} (a)\ \mathcal{U} = \mathcal{V} \oplus \mathcal{W}, \\ (b)\ \mathcal{V} = \mathcal{U} \oplus (-1)\mathcal{W}. \end{cases}$$

$\mathcal{U} \ominus_{gH} \mathcal{V}$ exists if and only if $\mathcal{U}^i \ominus_{gH} \mathcal{V}^i$ exists, that is,

$$\mathcal{U}^i \ominus_{gH} \mathcal{V}^i = \mathcal{W}^i \Leftrightarrow \begin{cases} (a_1)\ \mathcal{U}^i = \mathcal{V}^i \oplus \mathcal{W}^i, \\ (b_1)\ \mathcal{V}^i = \mathcal{U}^i \oplus (-1)\mathcal{W}^i. \end{cases}$$

Definition 2.4. Let $\mathcal{U} = \prec [\underline{\mathcal{U}}^i(\delta_i), \overline{\mathcal{U}}^i(\delta_i)] \succ$ and $\mathcal{V} = \prec [\underline{\mathcal{V}}^i(\delta_i), \overline{\mathcal{V}}^i(\delta_i)] \succ$, $0 \leq \delta_i \leq 1$, we define metric $\mathfrak{D} : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+ \cup \{0\}$ as follows

$$\mathfrak{D}(\mathcal{U}, \mathcal{V}) = \max_{0 \leq \delta_i \leq 1} \{|\underline{\mathcal{U}}^i(\delta_i) - \underline{\mathcal{V}}^i(\delta_i)|, |\overline{\mathcal{U}}^i(\delta_i) - \overline{\mathcal{V}}^i(\delta_i)|\}.$$

That is, $\mathfrak{D}_1(\mathcal{U}^1, \mathcal{V}^1) = \max_{0 \leq \delta_1 \leq 1} \{|\underline{\mathcal{U}}^1(\delta_1) - \underline{\mathcal{V}}^1(\delta_1)|, |\overline{\mathcal{U}}^1(\delta_1) - \overline{\mathcal{V}}^1(\delta_1)|\}$,
$\mathfrak{D}_2(\mathcal{U}^2, \mathcal{V}^2) = \max_{0 \leq \delta_2 \leq 1} \{|\underline{\mathcal{U}}^2(\delta_2) - \underline{\mathcal{V}}^2(\delta_2)|, |\overline{\mathcal{U}}^2(\delta_2) - \overline{\mathcal{V}}^2(\delta_2)|\}$ and
$\mathfrak{D}_m(\mathcal{U}^m, \mathcal{V}^m) = \max_{0 \leq \delta_m \leq 1} \{|\underline{\mathcal{U}}^m(\delta_m) - \underline{\mathcal{V}}^m(\delta_m)|, |\overline{\mathcal{U}}^m(\delta_m) - \overline{\mathcal{V}}^m(\delta_m)|\}$.
Thus $\mathfrak{D}(\mathcal{U}, \mathcal{V}) = \max\{\mathfrak{D}_1, \mathfrak{D}_2, \cdots, \mathfrak{D}_m\}$.

Based on [43], we can prove that $\mathfrak{D}$ has the following properties
(i). $\mathfrak{D}(\mathcal{U} \bigoplus \mathcal{W}, \mathcal{V} \bigoplus \mathcal{W}) = \mathfrak{D}(\mathcal{U}, \mathcal{V})$, $\forall \mathcal{U}, \mathcal{V}, \mathcal{W} \in \mathfrak{X}$.
(ii). $\mathfrak{D}(\lambda \bigodot \mathcal{U}, \lambda \bigodot \mathcal{V}) = |\lambda| \mathfrak{D}(\mathcal{U}, \mathcal{V})$, $\forall \mathcal{U}, \mathcal{V} \in \mathfrak{X}$, $\lambda \in \mathbb{R}$.
(iii). $\mathfrak{D}(\mathcal{U} \bigoplus \mathcal{V}, \mathcal{W} \bigoplus \mathcal{E}) \leq \mathfrak{D}(\mathcal{U}, \mathcal{W}) + \mathfrak{D}(\mathcal{V}, \mathcal{E})$, $\forall \mathcal{U}, \mathcal{V}, \mathcal{W}, \mathcal{E} \in \mathfrak{X}$.
(iv). $\mathfrak{D}(\mathcal{U} \ominus \mathcal{V}, \mathcal{W} \ominus \mathcal{E}) \leq \mathfrak{D}(\mathcal{U}, \mathcal{W}) + \mathfrak{D}(\mathcal{V}, \mathcal{E})$, when $\mathcal{U} \ominus \mathcal{V}$ and $\mathcal{W} \ominus \mathcal{E}$ both exist, $\forall \mathcal{U}, \mathcal{V}, \mathcal{W}, \mathcal{E} \in \mathfrak{X}$.
Where $\ominus$ is H-difference, that is $\mathcal{U} \ominus \mathcal{V} = \mathcal{W}$ if and only if $\mathcal{U} = \mathcal{V} \oplus \mathcal{W}$.
(v). $(\mathfrak{X}, \mathfrak{D})$ is a complete metric space.

Definition 2.5. [39] Let $\mathcal{F} : \mathcal{J} \subset \mathfrak{R} \to \mathfrak{X}$ and $\mathfrak{t}_0 \in \mathcal{J}$, then $\mathcal{F}$ is generalized Hukuhara differentiable (GHD) at $\mathfrak{t}_0 \in \mathcal{J}$ if there is $\mathcal{F}'(\mathfrak{t}_0) \in \mathfrak{X}$ such that $\lim_{k \to 0^+} \frac{\mathcal{F}(\mathfrak{t}_0 + k) \ominus_{gH} \mathcal{F}(\mathfrak{t}_0)}{k}$ exists and is equal to $\mathcal{F}'(\mathfrak{t}_0)$.

Definition 2.6. An m-polar FVF $\mathcal{F} : \mathcal{J} \subset \mathfrak{R} \to \mathfrak{X}$ is said to be

(i) 1-GHD if for $k > 0$, the differences $\mathcal{F}^i(\mathfrak{t}_0 + k) \ominus_{gH} \mathcal{F}^i(\mathfrak{t}_0)$ exist as case (a_1) of Definition 2.3, for all $i = 1, 2, \cdots, m$, and limits
$\lim_{k \to 0^+} \frac{\mathcal{F}^i(\mathfrak{t}_0 + k) \ominus_{gH} \mathcal{F}^i(\mathfrak{t}_0)}{k}$ exist and equal to $\mathcal{F}'^i_{1-GH}(\mathfrak{t}_0)$.

(ii) 2-GHD if for $k > 0$, the differences $\mathcal{F}^i(\mathfrak{t}_0 + k) \ominus_{gH} \mathcal{F}^i(\mathfrak{t}_0)$ exist as case (b_1) of Definition 2.3, for all $i = 1, 2, \cdots, m$, and limits
$\lim_{k \to 0^+} \frac{\mathcal{F}^i(\mathfrak{t}_0 + k) \ominus_{gH} \mathcal{F}^i(\mathfrak{t}_0)}{k}$ exist and equal to $\mathcal{F}'^i_{2-GH}(\mathfrak{t}_0)$.

(iii) (1,2)-GHD if for $k > 0$, the differences $\mathcal{F}^i(\mathfrak{t}_0 + k) \ominus_{gH} \mathcal{F}^i(\mathfrak{t}_0)$ and $\mathcal{F}^j(\mathfrak{t}_0 + k) \ominus_{gH} \mathcal{F}^j(\mathfrak{t}_0)$ exist as case (a_1) and (b_1) of Definition 2.3, respectively, for all $i, j = 1, 2, \cdots, m : i \neq j$, and limits $\lim_{k \to 0^+} \frac{\mathcal{F}^i(\mathfrak{t}_0 + k) \ominus_{gH} \mathcal{F}^i(\mathfrak{t}_0)}{k}$ and $\lim_{k \to 0^+} \frac{\mathcal{F}^j(\mathfrak{t}_0 + k) \ominus_{gH} \mathcal{F}^j(\mathfrak{t}_0)}{k}$ exist and equal to $\mathcal{F}'^i_{1-GH}(\mathfrak{t}_0)$ and $\mathcal{F}'^j_{2-GH}(\mathfrak{t}_0)$, respectively.

Theorem 2.7. Let $\mathcal{F}:\mathcal{J}\subset\mathfrak{R}\to\mathfrak{X}$ and

$$[\mathcal{F}(t)]^{\delta}=\prec[\underline{\mathcal{F}}^{i}(t;\delta_i),\overline{\mathcal{F}}^{i}(t;\delta_i)]\succ,\ \delta_i\in[0,1],\ i=1,2,\cdots,m.$$

(1) If $\mathcal{F}$ is 1-GHD then $\underline{\mathcal{F}}^{i}_{\delta_i},\ \overline{\mathcal{F}}^{i}_{\delta_i},\ i=1,2,\cdots,m$, all are differentiable and $\underline{\mathcal{F}'}^{i}(t;\delta_i)\le\overline{\mathcal{F}'}^{i}(t;\delta_i)$, and

$$[\mathcal{F}'(t)]^{\delta}=\prec[\underline{\mathcal{F}'}^{i}(t;\delta_i),\overline{\mathcal{F}'}^{i}(t;\delta_i)]\succ. \tag{2.1}$$

(2) If $\mathcal{F}$ is 2-GHD then $\underline{\mathcal{F}}^{i}_{\delta_i},\ \overline{\mathcal{F}}^{i}_{\delta_i},\ i=1,2,\cdots,m$, all are differentiable and $\overline{\mathcal{F}'}^{i}(t;\delta_i)\le\underline{\mathcal{F}'}^{i}(t;\delta_i)$, and

$$[\mathcal{F}'(t)]^{\delta}=\prec[\overline{\mathcal{F}'}^{i}(t;\delta_i),\underline{\mathcal{F}'}^{i}(t;\delta_i)]\succ. \tag{2.2}$$

(3) If $\mathcal{F}$ is (1,2)-GHD then $\underline{\mathcal{F}}^{i}_{\delta_i},\ \overline{\mathcal{F}}^{i}_{\delta_i},\underline{\mathcal{F}}^{j}_{\delta_j},\ \overline{\mathcal{F}}^{j}_{\delta_j},\ i,j=1,2,\cdots,m,i\neq j$, all are differentiable and $\underline{\mathcal{F}'}^{i}(t;\delta_i)\le\overline{\mathcal{F}'}^{i}(t;\delta_i),\ \overline{\mathcal{F}'}^{j}(t;\delta_j)\le\underline{\mathcal{F}'}^{j}(t;\delta_j)$ and

$$[\mathcal{F}'(t)]^{\delta}=\prec[\underline{\mathcal{F}'}^{i}(t;\delta_i),\overline{\mathcal{F}'}^{i}(t;\delta_i)],[\overline{\mathcal{F}'}^{j}(t;\delta_j),\underline{\mathcal{F}'}^{j}(t;\delta_j)]\succ, \tag{2.3}$$

$$[\mathcal{F}'(t)]^{\delta}=\prec[\overline{\mathcal{F}'}^{i}(t;\delta_i),\underline{\mathcal{F}'}^{i}(t;\delta_i)]\succ.$$

Definition 2.8. A point $t_0\in J$ is called a switching point for differentiability of the function $\mathcal{F}$ if in any neighborhood $\mathcal{U}$ of t_0, there are points $t_1<t_0<t_2$ such that
Type (I) At $t=t_1$, it is 1-GHD and is not 2-GHD and at $t=t_2$ it is 2-GHD and is not 1-GHD, or
Type (II) At $t=t_1$, it is 2-GHD and is not 1-GHD and at $t=t_2$, it is 1-GHD and is not 2-GHD, or
Type (III) At $t=t_1$, it is 1-GHD and is not (1,2)-GHD and at $t=t_2$, it is (1,2)-GHD and is not 1-GHD, or
Type (IV) At $t=t_1$ it is (1,2)-GHD and is not 1-GHD and at $t=t_2$, it is 1-GHD and is not (1,2)-GHD, or
Type (V) At $t=t_1$, it is 2-GHD and is not (1,2)-GHD and at $t=t_2$, it is (1,2)-GHD and is not 2-GHD, or
Type (VI) At $t=t_1$, it is (1,2)-GHD and is not 2-GHD and at $t=t_2$, it is 2-GHD and is not (1,2)-GHD.

Definition 2.9. Let $\mathcal{F}:(\mathfrak{c},\mathfrak{d})\to\mathfrak{X}$ and $\mathcal{F}(t)$ is GHD of order i, $i=1,2,\cdots,n-1$ at t_0 which has not any switching point over the interval $[\mathfrak{c},\mathfrak{d}]$, then $\mathcal{F}(t)$ is GHD of order n at t_0 if $\mathcal{F}^{(n)}(t_0)\in\mathfrak{X}$ and

$$\mathcal{F}^{(n)}(t_0)=\lim_{k\to0}\frac{\mathcal{F}^{(n-1)}(t_0+k)\ominus_{gH}\mathcal{F}^{(n-1)}(t_0)}{k}.$$

Throughout the rest of the paper, we represent $\mathcal{C}_{\mathcal{F}}([\mathfrak{c},\mathfrak{d}],\mathfrak{X})$, the set of all continuous m-polar FVFs in the interior of $[\mathfrak{c},\mathfrak{d}]$ and it is one-sided continuous at end-points $\mathfrak{c}$ and $\mathfrak{d}$. Let $\mathcal{C}^{k}_{gH}([\mathfrak{c},\mathfrak{d}],\mathfrak{X}),\ k\in\mathbb{N}$ denote the space of functions $\mathcal{F}$, such that $\mathcal{F}$ and its first k, gH-derivatives are in $\mathcal{C}_{\mathcal{F}}([\mathfrak{c},\mathfrak{d}],\mathfrak{X})$.

Lemma 2.10. Let $\mathcal{F}\in\mathcal{C}_{\mathcal{F}}([\mathfrak{c},\mathfrak{d}],\mathfrak{X}),\ k\in\mathbb{N}$, then the integrals

$$\int_{\mathfrak{c}}^{\mathfrak{u}_{k-1}}\mathcal{F}(\mathfrak{u}_k)d\mathfrak{u}_k,\ \int_{\mathfrak{c}}^{\mathfrak{u}_{k-2}}(\int_{\mathfrak{c}}^{\mathfrak{u}_{k-1}}\mathcal{F}(\mathfrak{u}_k)d\mathfrak{u}_k)d\mathfrak{u}_{k-1},\cdots,\int_{\mathfrak{c}}^{\mathfrak{u}}(\int_{\mathfrak{c}}^{\mathfrak{u}_1}\cdots(\int_{\mathfrak{c}}^{\mathfrak{u}_{k-2}}(\int_{\mathfrak{c}}^{\mathfrak{u}_{k-1}}\mathcal{F}(\mathfrak{u}_k)d\mathfrak{u}_k)d\mathfrak{u}_{k-1})\cdots)d\mathfrak{u}_1,$$

are continuous functions in $\mathfrak{u}_{k-1},\mathfrak{u}_{k-2},\cdots,\mathfrak{u}$, respectively. Here $\mathfrak{u}_{k-1},\mathfrak{u}_{k-2},\cdots,\mathfrak{u}\ge\mathfrak{c}$ all are real numbers.

3. Some Results of gH-differentiability

We present some properties of m-polar fuzzy Hukuhara gH-differentiability. We give some theorems by extending some results of [9]

Theorem 3.1. Let $\mathcal{F}:[\mathfrak{c},\mathfrak{d}]\to\mathfrak{X}$ and $\mathcal{F}\in\mathcal{C}^{n}_{gH}([\mathfrak{c},\mathfrak{d}],\mathfrak{X})$, then for all $s\in[\mathfrak{c},\mathfrak{d}]$,

(i) Let $\mathcal{F}^{(k)}_{gH}$, $k=1,2,\cdots,n$ are 1-GHD which have same kind of differentiability in $[\mathfrak{c},\mathfrak{d}]$, then

$$\mathcal{F}^{(k-1)}_{1-GH}(\mathfrak{u})=\mathcal{F}^{(k-1)}_{1-GH}(\mathfrak{c})\oplus\int_{\mathfrak{c}}^{\mathfrak{u}}\mathcal{F}^{(k)}_{1-GH}(t)dt.$$

(ii) Let $\mathcal{F}^{(k)}_{gH}$, $k = 1, 2, \cdots, n$ are 2-GHD which have same kind of differentiability in $[\mathfrak{c}, \mathfrak{d}]$, then

$$\mathcal{F}^{(k-1)}_{2-GH}(\mathfrak{u}) = \mathcal{F}^{(k-1)}_{2-GH}(\mathfrak{c}) \oplus \int_{\mathfrak{c}}^{\mathfrak{u}} \mathcal{F}^{(\mathrm{k})}_{2-\mathrm{GH}}(\mathrm{t})\mathrm{d}t.$$

(iii) Let $\mathcal{F}^{(k)}_{gH}$, $k = 1, 2, \cdots, n$ are (1,2)-GHD which have same kind of differentiability in $[a, b]$, then

$$\mathcal{F}^{(k-1)}_{(1,2).gH}(\mathfrak{u}) = \mathcal{F}^{(k-1)}_{(1,2).gH}(\mathfrak{c}) \oplus \int_{\mathfrak{c}}^{\mathfrak{u}} \mathcal{F}^{(\mathrm{k})}_{(1,2).\mathrm{gH}}(\mathrm{t})\mathrm{d}t.$$

(iv) Let $\mathcal{F}^{(k)}_{gH}$, $k = 2m' - 1,\ m' \in \mathbb{N}$ are 1-GHD and $\mathcal{F}^{(k)}_{gH}$, $k = 2m',\ m' \in \mathbb{N}$ are 2-GHD, then

$$\mathcal{F}^{(k-1)}_{1-GH}(\mathfrak{u}) = \mathcal{F}^{(k-1)}_{1-GH}(\mathfrak{c}) \ominus (-1) \int_{\mathfrak{c}}^{\mathfrak{u}} \mathcal{F}^{(k)}_{2-GH}(t)\mathrm{d}t.$$

(v) Let $\mathcal{F}^{(k)}_{gH}$, $k = 2m' - 1,\ m' \in \mathbb{N}$ are 2-GHD and $\mathcal{F}^{(k)}_{gH}$, $k = 2m',\ m' \in \mathbb{N}$ are 1-GHD, then

$$\mathcal{F}^{(k-1)}_{2-GH}(\mathfrak{u}) = \mathcal{F}^{(k-1)}_{2-GH}(\mathfrak{c}) \ominus (-1) \int_{\mathfrak{c}}^{\mathfrak{u}} \mathcal{F}^{(k)}_{1-GH}(t)\mathrm{d}t.$$

(vi) Let $\mathcal{F}^{(k)}_{gH}$, $k = 2m' - 1,\ m' \in \mathbb{N}$ are 1-GHD and $\mathcal{F}^{(k)}_{gH}$, $k = 2m',\ m' \in \mathbb{N}$ are (1,2)-GHD, then

$$\begin{aligned}\mathcal{F}^{(k-1)}_{1-GH}(\mathfrak{u}) = \prec (\mathcal{F}^{i}_{1-GH})^{(k-1)}(\mathfrak{c}) \oplus \int_{\mathfrak{c}}^{\mathfrak{u}} (\mathcal{F}^{i}_{1-GH})^{(k)}(t)\mathrm{d}t,\\ (\mathcal{F}^{j}_{1-GH})^{(k-1)}(\mathfrak{c}) \ominus (-1) \int_{\mathfrak{c}}^{\mathfrak{u}} (\mathcal{F}^{j}_{2-GH})^{(k)}(t)\mathrm{d}t \succ,\ i, j = 1, 2, \cdots, m;\ i \neq j.\end{aligned}$$

(vii) Let $\mathcal{F}^{(k)}_{gH}$, $k = 2m' - 1,\ m' \in \mathbb{N}$ are 2-GHD and $\mathcal{F}^{(k)}_{gH}$, $k = 2m',\ m' \in \mathbb{N}$ are (1,2)-GHD, then

$$\begin{aligned}\mathcal{F}^{(k-1)}_{2-GH}(\mathfrak{u}) = \prec (\mathcal{F}^{i}_{2-GH})^{(k-1)}(\mathfrak{c}) \ominus (-1) \int_{\mathfrak{c}}^{\mathfrak{u}} (\mathcal{F}^{i}_{1-GH})^{(k)}(t)\mathrm{d}t,\\ (\mathcal{F}^{j}_{2-GH})^{(k-1)}(\mathfrak{c}) \oplus \int_{\mathfrak{c}}^{\mathfrak{u}} (\mathcal{F}^{j}_{2-GH})^{(k)}(t)\mathrm{d}t \succ,\ i, j = 1, 2, \cdots, m;\ i \neq j.\end{aligned}$$

Proof. Since, by assumption $\mathcal{F} \in \mathcal{C}^{n}_{gH}([\mathfrak{c}, \mathfrak{d}], \mathfrak{X})$, therefore $\mathcal{F}^{(k)}$, $k = 1, 2, \cdots, n$ are m-polar fuzzy Reimann integrable.
(i). As $\mathcal{F}^{(k)}$, $k = 1, 2, \cdots, n$ are 1-GHD, we have

$$\begin{aligned}\int_{\mathfrak{c}}^{\mathfrak{u}} \mathcal{F}^{(k)}_{1-GH}(t; \delta)\mathrm{d}t &= \prec [\int_{\mathfrak{c}}^{\mathfrak{u}} (\underline{\mathcal{F}}^{i}_{1-GH})^{(k)}(t; \delta_i)\mathrm{d}t, \int_{\mathfrak{c}}^{\mathfrak{u}} (\overline{\mathcal{F}}^{i}_{1-GH})^{(k)}(t; \delta_i)\mathrm{d}t] \succ\\ &= \prec [(\underline{\mathcal{F}}^{i}_{1-GH})^{(k-1)}(\mathfrak{u}; \delta_i) - (\underline{\mathcal{F}}^{i}_{1-GH})^{(k-1)}(\mathfrak{c}; \delta_i), (\overline{\mathcal{F}}^{i}_{1-GH})^{(k-1)}(\mathfrak{u}; \delta_i) - (\overline{\mathcal{F}}^{i}_{1-GH})^{(k-1)}(\mathfrak{c}; \delta_i)] \succ\\ &= \prec [(\underline{\mathcal{F}}^{i}_{1-GH})^{(k-1)}(\mathfrak{u}; \delta_i), (\overline{\mathcal{F}}^{i}_{1-GH})^{(k-1)}(\mathfrak{u}; \delta_i)] \succ \ominus \prec [(\underline{\mathcal{F}}^{i}_{1-GH})^{(k-1)}(\mathfrak{c}; \delta_i), (\overline{\mathcal{F}}^{i}_{1-GH})^{(k-1)}(\mathfrak{c}; \delta_i)] \succ\\ &= \mathcal{F}^{(k-1)}_{1-GH}(\mathfrak{u}; \delta) \ominus \mathcal{F}^{(k-1)}_{1-GH}(\mathfrak{c}; \delta).\end{aligned}$$

Thus,

$$\mathcal{F}^{(k-1)}_{1-GH}(\mathfrak{u}; \delta) = \mathcal{F}^{(k-1)}_{1-GH}(\mathfrak{c}; \delta) \oplus \int_{\mathfrak{c}}^{\mathfrak{u}} \mathcal{F}^{(k)}_{1-GH}(t; \delta)\mathrm{d}t.$$

Hence,

$$\mathcal{F}^{(k-1)}_{1-GH}(\mathfrak{u}) = \mathcal{F}^{(k-1)}_{1-GH}(\mathfrak{c}) \oplus \int_{\mathfrak{c}}^{\mathfrak{u}} \mathcal{F}^{(k)}_{1-GH}(t)\mathrm{d}t.$$

(ii). As $\mathcal{F}^{(k)},\ k=1,2,\cdots,n$ are 2-GHD, we have

$$\begin{aligned}\int_{\mathfrak{c}}^{\mathfrak{u}} \mathcal{F}^{(k)}_{2-GH}(t;\delta)\mathrm{d}t &= \prec [\int_{\mathfrak{c}}^{\mathfrak{u}} (\overline{\mathcal{F}}^{i}_{2-GH})^{(k)}(t;\delta_i)\mathrm{d}t, \int_{\mathfrak{c}}^{\mathfrak{u}} (\underline{\mathcal{F}}^{i}_{2-GH})^{(k)}(t;\delta_i)\mathrm{d}t] \succ \\ &= \prec [(\overline{\mathcal{F}}^{i}_{2-GH})^{(k-1)}(\mathfrak{u};\delta_i) - (\overline{\mathcal{F}}^{i}_{2-GH})^{(k-1)}(\mathfrak{c};\delta_i), (\underline{\mathcal{F}}^{i}_{2-GH})^{(k-1)}(\mathfrak{u};\delta_i) - (\underline{\mathcal{F}}^{i}_{2-GH})^{(k-1)}(\mathfrak{c};\delta_i)] \succ \\ &= \prec ([\overline{\mathcal{F}}^{i}_{2-GH})^{(k-1)}(\mathfrak{u};\delta_i), (\underline{\mathcal{F}}^{i}_{2-GH})^{(k-1)}(\mathfrak{u};\delta_i)] \succ \ominus \prec [(\overline{\mathcal{F}}^{i}_{2-GH})^{(k-1)}(\mathfrak{c};\delta_i), (\underline{\mathcal{F}}^{i}_{2-GH})^{(k-1)}(\mathfrak{c};\delta_i)] \succ \\ &= \mathcal{F}^{(k-1)}_{2-GH}(\mathfrak{u};\delta) \ominus \mathcal{F}^{(k-1)}_{2-GH}(\mathfrak{c};\delta).\end{aligned}$$

Thus,

$$\mathcal{F}^{(k-1)}_{2-GH}(\mathfrak{u};\delta) = \mathcal{F}^{(k-1)}_{2-GH}(\mathfrak{c};\delta) \oplus \int_{\mathfrak{c}}^{\mathfrak{u}} \mathcal{F}^{(k)}_{2-GH}(t;\delta)\mathrm{d}t.$$

Hence,

$$\mathcal{F}^{(k-1)}_{1-GH}(\mathfrak{u}) = \mathcal{F}^{(k-1)}_{1-GH}(\mathfrak{c}) \oplus \int_{\mathfrak{c}}^{\mathfrak{u}} \mathcal{F}^{(k)}_{1-GH}(t)\mathrm{d}t.$$

□

Theorem 3.2. Let $\mathcal{F} \in \mathcal{C}^4_{gH}\big([\mathfrak{c},\mathfrak{d}],\mathfrak{X}\big)$, then

(i) Let $\mathcal{F}^{(k)}_{gH}$, $k=1,2,3,4$ are 1-GHD which has same kind of differentiability in $[\mathfrak{c},\mathfrak{d}]$, then

$$\mathcal{F}''_{1-GH}(\mathfrak{c}) \approx \frac{\mathcal{F}'_{1-GH}(\mathfrak{u}) \ominus \mathcal{F}'_{1-GH}(\mathfrak{c})}{\mathfrak{u}-\mathfrak{c}}.$$

(ii) Let $\mathcal{F}^{(k)}_{gH}$, $k=1,2,3,4$ are 2-GHD which has same kind of differentiability in $[\mathfrak{c},\mathfrak{d}]$, then

$$\mathcal{F}''_{2-GH}(\mathfrak{c}) \approx \frac{\mathcal{F}'_{2-GH}(\mathfrak{u}) \ominus \mathcal{F}'_{2-GH}(\mathfrak{c})}{\mathfrak{u}-\mathfrak{c}}.$$

(iii) Let $\mathcal{F}^{(k)}_{gH}$, $i=1,2,3,4$ are (1,2)-GHD which has same kind of differentiability in $[\mathfrak{c},\mathfrak{d}]$, then

$$\mathcal{F}''_{(1,2)}(\mathfrak{c}) \approx \frac{\mathcal{F}'_{(1,2)}sv) \ominus \mathcal{F}'_{(1,2)}(\mathfrak{c})}{\mathfrak{u}-\mathfrak{c}}.$$

(iv) Let $\mathcal{F}^{(k)}_{gH}$, $k=1,3$ are 1-GHD and $\mathcal{F}^{(k)}_{gH}$, $k=0,2,4$ are 2-GHD, then

$$\mathcal{F}''_{1-GH}(\mathfrak{c}) \approx \frac{\mathcal{F}'_{2-GH}(\mathfrak{u}) \ominus \mathcal{F}'_{2-GH}(\mathfrak{c})}{\mathfrak{u}-\mathfrak{c}}.$$

(v) Let $\mathcal{F}^{(k)}_{gH}$, $k=1,3$ are 2-GHD and $\mathcal{F}^{(k)}_{gH}$, $k=0,2,4$ are 1-GHD, then

$$\mathcal{F}''_{2-GH}(\mathfrak{c}) \approx \frac{\mathcal{F}'_{1-GH}(\mathfrak{u}) \ominus \mathcal{F}'_{1-GH}(\mathfrak{c})}{\mathfrak{u}-\mathfrak{c}}.$$

Proof. AS $\mathcal{F} \in \mathcal{C}^4\big([\mathfrak{c},\mathfrak{d}],\mathfrak{X}\big)$, so $\mathcal{F}^{(k)}_{gH},\ k=0,1,2,3,4,$ are integrable.

(i). Let $\mathcal{F}^{(k)}_{gH}$ be 1-GHD; by Theorem 3.1, we have

$$\mathcal{F}'_{1-GH}(\mathfrak{u}) = \mathcal{F}'_{1-GH}(\mathfrak{c}) \oplus \int_{\mathfrak{c}}^{\mathfrak{u}} \mathcal{F}''_{1-\mathrm{GH}}(\mathfrak{u}_1)\mathrm{d}\mathfrak{u}_1,$$

and

$$\mathcal{F}''_{1-GH}(\mathfrak{u}_1) = \mathcal{F}''_{1-GH}(\mathfrak{c}) \oplus \int_{\mathfrak{c}}^{\mathfrak{u}_1} \mathcal{F}'''_{1-\mathrm{GH}}(\mathfrak{u}_2)\mathrm{d}\mathfrak{u}_2.$$

By apply integration, we have

$$\int_{\mathfrak{c}}^{\mathfrak{u}} \mathcal{F}''_{1-\mathrm{GH}}(\mathfrak{u}_1)d\mathfrak{u}_1 = \mathcal{F}''_{1-GH}(\mathfrak{c}) \odot (\mathfrak{u}-\mathfrak{c}) \oplus \int_{\mathfrak{c}}^{\mathfrak{u}} \big(\int_{\mathfrak{c}}^{\mathfrak{u}_1} \mathcal{F}'''_{1-\mathrm{GH}}(\mathfrak{u}_2)d\mathfrak{u}_2\big)d\mathfrak{u}_1,$$

where $\int_{\mathfrak{c}}^{\mathfrak{u}} \big(\int_{\mathfrak{c}}^{\mathfrak{u}_1} \mathcal{F}'''_{1-\mathrm{GH}}(\mathfrak{u}_2)d\mathfrak{u}_2\big)d\mathfrak{u}_1 \in \mathfrak{X}$. Thus

$$\mathcal{F}'_{1-GH}(\mathfrak{u}) = \mathcal{F}'_{1-GH}(\mathfrak{c}) \oplus \mathcal{F}''_{1-GH}(\mathfrak{c}) \odot (\mathfrak{u}-\mathfrak{c}) \oplus \int_{\mathfrak{c}}^{\mathfrak{u}} \big(\int_{\mathfrak{c}}^{\mathfrak{u}_1} \mathcal{F}'''_{1-\mathrm{GH}}(\mathfrak{u}_2)d\mathfrak{u}_2\big)d\mathfrak{u}_1.$$

Again, by Theorem 3.1, we have

$$\mathcal{F}'''_{1-GH}(\mathfrak{u}_2) = \mathcal{F}'''_{1-GH}(\mathfrak{c}) \oplus \int_{\mathfrak{c}}^{\mathfrak{u}_2} \mathcal{F}^{(\mathrm{iv})}_{(1-\mathrm{GH}}(\mathfrak{u}_3)d\mathfrak{u}_3.$$

Apply integration, we get

$$\int_{\mathfrak{c}}^{\mathfrak{u}_1} \mathcal{F}'''_{1-\mathrm{GH}}(\mathfrak{u}_2)d\mathfrak{u}_2 = \mathcal{F}'''_{1-GH}(\mathfrak{c}) \odot (\mathfrak{u}_1-\mathfrak{c}) \oplus \int_{\mathfrak{c}}^{\mathfrak{u}_1} \big(\int_{\mathfrak{c}}^{\mathfrak{u}_2} \mathcal{F}^{(\mathrm{iv})}_{1-\mathrm{GH}}(\mathfrak{u}_3)d\mathfrak{u}_3\big)d\mathfrak{u}_2,$$

furthermore

$$\int_{\mathfrak{c}}^{\mathfrak{u}} \big(\int_{\mathfrak{c}}^{\mathfrak{u}_1} \mathcal{F}'''_{1-\mathrm{GH}}(\mathfrak{u}_2)d\mathfrak{u}_2\big)d\mathfrak{u}_1 = \mathcal{F}'''_{1-GH}(\mathfrak{c}) \odot \frac{(\mathfrak{u}-\mathfrak{c})^2}{2} \oplus \int_{\mathfrak{c}}^{\mathfrak{u}} \big(\int_{\mathfrak{c}}^{\mathfrak{u}_1} \big(\int_{\mathfrak{c}}^{\mathfrak{u}_2} \mathcal{F}^{(\mathrm{iv})}_{1-\mathrm{GH}}(\mathfrak{u}_3)d\mathfrak{u}_3\big)d\mathfrak{u}_2\big)d\mathfrak{u}_1,$$

where $\int_{\mathfrak{c}}^{\mathfrak{u}} \big(\int_{\mathfrak{c}}^{\mathfrak{u}_1} \big(\int_{\mathfrak{c}}^{\mathfrak{u}_2} \mathcal{F}^{(\mathrm{iv})}_{1-\mathrm{GH}}(\mathfrak{u}_3)d\mathfrak{u}_3\big)d\mathfrak{u}_2\big)d\mathfrak{u}_1 \in \mathfrak{X}$.
Hence

$$\mathcal{F}'_{1-GH}(\mathfrak{u}) = \mathcal{F}'_{1-GH}(\mathfrak{c}) \oplus \mathcal{F}''_{1-GH}(\mathfrak{c}) \odot (\mathfrak{u}-\mathfrak{c}) \oplus \mathcal{F}'''_{1-GH}(\mathfrak{c}) \odot \frac{(\mathfrak{u}-\mathfrak{c})^2}{2} \oplus \int_{\mathfrak{c}}^{\mathfrak{u}} \big(\int_{\mathfrak{c}}^{\mathfrak{u}_1} \big(\int_{\mathfrak{c}}^{\mathfrak{u}_2} \mathcal{F}^{(\mathrm{iv})}_{1-\mathrm{GH}}(\mathfrak{u}_3)d\mathfrak{u}_3\big)d\mathfrak{u}_2\big)d\mathfrak{u}_1.$$

If $\mathfrak{u} \to \mathfrak{c}$, that is, $(\mathfrak{u}-\mathfrak{c}) \to 0$, then

$$\mathfrak{D}\big(\mathcal{F}'_{1-GH}(\mathfrak{u}), \mathcal{F}'_{1-GH}(\mathfrak{c}) \oplus \mathcal{F}''_{1-GH}(\mathfrak{c}) \odot (\mathfrak{u}-\mathfrak{c})\big) \to 0,$$

thus

$$\mathfrak{D}\big(\mathcal{F}'_{1-GH}(\mathfrak{u}) \ominus \mathcal{F}'_{1-GH}(\mathfrak{c}), \mathcal{F}''_{1-GH}(\mathfrak{c}) \odot (\mathfrak{u}-\mathfrak{c})\big) \to 0.$$

Hence

$$\mathcal{F}''_{1-GH}(\mathfrak{c}) \approx \frac{\mathcal{F}'_{1-GH}(\mathfrak{u}) \ominus \mathcal{F}'_{1-GH}(\mathfrak{c})}{\mathfrak{u}-\mathfrak{c}}.$$

The remaining parts of the theorem can be proved in similar manners. □

4. Taylor theorem for m-polar fuzzy valued function

We present Taylor expansion for m-polar FVFs for several cases by using gH-differentiability concept.

Theorem 4.1. Let $I = [\mathfrak{c}, \mathfrak{c}+h] \subseteq \mathfrak{R}$, $h > 0$ and $\mathcal{F} \in \mathcal{C}^n_{gH}([\mathfrak{c}, \mathfrak{d}], \mathfrak{X})$. For $\mathfrak{u} \in I$

(i) Let $\mathcal{F}^{(k)}_{gH}$, $k = 0, 1, \cdots, n-1$ are 1-GHD which have same kind of differentiability over $[\mathfrak{c}, \mathfrak{d}]$, then

$$\mathcal{F}(\mathfrak{u}) = \mathcal{F}(\mathfrak{c}) \oplus (\mathcal{F}_{1-GH})'(\mathfrak{c}) \odot (\mathfrak{u}-\mathfrak{c}) \oplus (\mathcal{F}_{1-GH})''(\mathfrak{c}) \odot \frac{(\mathfrak{u}-\mathfrak{c})^2}{2!} \oplus \cdots$$
$$\oplus (\mathcal{F}_{1-GH})^{(n-1)}(\mathfrak{c}) \odot \frac{(\mathfrak{u}-\mathfrak{c})^{n-1}}{(n-1)!} \oplus R_n(\mathfrak{c}, \mathfrak{u}),$$

where $R_n(\mathfrak{c}, \mathfrak{u}) = \int_{\mathfrak{c}}^{\mathfrak{u}}(\int_{\mathfrak{c}}^{\mathfrak{u}_1} \cdots (\int_{\mathfrak{c}}^{\mathfrak{u}_{k-2}}(\int_{\mathfrak{c}}^{\mathfrak{u}_{k-1}} (\mathcal{F}_{1-\mathrm{GH}})^{(\mathrm{n})}(\mathfrak{u}_{\mathrm{k}})d\mathfrak{u}_k)d\mathfrak{u}_{k-1}) \cdots)d\mathfrak{u}_1$.

(ii) Let $\mathcal{F}_{gH}^{(k)}$, $k=0,1,\cdots,n-1$ are 2-GHD which have same kind of differentiability over $[\mathfrak{c},\mathfrak{d}]$, then

$$\begin{aligned}\mathcal{F}(\mathfrak{u}) =&\mathcal{F}(\mathfrak{c}) \ominus (-1)(\mathcal{F}_{2-GH})'(\mathfrak{c}) \odot (\mathfrak{u}-\mathfrak{c}) \ominus (-1)(\mathcal{F}_{2-GH})''(\mathfrak{c}) \odot \frac{(\mathfrak{u}-\mathfrak{c})^2}{2!}\\ &\ominus (-1)\cdots \ominus (-1)(\mathcal{F}_{2-GH})^{(n-1)}(\mathfrak{c}) \odot \frac{(\mathfrak{u}-\mathfrak{c})^{n-1}}{(n-1)!} \ominus (-1)R_n(\mathfrak{c},\mathfrak{u}),\end{aligned}$$

where $R_n(\mathfrak{c},\mathfrak{u})=\int_{\mathfrak{c}}^{\mathfrak{u}}(\int_{\mathfrak{c}}^{\mathfrak{u}_1}\cdots(\int_{\mathfrak{c}}^{\mathfrak{u}_{k-2}}(\int_{\mathfrak{c}}^{\mathfrak{u}_{k-1}}(\mathcal{F}_{2-\mathrm{GH}})^{(\mathrm{n})}(\mathfrak{u}_\mathrm{k})d\mathfrak{u}_k)d\mathfrak{u}_{k-1})\cdots)d\mathfrak{u}_1$.

(iii) Let $\mathcal{F}_{gH}^{(k)}$, $k=0,1,\cdots,n-1$ are (1,2)-GHD which have same kind of differentiability over $[\mathfrak{c},\mathfrak{d}]$, then

$$\begin{aligned}\mathcal{F}(\mathfrak{u}) =&\prec\mathcal{F}^i(\mathfrak{c}) \oplus (\mathcal{F}^i_{1-GH})'(\mathfrak{c}) \odot (\mathfrak{u}-\mathfrak{c}) \oplus (\mathcal{F}^i_{1-GH})''(\mathfrak{c}) \odot \frac{(\mathfrak{u}-\mathfrak{c})^2}{2!} \oplus\cdots\\ &\oplus (\mathcal{F}^i_{1-GH})^{(n-1)}(\mathfrak{c}) \odot \frac{(\mathfrak{u}-\mathfrak{c})^{n-1}}{(n-1)!} \oplus R^i_n(\mathfrak{c},\mathfrak{u}),\\ &\mathcal{F}^j(\mathfrak{c}) \ominus (-1)(\mathcal{F}^j_{2-GH})'(\mathfrak{c}) \odot (\mathfrak{u}-\mathfrak{c}) \ominus (-1)(\mathcal{F}^j_{2-GH})''(\mathfrak{c}) \odot \frac{(\mathfrak{u}-\mathfrak{c})^2}{2!}\\ &\ominus(-1)\cdots \ominus (-1)(\mathcal{F}^j_{2-GH})^{(n-1)}(\mathfrak{c}) \odot \frac{(\mathfrak{u}-\mathfrak{c})^{n-1}}{(n-1)!} \ominus (-1)R^j_n(\mathfrak{c},\mathfrak{u}) \succ\\ &i,j=1,2,\cdots,m;\ i\neq j,\end{aligned}$$

where $R^i_n(\mathfrak{c},\mathfrak{u})=\int_{\mathfrak{c}}^{\mathfrak{u}}(\int_{\mathfrak{c}}^{\mathfrak{u}_1}\cdots(\int_{\mathfrak{c}}^{\mathfrak{u}_{k-2}}(\int_{\mathfrak{c}}^{\mathfrak{u}_{k-1}}(\mathcal{F}_{(1-\mathrm{GH})})^{(\mathrm{n})}(\mathfrak{u}_\mathrm{k})d\mathfrak{u}_k)d\mathfrak{u}_{k-1})\cdots)d\mathfrak{u}_1$ and
$R^j_n(\mathfrak{c},\mathfrak{u})=\int_{\mathfrak{c}}^{\mathfrak{u}}(\int_{\mathfrak{c}}^{\mathfrak{u}_1}\cdots(\int_{\mathfrak{c}}^{\mathfrak{u}_{k-2}}(\int_{\mathfrak{c}}^{\mathfrak{u}_{k-1}}(\mathcal{F}_{2-\mathrm{GH}})^{(\mathrm{n})}(\mathfrak{u}_\mathrm{k})d\mathfrak{u}_k)d\mathfrak{u}_{k-1})\cdots)d\mathfrak{u}_1$.

(iv) Let $\mathcal{F}_{gH}^{(k)}$, $k=2m'-1,\ m'\in\mathbb{N}$ are 1-GHD and $\mathcal{F}_{gH}^{(k)}$, $k=2m',\ m'\in\mathbb{N}\cup\{0\}$ are 2-GHD, then

$$\begin{aligned}\mathcal{F}(\mathfrak{u}) =&\mathcal{F}(\mathfrak{c}) \ominus (-1)(\mathcal{F}_{2-GH})'(\mathfrak{c}) \odot (\mathfrak{u}-\mathfrak{c}) \oplus (\mathcal{F}_{2-GH})''(\mathfrak{c}) \odot \frac{(\mathfrak{u}-\mathfrak{c})^2}{2!}\\ &\ominus (-1)\cdots \ominus (-1)(\mathcal{F}_{2-GH})^{(\frac{k-1}{2})}(\mathfrak{c}) \odot \frac{(\mathfrak{c}-\mathfrak{u})^{\frac{k-1}{2}}}{(\frac{k-1}{2})!} \oplus (\mathcal{F}_{1-GH})^{(\frac{k}{2})}(\mathfrak{c})\\ &\odot \frac{(\mathfrak{c}-\mathfrak{u})^{\frac{k}{2}}}{(\frac{k}{2})!} \ominus (-1)R_n(\mathfrak{c},\mathfrak{u}),\end{aligned}$$

where $R_n(\mathfrak{c},\mathfrak{u})=\int_{\mathfrak{c}}^{\mathfrak{u}}(\int_{\mathfrak{c}}^{\mathfrak{u}_1}\cdots(\int_{\mathfrak{c}}^{\mathfrak{u}_{k-2}}(\int_{\mathfrak{c}}^{\mathfrak{u}_{k-1}}(\mathcal{F}_{1-\mathrm{GH}})^{(\mathrm{n})}(\mathfrak{u}_\mathrm{k})d\mathfrak{u}_k)d\mathfrak{u}_{k-1})\cdots)d\mathfrak{u}_1$.

(v) Let $\mathcal{F}_{gH}^{(k)}$, $k=2m'-1,\ m'\in\mathbb{N}$ are 1-GHD and $\mathcal{F}_{gH}^{(k)}$, $k=2m',\ m'\in\mathbb{N}\cup\{0\}$ are (1,2)-GHD, then

$$\begin{aligned}\mathcal{F}(\mathfrak{u}) =&\prec\mathcal{F}^i(\mathfrak{c}) \oplus (\mathcal{F}^i_{1-GH})'(\mathfrak{c}) \odot (\mathfrak{u}-\mathfrak{c}) \oplus (\mathcal{F}^i_{1-GH})''(\mathfrak{c}) \odot \frac{(\mathfrak{u}-\mathfrak{c})^2}{2!} \oplus\cdots\\ &\oplus (\mathcal{F}^i_{1-GH})^{(n-1)}(\mathfrak{c}) \odot \frac{(\mathfrak{u}-\mathfrak{c})^{n-1}}{(n-1)!} \oplus R^i_n(\mathfrak{c},\mathfrak{u}),\\ &\mathcal{F}^j(\mathfrak{c}) \ominus (-1)(\mathcal{F}^j_{2-GH})'(\mathfrak{c}) \odot (\mathfrak{u}-\mathfrak{c}) \oplus (\mathcal{F}^j_{1-GH})''(\mathfrak{c}) \odot \frac{(\mathfrak{u}-\mathfrak{c})^2}{2!}\\ &\ominus (-1)\cdots \ominus (-1)(\mathcal{F}^j_{2-GH})^{(\frac{i-1}{2})}(\mathfrak{c}) \odot \frac{(\mathfrak{c}-\mathfrak{u})^{\frac{i-1}{2}}}{(\frac{i-1}{2})!} \oplus (\mathcal{F}^j_{1-GH})^{(\frac{i}{2})}(\mathfrak{c})\\ &\odot \frac{(\mathfrak{c}-\mathfrak{u})^{\frac{i}{2}}}{(\frac{i}{2})!} \ominus (-1)R^j_n(\mathfrak{c},\mathfrak{u}) \succ, \quad i,j=1,2,\cdots,m;\ i\neq j,\end{aligned}$$

where $R^i_n(\mathfrak{c},\mathfrak{u})=\int_{a}^{\mathfrak{u}}(\int_{\mathfrak{c}}^{\mathfrak{u}_1}\cdots(\int_{\mathfrak{c}}^{\mathfrak{u}_{k-2}}(\int_{\mathfrak{c}}^{\mathfrak{u}_{k-1}}(\mathcal{F}^{\mathrm{i}}_{(1-\mathrm{GH})})^{(\mathrm{n})}(\mathfrak{u}_\mathrm{k})d\mathfrak{u}_k)d\mathfrak{u}_{k-1})\cdots)d\mathfrak{u}_1$
and
$R^j_n(\mathfrak{c},\mathfrak{u})=\int_{\mathfrak{c}}^{\mathfrak{u}}(\int_{\mathfrak{c}}^{\mathfrak{u}_1}\cdots(\int_{\mathfrak{c}}^{\mathfrak{u}_{k-2}}(\int_{\mathfrak{c}}^{\mathfrak{u}_{k-1}}(\mathcal{F}^{\mathrm{j}}_{1-\mathrm{GH}})^{(\mathrm{n})}(\mathfrak{u}_\mathrm{k})d\mathfrak{u}_k)d\mathfrak{u}_{k-1})\cdots)d\mathfrak{u}_1$.

(vi) Let $\mathcal{F}_{gH}^{(k)}$, $k=2m'-1,\ m'\in\mathbb{N}$ are 2-GHD and $\mathcal{F}_{gH}^{(k)}$, $k=2m',\ m'\in\mathbb{N}\cup\{0\}$ are (1,2)-GHD, then

$$\begin{aligned}\mathcal{F}(\mathfrak{u}) =\prec & \mathcal{F}^i(\mathfrak{c})\ominus(-1)(\mathcal{F}^i_{2-GH})'(\mathfrak{c})\odot(\mathfrak{u}-\mathfrak{c})\oplus(\mathcal{F}^i_{1-GH})''(\mathfrak{c})\\ & \odot\frac{(\mathfrak{u}-\mathfrak{c})^2}{2!}\ominus(-1)\cdots\ominus(-1)(\mathcal{F}^i_{2-GH})^{(\frac{k-1}{2})}(\mathfrak{c})\odot\\ & \frac{(\mathfrak{c}-\mathfrak{u})^{\frac{k-1}{2}}}{(\frac{k-1}{2})!}\oplus(\mathcal{F}^i_{1-GH})^{(\frac{k}{2})}(\mathfrak{c})\odot\frac{(\mathfrak{c}-\mathfrak{u})^{\frac{k}{2}}}{(\frac{k}{2})!}\ominus(-1)R^i_n(a,\mathfrak{u}),\\ & \mathcal{F}^j(\mathfrak{c})\ominus(-1)(\mathcal{F}^j_{2-GH})'(\mathfrak{u})\odot(\mathfrak{u}-\mathfrak{c})\ominus(-1)(\mathcal{F}^j_{2-GH})''(\mathfrak{u})\odot\frac{(\mathfrak{u}-\mathfrak{c})^2}{2!}\\ & \ominus(-1)\cdots\ominus(-1)(\mathcal{F}^j_{2-GH})^{(n-1)}(\mathfrak{u})\odot\frac{(\mathfrak{u}-\mathfrak{c})^{n-1}}{(n-1)!}\ominus(-1)R^j_n(\mathfrak{c},\mathfrak{u})\succ,\quad i,j=1,2,\cdots,m;\ i\neq j,\end{aligned}$$

where $R^i_n(\mathfrak{c},\mathfrak{u})=\int_{\mathfrak{c}}^{\mathfrak{u}}(\int_{\mathfrak{c}}^{\mathfrak{u}_1}\cdots(\int_{\mathfrak{c}}^{\mathfrak{u}_{k-2}}(\int_{\mathfrak{c}}^{\mathfrak{u}_{k-1}}(\mathcal{F}^{\mathrm{i}}_{1-\mathrm{GH}})^{(\mathrm{n})}(\mathfrak{u}_\mathrm{k})d\mathfrak{u}_k)d\mathfrak{u}_{k-1})\cdots)d\mathfrak{u}_1$, and
$R^j_n(\mathfrak{c},\mathfrak{u})=\int_{\mathfrak{c}}^{\mathfrak{u}}(\int_{\mathfrak{c}}^{\mathfrak{u}_1}\cdots(\int_{\mathfrak{c}}^{\mathfrak{u}_{k-2}}(\int_{\mathfrak{c}}^{\mathfrak{u}_{k-1}}(\mathcal{F}^{\mathrm{j}}_{2-\mathrm{GH}})^{(\mathrm{n})}(\mathfrak{u}_\mathrm{k})d\mathfrak{u}_k)d\mathfrak{u}_{k-1})\cdots)d\mathfrak{u}_1$.

Proof. **(i)**. Since $\mathcal{F}$ is 1-GHD, by Theorem 2.7, we have

$$\mathcal{F}(\mathfrak{u})=\mathcal{F}(\mathfrak{c})\oplus\int_{\mathfrak{c}}^{\mathfrak{u}}(\mathcal{F}_{1-GH})'(\mathfrak{u}_1)d\mathfrak{u}_1,$$

by Theorem 3.1, we have

$$(\mathcal{F}_{1-GH})'(\mathfrak{u}_1)=(\mathcal{F}_{1-GH})'(\mathfrak{c})\oplus\int_{\mathfrak{c}}^{\mathfrak{u}_1}(\mathcal{F}_{1-GH})''(\mathfrak{u}_2)d\mathfrak{u}_2,$$

thus

$$\begin{aligned}\int_{\mathfrak{c}}^{\mathfrak{u}}(\mathcal{F}_{1-GH})'(\mathfrak{u}_1)d\mathfrak{u}_1 &=\int_{\mathfrak{c}}^{\mathfrak{u}}(\mathcal{F}_{1-GH})'(\mathfrak{c})d\mathfrak{u}_1\oplus\int_{\mathfrak{c}}^{\mathfrak{u}}(\int_{\mathfrak{c}}^{\mathfrak{u}_1}(\mathcal{F}_{1-GH})''(\mathfrak{u}_2)d\mathfrak{u}_2)d\mathfrak{u}_1\\ &=(\mathcal{F}_{1-GH})'(\mathfrak{c})\odot(\mathfrak{u}-\mathfrak{c})\oplus\int_{\mathfrak{c}}^{\mathfrak{u}}(\int_{\mathfrak{c}}^{\mathfrak{u}_1}(\mathcal{F}_{1-GH})''(\mathfrak{u}_2)d\mathfrak{u}_2)d\mathfrak{u}_1.\end{aligned}$$

Now by Lemma 2.10, $\int_{\mathfrak{c}}^{\mathfrak{u}}(\int_{\mathfrak{c}}^{\mathfrak{u}_1}(\mathcal{F}_{1-GH})''(\mathfrak{u}_2)d\mathfrak{u}_2)d\mathfrak{u}_1\in\mathfrak{X}$, so

$$\mathcal{F}(\mathfrak{u})=\mathcal{F}(\mathfrak{c})\oplus(\mathcal{F}_{1-GH})'(a)\odot(\mathfrak{u}-\mathfrak{c})\oplus\int_{\mathfrak{c}}^{\mathfrak{u}}(\int_{\mathfrak{c}}^{\mathfrak{u}_1}(\mathcal{F}_{1-GH})''(\mathfrak{u}_2)d\mathfrak{u}_2)d\mathfrak{u}_1.$$

Again, using Theorem 3.1, we have

$$(\mathcal{F}_{1-GH})''(\mathfrak{u}_2)=(\mathcal{F}_{1-GH})''(\mathfrak{c})\oplus\int_{\mathfrak{c}}^{\mathfrak{u}_2}(\mathcal{F}_{1-GH})'''(\mathfrak{u}_3)d\mathfrak{u}_3,$$

Applying m-polar fuzzy Riemann operator, we have

$$\begin{aligned}\int_{\mathfrak{c}}^{\mathfrak{u}_1}(\mathcal{F}_{1-GH})''(\mathfrak{u}_2)d\mathfrak{u}_2 &=\int_{\mathfrak{c}}^{\mathfrak{u}_1}(\mathcal{F}_{1-GH})''(a)d\mathfrak{u}_2\oplus\int_{\mathfrak{c}}^{\mathfrak{u}_1}(\int_{\mathfrak{c}}^{\mathfrak{u}_2}(\mathcal{F}_{1-GH})'''(\mathfrak{u}_3)d\mathfrak{u}_3)d\mathfrak{u}_2\\ &=(\mathcal{F}_{1-GH})''(a)\odot(\mathfrak{u}_1-a)\oplus\int_{\mathfrak{c}}^{\mathfrak{u}_1}(\int_{\mathfrak{c}}^{\mathfrak{u}_2}(\mathcal{F}_{1-GH})'''(\mathfrak{u}_3)d\mathfrak{u}_3)d\mathfrak{u}_2,\end{aligned}$$

furthermore

$$\int_{\mathfrak{c}}^{\mathfrak{u}}(\int_{\mathfrak{c}}^{\mathfrak{u}_1}(\mathcal{F}_{1-GH})''(\mathfrak{u}_2)d\mathfrak{u}_2)d\mathfrak{u}_1=(\mathcal{F}_{1-GH})''(\mathfrak{c})\odot\int_{\mathfrak{c}}^{\mathfrak{u}}(\mathfrak{u}_1-\mathfrak{c})d\mathfrak{u}_1\oplus\int_{\mathfrak{c}}^{\mathfrak{u}}(\int_{\mathfrak{c}}^{\mathfrak{u}_1}(\int_{\mathfrak{c}}^{\mathfrak{u}_2}(\mathcal{F}_{1-GH})'''(\mathfrak{u}_3)d\mathfrak{u}_3)d\mathfrak{u}_2)d\mathfrak{u}_1.$$

By Lemma 2.10, $\int_{\mathfrak{c}}^{\mathfrak{u}}(\int_{\mathfrak{c}}^{\mathfrak{u}_1}(\int_{\mathfrak{c}}^{\mathfrak{u}_2}(\mathcal{F}_{1-GH})'''(\mathfrak{u}_3)d\mathfrak{u}_3)d\mathfrak{u}_2)d\mathfrak{u}_1\in\mathfrak{X}$.
Thus

$$\mathcal{F}(\mathfrak{u})=\mathcal{F}(\mathfrak{c})\oplus(\mathcal{F}_{1-GH})'(\mathfrak{c})\odot(\mathfrak{u}-\mathfrak{c})\oplus(\mathcal{F}_{1-GH})''(\mathfrak{c})\odot\frac{(\mathfrak{u}-\mathfrak{c})^2}{2!}\oplus\int_{\mathfrak{c}}^{\mathfrak{u}}(\int_{\mathfrak{c}}^{\mathfrak{u}_1}(\int_{\mathfrak{c}}^{\mathfrak{u}_1}(\mathcal{F}_{1-GH})'''(\mathfrak{u}_2)d\mathfrak{u}_3)d\mathfrak{u}_2)d\mathfrak{u}_1.$$

□

5. m-polar Fuzzy Cauchy Problem

Following the idea of Ma et. al [32], an m-polar fuzzy cauchy problem is defined as follows:

$$x'_{gH}(t) = \mathcal{F}(t, x(t));\ x(t_0) = x_0 \text{ and } t \in \mathcal{J} = [\mathfrak{c}, \mathfrak{d}], \tag{5.1}$$

where $x_0 \in \mathfrak{X},\ x : \mathcal{J} \to \mathfrak{X},$ and $\mathcal{F} : \mathcal{J} \times \mathfrak{X} \to \mathfrak{X}$.

Lemma 5.1. (i). A mapping $x : \mathcal{J} \to \mathfrak{X}$ is a (1)-solution of (5.1) iff it is continuous and satisfies

$$x(t) = y_0 \oplus \int_{t_0}^{t} \mathcal{F}(u, x(u))du;\ y(t_0) = x_0.$$

(ii). A mapping $x : J \to \mathfrak{X}$ is a (1,2)-solution of (5.1) iff it is continuous and satisfies

$$x^i(t) = x_0^i \oplus \int_{t_0}^{t} \mathcal{F}^i(u, x^i(u))du;\ x^i(t_0) = x_0^i,$$

$$x^j(t) = x_0^j \ominus (-1) \int_{t_0}^{t} \mathcal{F}^j(u, x^j(u))du;\ x^j(t_0) = x_0^j,\ i, j = 1, 2, \cdots, m; i \neq j.$$

(iii). A mapping $x : J \to \mathfrak{X}$ is a (2)-solution of (5.1) iff it is continuous and satisfies

$$x(t) = y_0 \ominus (-1) \int_{t_0}^{t} \mathcal{F}(u, x(u))du;\ x(t_0) = x_0.$$

Based on [31].

Theorem 5.2. Let $\mathcal{F} : \mathcal{J} \times \mathfrak{X} \to \mathfrak{X}$ be continuous and assume there is a $\mu > 0$ such that

$$\mathfrak{D}((\mathcal{F}(t, \mathcal{U}), \mathcal{F}(t, \mathcal{V})) \leq \mu \mathfrak{D}(\mathcal{U}, \mathcal{V}),$$

for all $t \in T,\ \mathcal{U}, \mathcal{V} \in \mathfrak{X}$. Then the problem (5.1) has a unique solution on $\mathcal{J}$.

To prove the equivalence between a m-polar fuzzy differential equation and system of m real differential equations, we need a Characteristic theorem.

Theorem 5.3. Characteristic Theorem
If a function $\mathcal{F} : \mathcal{J} \times \mathfrak{X} \to \mathfrak{X}$ is continuous, m-polar FVF and GHD that satisfies the following differential equation

$$x'_{gH}(t) = \mathcal{F}(t, x(t)),\ x(t_0) = x_0 \in \mathfrak{X} \text{ and } t \in \mathcal{J} = [\mathfrak{c}, \mathfrak{d}],$$

Also suppose the following conditions

- $[\mathcal{F}(t, x(t))]^\delta = \prec [\underline{\mathcal{F}}^i(t, \underline{x}^i(t;\delta_i), \overline{x}^i(t;\delta_i)), \overline{\mathcal{F}}^i(t, \underline{x}^i(t;\delta_i), \overline{x}^i(t;\delta_i))] \succ,$
  $\delta = (\delta_1, \delta_2, \cdots, \delta_m) \in [0, 1]^m,\ i = 1, 2, \cdots, m.$
- Each $\underline{\mathcal{F}}^i(t, \underline{x}^i(t;\delta_i), \overline{x}^i(t;\delta_i)), \overline{\mathcal{F}}^i(t, \underline{x}^i(t;\delta_i), \overline{x}^i(t;\delta_i)),\ i = 1, 2, \cdots, m$ are equicontinuous. That is for $\epsilon > 0$ and any point $(t, \mathcal{U}, \mathcal{V}) \in \mathcal{J} \times \mathfrak{R}^2$, if $||(t, \mathcal{U}, \mathcal{V}) - (t, \mathcal{U}_1, \mathcal{V}_1)|| < \delta$, for all $\delta_i \in [0, 1]$, we have

  $$|\underline{\mathcal{F}}^i_{\delta_i}(t, \underline{x}^i(t;\ \delta_i), \overline{x}^i(t;\ \delta_i)) - \underline{\mathcal{F}}^i_{\delta_i}(t, \underline{x}^i(t;\ \delta_i), \overline{x}^i(t;\ \delta_i))| < \epsilon,$$

  $$|\underline{\mathcal{F}}^i_{\delta_i}(t, \underline{x}^i(t;\ \delta_i), \overline{x}^i(t;\ \delta_i)) - \overline{\mathcal{F}}^i_{\delta_i}(t, \underline{x}^i(t;\ \delta_i), \overline{x}^i(t;\ \delta_i))| < \epsilon,$$

  $$|\overline{\mathcal{F}}^i_{\delta_i}(t, \underline{x}^i(t;\ \delta_i), \overline{x}^i(t;\ \delta_i)) - \underline{\mathcal{F}}^i_{\delta_i}(t, \underline{x}^i(t;\ \delta_i), \overline{x}^i(t;\ \delta_i))| < \epsilon,$$

  $$|\overline{\mathcal{F}}^i_{\delta_i}(t, \underline{x}^i(t;\ \delta_i), \overline{x}^i(t;\ \delta_i)) - \overline{\mathcal{F}}^i_{\delta_i}(t, \underline{x}^i(t;\ \delta_i), \overline{x}^i(t;\ \delta_i))| < \epsilon.$$

- Each $\underline{\mathcal{F}}^i_{\delta_i}(t, \underline{x}^i(t;\ \delta_i)), \overline{x}^i(t;\ \delta_i), \overline{\mathcal{F}}^i_{\delta_i}(t, \underline{x}^i(t;\ \delta_i), \overline{x}^i(t;\ \delta_i)),\ i = 1, 2, \cdots, m,$ are bounded on any bounded set.

- Each $\underline{\mathcal{F}}^i_{\delta_i}(t, \underline{x}^i(t;\ \delta_i), \overline{x}^i(t;\ \delta_i), \overline{\mathcal{F}}^i_{\delta_i}(t, \underline{x}^i(t;\ \delta_i), \overline{x}^i(t;\ \delta_i)),\ i = 1, 2, \cdots, m$ satisfy Lipschitz condition, that is for all $\delta_i \in [0, 1]$, there exists $L > 0$ such that

  $|\underline{\mathcal{F}}^i_{\delta_i}(t, \mathcal{U}_1, \mathcal{V}_1) - \underline{\mathcal{F}}^i_{\delta_i}(t, \mathcal{U}_2, \mathcal{V}_2)| \leq Lmax\{|\mathcal{U}_1 - \mathcal{U}_2|, |\mathcal{V}_1 - \mathcal{V}_2|\},$

  $|\underline{\mathcal{F}}^i_{\delta_i}(t, \mathcal{U}_1, \mathcal{V}_1) - \overline{\mathcal{F}}^i_{\delta_i}(t, \mathcal{U}_2, \mathcal{V}_2)| \leq Lmax\{|\mathcal{U}_1 - \mathcal{U}_2|, |\mathcal{V}_1 - \mathcal{V}_2|\},$

  $|\overline{\mathcal{F}}^i_{\delta_i}(t, \mathcal{U}_1, \mathcal{V}_1) - \underline{\mathcal{F}}^i_{\delta_i}(t, \mathcal{U}_2, \mathcal{V}_2)| \leq Lmax\{|\mathcal{U}_1 - \mathcal{U}_2|, |\mathcal{V}_1 - \mathcal{V}_2|\},$

  $|\overline{\mathcal{F}}^i_{\delta_i}(t, \mathcal{U}_1, \mathcal{V}_1) - \overline{\mathcal{F}}^i_{\delta_i}(t, \mathcal{U}_2, \mathcal{V}_2)| \leq Lmax\{|\mathcal{U}_1 - \mathcal{U}_2|, |\mathfrak{v}_1 - \mathfrak{v}_2|\},$

then the m-polar fuzzy differential equation is equivalent to one of the system of real differential equations in cone

$$\begin{cases} \underline{x'}^i(t;\ \delta_i) = \underline{\mathcal{F}}^i(t, \underline{x}^i(t;\ \delta_i), \overline{x}^i(t;\ \delta_i));\ \underline{x}^i(t_0;\ \delta_i) = \underline{x}^i_0(\delta_i), \\ \overline{x'}^i(t;\ \delta_i) = \overline{v}^i(t, \underline{x}^i(t;\ \delta_i), \overline{x}^i(t;\ \delta_i));\ \overline{x}^i(t_0;\ \delta_i) = \overline{x}^i_0(\delta_i), \\ i = 1, 2, \cdots, m. \end{cases}$$

$$\begin{cases} \underline{x'}^i(t;\ \delta_i) = \overline{\mathcal{F}}^i(t, \underline{x}^i(t;\ \delta_i), \overline{x}^i(t;\ \delta_i));\ \underline{x}^i(t_0;\ \delta_i) = \underline{x}^i_0(\delta_i), \\ \overline{x'}^i(t;\ \delta_i) = \underline{\mathcal{F}}^i(t, \underline{x}^i(t;\ \delta_i), \overline{x}^i(t;\ \delta_i));\ \overline{x}^i(t_0;\ \delta_i) = \overline{x}^i_0(\delta_i), \\ i = 1, 2, \cdots, m. \end{cases}$$

6. Proposed methods

In this section, we demonstrate Euler and modified Euler method for m-polar fuzzy IVPs.

6.1. **Euler method.** To derive the Euler method, we subdivide the interval $[0, T]$ into partition $P = \{t_0 = 0 < t_1 < \cdots < t_{\mathcal{N}} = T\}$, where $t_n = nh,\ n = 0, 1, \cdots, \mathcal{N}$.
Under the assumption that second order gH-derivative of $x(t)$ exists, we examine the solution of m-polar fuzzy IVP (5.1).
Case 1.
Suppose the unique solution of m-polar fuzzy IVP (5.1), $x(t)$ is 1-GHD and belongs to $\mathcal{C}^2_{gH}\big([0, T], \mathfrak{X}\big)$ which has same kind of differentiability over $[0, T]$. Using Taylor expansion of unknown m-polar fuzzy-valued function $x(t)$ about t_n, for each $k = 0, 1, \cdots, \mathcal{N}$, we have

$$x(t_{n+1}) = x(t_n) \oplus (t_{n+1} - t_n) \odot (x_{1-GH})'(t_n) \oplus \frac{(t_{n+1} - t_n)^2}{2!} \odot (x_{1-GH})''(\xi_k),\ \text{where } t_n < \xi_n < t_{n+1},$$

$$x(t_{n+1}) = x(t_n) \oplus h \odot \mathcal{F}(t_n, x(t_n)) \oplus \frac{h^2}{2!} \odot (x_{1-GH})''(\xi_n).$$

Moreover, we have

$$\mathfrak{D}\big(x(t_{n+1}), x(t_n) \oplus h \odot \mathcal{F}(t_n, x(t_n)) \oplus \frac{h^2}{2!} \odot (x_{1-GH})''(\xi_k)\big)$$
$$\leq \mathfrak{D}\big(x(t_{n+1}), x(t_n) \oplus h \odot \mathcal{F}(t_n, x(t_n))\big) + \mathfrak{D}\big(0, \frac{h^2}{2!} \odot (x_{1-GH})''(\xi_n)\big) \to 0,$$

as $h \to 0$, since

$$\mathfrak{D}\big(x(t_{n+1}), x(t_n) \oplus h \odot \mathcal{F}(t_n, x(t_n))\big) \to 0,$$
$$\mathfrak{D}\big(0, \frac{h^2}{2!} \odot (x_{1-GH})''(\xi_n)\big) \to 0.$$

Hence, for sufficiently small h, we have

$$x(t_{k+1}) \approx x(t_n) \oplus h \odot \mathcal{F}(t_n, x(t_n)).$$

Let, the approximated value of $x(t_{n+1}) =\prec x^1(t_{n+1}), x^2(t_{n+1}), \cdots, x^m(t_{n+1}) \succ$ be $x_{n+1} =\prec x^1_{n+1}, x^2_{n+1}, \cdots, x^m_{n+1} \succ$, then we have Euler method as follows;

$$\begin{cases} x(t_0) &= x_0, \\ x_{n+1} &= x_n \oplus h \odot \mathcal{F}(t_n, x(t_n)). \end{cases} \tag{6.1}$$

Case 2.
When $x(t)$ is 2-GHD and kind of differentiability does not change over $[0, T]$, we have the Euler method as follows;

$$\begin{cases} x(t_0) = x_0, \\ x_{n+1} = x_n \ominus (-1)h \odot \mathcal{F}(t_n, x(t_n)). \end{cases} \tag{6.2}$$

Case 3.
When $x(t)$ is $[(1,2) - gH]$-differentiable and kind of differentiability do not change over $[0, T]$, then $x_n =\prec x^i_n, x^j_n \succ$, $i, j = 1, 2, \cdots, m; i \neq j$, is approximated as

$$\begin{cases} x^i_0 &= x^i_0, \\ x^j_0 &= x^j_0, \\ x^i_{n+1} &= x^i_n \oplus h \odot \mathcal{F}^i(t_n, x^i_n), \\ x^j_{n+1} &= x^j_n \ominus (-1)h \odot \mathcal{F}^j(t_n, x^j_n), \quad i, j = 1, 2, \cdots, m; i \neq j. \end{cases} \tag{6.3}$$

Case 4.
When $x(t)$ has switching point of type **I** at $\zeta \in [0, T]$ and $t_0 = 0, t_1, \cdots, t_{k-1}$, $\zeta, t_{k+1}, \cdots, t_{\mathcal{N}} = T$ be partition of interval $[0, T]$, then we have

$$\begin{cases} x_0 &= x_0, \\ x_{n+1} &= x_n \oplus h \odot \mathcal{F}(t_n, x_n), \;\; n = 0, 1, \cdots, k-1, \\ x_{n+1} &= x_n \ominus (-1)h \odot \mathcal{F}(t_n, x_n), \;\; n = k+1, k+2, \cdots, \mathcal{N}-1. \end{cases} \tag{6.4}$$

Case 5.
When $x(t)$ has switching point of type **II** at $\zeta \in [0, T]$ and $t_0 = 0, t_1, \cdots, t_{k-1}, \zeta$, $t_{k+1}, \cdots, t_{\mathcal{N}} = T$ be partition of interval $[0, T]$, then we have

$$\begin{cases} x_0 &= x_0, \\ x_{n+1} &= x_n \ominus (-1)h \odot \mathcal{F}(t_n, x_n), \;\; n = 0, 1, \cdots, k-1, \\ x_{n+1} &= x_n \oplus h \odot \mathcal{F}(t_n, x_n), \;\; n = k+1, k+2, \cdots, \mathcal{N}-1. \end{cases} \tag{6.5}$$

Case 6.
When $x(t)$ has switching point of type **III** at $\zeta \in [0, T]$ and $t_0 = 0, t_1, \cdots, t_{k-1}, \zeta$, $t_{k+1}, \cdots, t_{\mathcal{N}} = T$ be partition of interval $[0, T]$, then we have

$$\begin{cases} x^i_0 &= x^i_0, \\ x^j_0 &= x^j_0, \\ x_{n+1} &= x_n \oplus h \odot \mathcal{F}(t_n, x_n), \;\; n = 0, 1, \cdots, k-1, \\ x^i_{n+1} &= x^i_n \oplus h \odot \mathcal{F}^i(t_n, x^i_n), \;\; n = k+1, k+2, \cdots, \mathcal{N}-1, \\ x^j_{n+1} &= x^j_n \ominus (-1)h \odot \mathcal{F}^j(t_n, x^j_n), \;\; n = k+1, k+2, \cdots, \mathcal{N}-1, \quad i, j = 1, 2, \cdots, m; i \neq j. \end{cases} \tag{6.6}$$

Case 7.
When $x(t)$ has switching point of type **IV** at $\zeta \in [0, T]$ and $t_0 = 0, t_1, \cdots, t_{k-1}, \zeta$,

$t_{k+1}, \cdots, t_{\mathcal{N}} = T$ be partition of interval $[0, T]$, then we have

$$\begin{cases} x_0^i &= x_0^i, \\ x_0^j &= x_0^j, \\ x_{n+1}^i &= x_n^i \oplus h \odot \mathcal{F}^i(t_n, x_n^i), \; n = 1, 2, \cdots, k-1, \\ x_{n+1}^j &= x_n^j \ominus (-1)h \odot \mathcal{F}^j(t_n, x_n^j), \; n = 1, 2, \cdots, k-1, \\ x_{n+1} &= x_n \oplus h \odot \mathcal{F}(t_n, x_n), \; n = k+1, k+2, \cdots, \mathcal{N}-1; \quad i, j = 1, 2, \cdots, m; i \neq j. \end{cases} \tag{6.7}$$

Case 8.
When $x(t)$ has a switching point of type **V** at $\zeta \in [0, T]$ and $t_0 = 0, t_1, \cdots, t_{k-1}, \zeta$, $t_{k+1}, \cdots, t_{\mathcal{N}} = T$ be partition of interval $[0, T]$, then we have

$$\begin{cases} x_0^i &= x_0^i, \\ x_0^j &= x_0^j, \\ x_{n+1} &= x_n \ominus (-1)h \odot \mathcal{F}(t_n, x_n), \; n = 0, 1, \cdots, k-1, \\ x_{n+1}^i &= x_n^i \oplus h \odot \mathcal{F}^i(t_n, x_n^i), \; n = k+1, k+2, \cdots, \mathcal{N}-1, \\ x_{n+1}^j &= x_n^j \ominus (-1)h \odot \mathcal{F}^j(t_n, x_n^j), \; n = k+1, k+2, \cdots, \mathcal{N}-1, \quad i, j = 1, 2, \cdots, m; i \neq j. \end{cases} \tag{6.8}$$

Case 9.
When $x(t)$ has switching point of type **VI** at $\zeta \in [0, T]$ and $t_0 = 0, t_1, \cdots, t_{k-1}, \zeta$, $t_{k+1}, \cdots, t_{\mathcal{N}} = T$ be partition of interval $[0, T]$, then we have

$$\begin{cases} x_0^i &= x_0^i, \\ x_0^j &= x_0^j, \\ x_{n+1}^i &= x_n^i \oplus h \odot \mathcal{F}^i(t_n, x_n^i), \; n = 1, 2, \cdots, k-1, \\ x_{n+1}^j &= x_n^j \ominus (-1)h \odot \mathcal{F}^j(t_n, x_n^j), \; n = 1, 2, \cdots, k-1, \\ x_{n+1} &= x_n \ominus (-1)h \odot \mathcal{F}(t_n, x_n), \; n = k+1, k+2, \cdots, \mathcal{N}-1; \quad i, j = 1, 2, \cdots, m; i \neq j. \end{cases} \tag{6.9}$$

6.2. **Modified Euler method.** To derive Modified Euler method, we subdivide the interval $[0, T]$ into partition $P = \{t_0 = 0 < t_1 < \cdots < t_{\mathcal{N}} = T\}$, where $t_n = nh, \; n = 0, 1, \cdots, \mathcal{N}$.
Case 1.
Suppose the unique solution of m-polar fuzzy IVP (5.1), $x(t) = \prec x^i(t) \succ, \; i = 1, 2, \cdots, m$ is 1-GHD and belongs to $\mathcal{C}^4_{gH}([0, T], \mathfrak{X})$ such that type of differentiability do not change over $[0, T]$. Using Taylor expansion of unknown m-polar fuzzy-valued function $x(t)$ about t_n, for each $k = 0, 1, \cdots, N$, we have

$$\begin{aligned} x(t_{n+1}) =& x(t_n) \oplus (t_{n+1} - t_n) \odot (x_{(1-GH)})'(t_n) \oplus \frac{(t_{n+1} - t_n)^2}{2!} \odot (x_{1-GH})''(t_n) \\ & \oplus \frac{(t_{n+1} - t_n)^3}{3!} \odot (x_{1-GH})'''(\xi_k), \text{ where } t_n < \xi_n < t_{n+1}. \\ x(t_{n+1}) =& x(t_n) \oplus h \odot \mathcal{F}(t_n, x(t_n)) \oplus \frac{h^2}{2!} \odot (x_{1-GH})''(t_n) \oplus \frac{h^3}{3!} \odot (x_{1-gH})'''(\xi_n). \end{aligned}$$

By Theorem 4.1, we get

$$x(t_{n+1}) = x(t_n) \oplus h \odot \mathcal{F}(t_n, x(t_n)) \oplus \frac{h^2}{2!} \odot \left(\frac{x'_{1-GH}(t_{n+1}) \ominus x'_{1-GH}(t_n)}{t_{n+1} - t_n}\right) \oplus \frac{h^3}{3!} \odot (x_{1-GH})'''(\xi_n).$$

Thus

$$x(t_{n+1}) = x(t_n) \oplus h \odot \mathcal{F}(t_n, x(t_n)) \oplus \frac{h^2}{2!} \odot \left(\frac{\mathcal{F}(t_{n+1}, x(t_{n+1})) \ominus \mathcal{F}(t_n, x(t_n))}{h}\right) \oplus \frac{h^3}{3!} \odot (x_{1-GH})'''(\xi_k).$$

Hence, for sufficiently small h, we have

$$x(t_{n+1}) \approx x(t_n) \oplus h \odot \mathcal{F}(t_n, x(t_n)) \oplus \frac{h^2}{2!} \odot \big(\frac{\mathcal{F}(t_{n+1}, x(t_{n+1})) \ominus \mathcal{F}(t_n, x(t_n))}{h}\big). \tag{6.10}$$

For an iterative method, first we predict the value of $x(t_{k+1})$ by Euler method and then this predicted value of $x(t_{n+1})$ is used in (6.10).
Let, the approximated value of $x(t_{n+1}) = \prec x^1(t_{n+1}), x^2(t_{n+1}), \cdots, x^m(t_{n+1}) \succ$ be $x_{n+1} = \prec x^1_{n+1}, x^2_{n+1}, \cdots, x^m_{n+1} \succ$, then we have the Modified Euler method as follows;

$$\begin{cases} x(t_0) & = x_0, \\ x*_{n+1} & = x_n \oplus \mathcal{F}(t_n, x_n), \\ x_{n+1} & = x_n \oplus \frac{h}{2} \odot [\mathcal{F}(t_n, x_n) \oplus \mathcal{F}(t_n + h, x*_{n+1})]. \end{cases} \tag{6.11}$$

Case 2.
When $x(t)$ is 2-GHD and type of differentiability do not change over $[0, T]$, we have the Euler method as follows;

$$\begin{cases} x(t_0) = x_0, \\ x*_{n+1} = x_n \ominus (-1)h \odot \mathcal{F}(t_n, x_n), \\ x_{n+1} = x_n \ominus (-1)\frac{h}{2} \odot [\mathcal{F}(t_n, x_n) \oplus \mathcal{F}(t_n + h, x*_{n+1})]. \end{cases} \tag{6.12}$$

Case 3.
When $x(t)$ is $[(1, 2) - gH]$-differentiable and type of differentiability do not change over $[0, T]$, then $x_n = \prec x^i_n, x^j_n \succ$ $, i, j = 1, 2, \cdots, m; i \neq j$, is approximated as

$$\begin{cases} x^i_0 & = x^i_0, \\ x^j_0 & = x^j_0, \\ x*^i_{n+1} & = x^i_n \oplus h \odot \mathcal{F}^i(t_n, x^i_n), \\ x*^j_{n+1} & = x^j_n \ominus (-1)h \odot \mathcal{F}^j(t_n, x^j_n), \\ x^i_{n+1} & = x^i_n \oplus \frac{h}{2} \odot [\mathcal{F}^i(t_n, x^i_n) \oplus \mathcal{F}^i(t_n + h, x*^i_{n+1})], \\ x^j_{n+1} & = x^j_n \ominus (-1)\frac{h}{2} \odot [\mathcal{F}^j(t_n, x^j_n) \oplus \mathcal{F}^j(t_n + h, x*^j_{n+1})], \quad i, j = 1, 2, \cdots, m; i \neq j. \end{cases} \tag{6.13}$$

Case 4.
When $x(t)$ has switching point of type **I** at $\zeta \in [0, T]$ and $t_0 = 0, t_1, \cdots, t_{k-1}, \zeta,$
$t_{k+1}, \cdots, t_{\mathcal{N}} = T$ be partition of interval $[0, T]$, then we have

$$\begin{cases} x_0 & = x_0, \\ x*_{n+1} & = x_n \oplus h \odot \mathcal{F}(t_n, x_n), \\ x_{n+1} & = x_n \oplus \frac{h}{2} \odot [\mathcal{F}(t_n, x_n) \oplus \mathcal{F}(t_n + h, x*_{n+1})] \ n = 0, 1, \cdots, k-1, \\ x*_{n+1} & = x_n \ominus (-1)h \odot \mathcal{F}(t_n, x_n), \\ x_{n+1} & = x_n \ominus (-1)\frac{h}{2} \odot [\mathcal{F}(t_n, x_n) \oplus \mathcal{F}(t_n + h, x*_{n+1})], \quad n = k+1, k+2, \cdots, \mathcal{N}-1. \end{cases} \tag{6.14}$$

Case 5.
When $x(t)$ has switching point of type **II** at $\zeta \in [0, T]$ and $t_0 = 0, t_1, \cdots, t_{k-1}, \zeta,$
$t_{k+1}, \cdots, t_{\mathcal{N}} = T$ be partition of interval $[0, T]$, then we have

$$\begin{cases} x_0 & = x_0, \\ x*_{n+1} & = x_n \ominus (-1)h \odot \mathcal{F}(t_n, x_n), \\ x_{n+1} & = x_n \ominus (-1)\frac{h}{2} \odot [\mathcal{F}(t_n, x_n) \oplus \mathcal{F}(t_n + h, x*_{n+1})], \quad n = 0, 1, \cdots, k-1, \\ x*_{n+1} & = x_n \oplus h \odot \mathcal{F}(t_n, x_n), \\ x_{n+1} & = x_n \oplus \frac{h}{2} \odot [\mathcal{F}(t_n, x_n) \oplus \mathcal{F}(t_n + h, x*_{n+1})], \ n = k+1, k+2, \cdots, \mathcal{N}-1. \end{cases} \tag{6.15}$$

Case 6.
When $x(t)$ has switching point of type **III** at $\zeta \in [0,T]$ and $t_0 = 0, t_1, \cdots, t_{k-1}, \zeta,$ $t_{k+1}, \cdots, t_N = T$ be partition of interval $[0,T]$, then we have

$$\begin{cases} x_0^i & = x_0^i, \\ x_0^j & = x_0^j, \\ x*_{n+1} & = x_n \oplus h \odot \mathcal{F}(t_n, x_n), \\ x_{n+1} & = x_n \oplus \frac{h}{2} \odot [\mathcal{F}(t_n, x_n) \oplus \mathcal{F}(t_n + h, x*_{n+1})], \quad n = 0, 1, \cdots, k-1; \\ x*_{n+1}^i & = x_n^i \oplus h \odot \mathcal{F}^i(t_n, x_n^i), \\ x_{n+1}^i & = x_n^i \oplus \frac{h}{2} \odot [\mathcal{F}^i(t_n, x_n^i) \oplus \mathcal{F}^I(t_n + h, x*_{n+1}^i)], \quad n = k+1, k+2, \cdots, \mathcal{N}-1, \\ x*_{n+1}^j & = x_n^j \ominus (-1)h \odot \mathcal{F}^j(t_n, x_n^j), \\ x_{n+1}^j & = x_n^J \ominus (-1)\frac{h}{2} \odot [\mathcal{F}^j(t_n, x_n^j) \oplus \mathcal{F}^j(t_n + h, x*_{n+1}^j)], \\ & n = k+1, k+2, \cdots, \mathcal{N}-1, \quad i, j = 1, 2, \cdots, m; i \neq j. \end{cases} \tag{6.16}$$

Case 7.
When $x(t)$ has switching point of type **IV** at $\zeta \in [0,T]$ and $t_0 = 0, t_1, \cdots, t_{k-1}, \zeta,$ $t_{k+1}, \cdots, t_v = T$ be partition of interval $[0,T]$, then we have

$$\begin{cases} x_0^i & = x_0^i, \\ x_0^j & = x_0^j, \\ x*_{n+1}^i & = x_n^i \oplus h \odot \mathcal{F}^i(t_n, x_n^i), \\ x_{n+1}^i & = x_n^i \oplus \frac{h}{2} \odot [\mathcal{F}^i(t_n, x_n^i) \oplus \mathcal{F}^i(t_n + h, x*_{n+1}^i)], \quad n = 1, 2, \cdots, k-1, \\ x*_{n+1}^j & = x_n^j \ominus (-1)h \odot \mathcal{F}^j(t_n, x_n^j), \\ x_{n+1}^j & = x_n \ominus (-1)\frac{h}{2} \odot [\mathcal{F}^j(t_n, x_n^j) \oplus \mathcal{F}^j(t_n + h, x*_{n+1}^j)], \quad n = 1, 2, \cdots, k-1, \\ x*_{n+1} & = x_n \oplus h \odot \mathcal{F}(t_n, x_n), \\ x_{n+1} & = x_n \oplus \frac{h}{2} \odot [\mathcal{F}(t_n, x_n) \oplus \mathcal{F}(t_n + h, x*_{n+1})], \\ & n = k+1, k+2, \cdots, \mathcal{N}-1, \quad i, j = 1, 2, \cdots, m; \ i \neq j. \end{cases} \tag{6.17}$$

Case 8.
When $x(t)$ has switching point of type **V** at $\zeta \in [0,T]$ and $t_0 = 0, t_1, \cdots, t_{k-1}, \zeta,$ $t_{k+1}, \cdots, t_{\mathcal{N}} = T$ be partition of interval $[0,T]$, then we have

$$\begin{cases} x_0^i & = x_0^i, \\ x_0^j & = x_0^j, \\ x*_{n+1} & = x_n \ominus (-1)h \odot \mathcal{F}(t_n, x_n), \\ x_{n+1} & = x_n \ominus (-1)\frac{h}{2} \odot [\mathcal{F}(t_n, x_n) \oplus \mathcal{F}(t_n + h, x*_{n+1})], \quad n = 0, 1, \cdots, k-1, \\ x*_{n+1}^i & = x_n^i \oplus h \odot \mathcal{F}^i(t_n, x_n^i), \\ x_{n+1}^i & = x_n^i \oplus \frac{h}{2} \odot [\mathcal{F}^i(t_n, x_n^i) \oplus \mathcal{F}^i(t_n + h, x*_{n+1}^i)], \quad n = k+1, k+2, \cdots, \mathcal{N}-1, \\ x*_{n+1}^j & = x_n^j \ominus (-1)h \odot \mathcal{F}^j(t_n, x_n^j), \\ x_{n+1}^j & = x_n^j \ominus \frac{h}{2} \odot [\mathcal{F}^j(t_n, x_n^j) \oplus \mathcal{F}^j(t_n + h, x*_{n+1}^j)], \\ & n = k+1, k+2, \cdots, \mathcal{N}-1, \quad i, j = 1, 2, \cdots, m; \ i \neq j. \end{cases} \tag{6.18}$$

Case 9.
When $x(t)$ has switching point of type **VI** at $\zeta \in [0,T]$ and $t_0 = 0, t_1, \cdots, t_{k-1}, \zeta,$

$t_{k+1}, \cdots, t_{\mathcal{N}} = T$ be partition of interval $[0, T]$, then we have

$$\begin{cases} x_0^i &= x_0^i, \\ x_0^j &= x_0^j, \\ x*_{n+1}^i &= x_n^i \oplus h \odot \mathcal{F}^i(t_n, x_n^i), \\ x_{n+1}^i &= x_n^i \oplus \frac{h}{2} \odot [\mathcal{F}^i(t_n, x_n^i) \oplus \mathcal{F}(t_n + h, x*_{n+1}^i)], \quad n = 1, 2, \cdots, k-1, \\ x*_{n+1}^j &= x_n^j \ominus (-1)h \odot \mathcal{F}^j(t_n, x_n^j), \\ x_{n+1}^j &= x_n^j \ominus (-1)\frac{h}{2} \odot [\mathcal{F}^j(t_n, x_n^j) \oplus \mathcal{F}^j(t_n + h, x*_{n+1}^j)], \quad n = 1, 2, \cdots, k-1, \\ x_{n+1} &= x_n \ominus (-1)h \odot \mathcal{F}(t_n, x_n), \\ x_{n+1} &= x_n \ominus (-1)\frac{h}{2} \odot [\mathcal{F}(t_n, x_n) \oplus \mathcal{F}(t_n + h, x*_{n+1})], \\ & n = k+1, k+2, \cdots, \mathcal{N}-1, \quad i, j = 1, 2, \cdots, m; \ i \neq j. \end{cases} \tag{6.19}$$

7. Consistency, Stability and Convergence analysis

We discuss consistency, stability, and convergence of the Euler and modified Euler methods for m-polar fuzzy IVPs. Our results are extensions of definitions and results presented in [7].

7.1. **Consistency.**

Definition 7.1. For numerical methods described in (6.1), (6.4), (6.11), and (6.14), we define residual $\mathcal{R}_n = \prec \mathcal{R}_n^i \succ$, $i = 1, 2, \cdots, m$ respectively, as

$$\mathcal{R}_n = x(t_{n+1}) \ominus_{gH} \big(x(t_n) \oplus h \odot \mathcal{F}(t_n, x(t_n))\big),$$

$$\mathcal{R}_n = x(t_{n+1}) \ominus_{gH} \big(x(t_n) \ominus (-1)h \odot \mathcal{F}(t_n, x(t_n))\big),$$

$$\mathcal{R}_n = x(t_{n+1}) \ominus_{gH} \left[x(t_n) \oplus \frac{h}{2} \odot \left(\mathcal{F}\big(t_{n+1}, x(t_n) \oplus h \odot \mathcal{F}(t_n, x(t_n))\big) \oplus \mathcal{F}\big(t_n, x(t_n)\big)\right)\right],$$

and

$$\mathcal{R}_n = x(t_{n+1}) \ominus_{gH} \left[x(t_n) \ominus (-1)\frac{h}{2} \odot \left(\mathcal{F}\big(t_{n+1}, x(t_n) \oplus h \odot \mathcal{F}(t_n, x(t_n))\big) \oplus \mathcal{F}\big(t_n, x(t_n)\big)\right)\right].$$

The residual for remaining cases can be written in a similar way.

Definition 7.2. The local truncation error $\tau_n = \prec \tau_n^i \succ, \ i = 1, 2, \cdots, m$ is defined as

$$\tau_n = \frac{1}{h}\mathcal{R}_n,$$

and the method is consistent if

$$\lim_{h \to 0} \max_{t \leq b_n} \mathfrak{D}(\tau_n, 0) = 0.$$

Theorem 7.3. The Euler method is consistent.

Proof. When $x(t)$ is 1-GHD, let $\mathfrak{D}\big((x_{1-GH})''(\xi_n), 0\big) \leq \mathcal{M}_1$, then

$$\begin{aligned} \lim_{h \to 0} \max_{t_n \leq T} \mathfrak{D}(\tau_n, 0) &= \lim_{h \to 0} \max_{t_n \leq T} \mathfrak{D}\Big(\frac{h}{2} \odot (x_{1-GH})''(\xi_n), 0\Big) \\ &= \lim_{h \to 0} \frac{h}{2} \max_{t_n \leq T} \mathfrak{D}\big((x_{1-GH})''(\xi_n), 0\big) \leq \lim_{h \to 0} \frac{h}{2}\mathcal{M}_1 = 0. \end{aligned}$$

Thus, the Euler method is consistent in this case.
When $x(t)$ is 2-GHD, let $\mathfrak{D}\big((x_{2-GH})''(\xi_n),0\big) \leq \mathcal{N}_1$, then

$$\begin{aligned}\lim_{h\to 0}\max_{t_n\leq T}\mathfrak{D}(\tau_k,0) &= \lim_{h\to 0}\max_{t_n\leq T}\mathfrak{D}\big(\ominus(-1)\frac{h}{2}\odot(x_{2-GH})''(\xi_n),0\big)\\ &= \lim_{h\to 0}|(-1)\frac{h}{2}|\max_{t_n\leq T}\mathfrak{D}\big(\ominus(x_{2-GH})''(\xi_n),0\big)\\ &= \lim_{h\to 0}\frac{h}{2}\max_{t_n\leq T}\mathfrak{D}\big((x_{2-GH})''(\xi_n),0\big) \leq \lim_{h\to 0}\frac{h}{2}\mathcal{N}_1 = 0.\end{aligned}$$

Hence, the Euler method is consistent. The consistency of the Euler method for the other cases can be discussed in similar way. □

Theorem 7.4. The Modified Euler method is consistent.

Proof. When $x(t)$ is 1-GHD, let $\mathfrak{D}\big((x_{1-GH})'''(\xi_n),0\big) \leq \mathcal{M}_1$, then

$$\begin{aligned}\lim_{h\to 0}\max_{t_n\leq T}\mathfrak{D}(\tau_k,0) &= \lim_{h\to 0}\max_{t_n\leq T}\mathfrak{D}\big(\frac{h^2}{3!}\odot(x_{1-GH})'''(\xi_n),0\big)\\ &= \lim_{h\to 0}\frac{h^2}{3!}\max_{t_n\leq T}\mathfrak{D}\big((x_{1-GH})'''(\xi_n),0\big) \leq \lim_{h\to 0}\frac{h^2}{3!}\mathcal{M}_1 = 0.\end{aligned}$$

Thus, the Modified Euler method is consistent in this case.
When $x(t)$ is 2-GHD, let $\mathfrak{D}\big((x_{2-GH})''(\xi_n),0\big) \leq \mathcal{N}_1$, then

$$\begin{aligned}\lim_{h\to 0}\max_{t_n\leq T}\mathfrak{D}(\tau_k,0) &= \lim_{h\to 0}\max_{t_n\leq T}\mathfrak{D}\big(\ominus(-1)\frac{h^2}{3!}\odot(x_{2-GH})'''(\xi_n),0\big)\\ &= \lim_{h\to 0}|(-1)\frac{h^2}{3!}|\max_{t_n\leq T}\mathfrak{D}\big(\ominus(x_{2-GH})'''(\xi_n),0\big)\\ &= \lim_{h\to 0}\frac{h^2}{3!}\max_{t_n\leq T}\mathfrak{D}\big((x_{2-GH})''(\xi_n),0\big) \leq \lim_{h\to 0}\frac{h^2}{3!}\mathcal{N}_1 = 0.\end{aligned}$$

Hence, the Modified Euler method is consistent. The consistency of the Modified Euler method for the other cases can be discussed in a similar way. □

7.2. **Convergence.**

Definition 7.5. [8] The global truncation error, e_{n+1}, at time t_{n+1} is defined by:

$$e_{n+1} = x(t_{n+1}) \ominus_{gH} x_{n+1}.$$

Definition 7.6. [8] The numerical method is convergent

$$\begin{aligned}&\lim_{h\to 0}\max_{n} D(e_{n+1},0) = 0,\\ \Rightarrow &\lim_{h\to 0}\max_{n} D(x(t_{n+1}),x_{n+1}) = 0.\end{aligned}$$

Lemma 7.7. [23] For all real x

$$1 + x \leq e^x.$$

Theorem 7.8. Let $x''_{gH}(t)$ exists and $\mathcal{F}(t,x(t))$ satisfies Lipschitz condition on $\{\mathcal{F}(t,x(t)), t\in[0,\mathcal{P}], x\in\overline{B}(x_0,\mathcal{Q}),\mathcal{P},\mathcal{Q}>0\}$, then the proposed Euler method converges to the solution of bipolar fuzzy IVPs (5.1).

Proof. **Case 1**.
When $x(t)$ is 1-GHD. Let suppose $\mathcal{C}_n = \frac{h^2}{2}\odot(x_{1-GH}(t_n))''$, so by equation (6.1) and $\mathcal{C}_n$, the exact solution $x(t) = \prec x^i(t)\succ,\ i = 1,2,\cdots,m$ of (5.1) satisfies

$$x(t_{n+1}) = x(t_n)\oplus h\odot\mathcal{F}(t_n,x(t_n))\oplus\mathcal{C}_n.$$

Now

$$\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) = \mathfrak{D}\big(x(t_n) \oplus h \odot \mathcal{F}(t_n, x(t_n)) \oplus \mathcal{C}_n, x_n \oplus h \odot \mathcal{F}(t_n, x_n)\big)$$
$$\leq \mathfrak{D}\big(x(t_n), x_n\big) + h\mathfrak{D}\big(\mathcal{F}(t_n, x(t_n), \mathcal{F}(t_n, x_n)\big) + \mathfrak{D}(\mathcal{C}_n, 0).$$

Since, $\mathcal{F}$ satisfies Lipschitz condition, so there exists $\mathcal{L}_n > 0$ such that

$$\mathfrak{D}\big(\mathcal{F}(t_n, x(t_n), \mathcal{F}(t_n, x_n)\big) \leq \mathcal{L}_n \mathfrak{D}(x(t_n), x_n).$$

Thus

$$\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) \leq (1 + h\mathcal{L}_n)\mathfrak{D}\big(x(t_n), x_n\big) + \mathfrak{D}(\mathcal{C}_n, 0). \tag{7.1}$$

Suppose $\mathcal{L} = \max\limits_{0 \leq n \leq \mathcal{N}} \mathcal{L}_n$ and $\mathcal{C} = \max\limits_{0 \leq n \leq N} \mathfrak{D}_1(\mathcal{C}_n, 0)$, then (7.6) can be written as

$$\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) \leq (1 + h\mathcal{L})\mathfrak{D}\big(x(t_n), x_n\big) + \mathcal{C}.$$

Continuing in this way and after substitution, we have

$$\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) \leq (1 + h\mathcal{L})^{k+1}\mathfrak{D}\big(x(t_0), x_0\big) + \{1 + (1 + h\mathcal{L}) + \cdots + (1 + h\mathcal{L})^k\}\mathcal{C}$$
$$= (1 + h\mathcal{L})^{n+1}\mathfrak{D}\big(x(t_0), x_0\big) + \{\frac{(1 + h\mathcal{L})^{n+1} - 1}{h\mathcal{L}}\}\mathcal{C}.$$

Now, $0 \leq (n+1)h \leq T$ for $(n+1) \leq (\mathcal{N} - 1)$ and by Lemma 7.7, we have

$$\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) \leq e^{\mathcal{L}T}\mathfrak{D}\big(x(t_0), x_0\big) + \frac{d}{h\mathcal{L}}[e^{\mathcal{L}T} - 1].$$

Moreover, $\mathcal{C} = \max\limits_{0 \leq n \leq \mathcal{N}-1} \mathfrak{D}(\mathcal{C}_n, 0) = \max\limits_{0 \leq n \leq N} \mathfrak{D}(\frac{h^2}{2} \odot (x_{1-gH}(t_n))'', 0)$ and $\mathfrak{D}\big(x(t_0), x_0\big) = 0$, so

$$\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) \leq \frac{h}{2\mathcal{L}}[e^{\mathcal{L}T} - 1] \max_{0 \leq t \leq T} \mathfrak{D}(x_{1-GH}(t_n))'', 0).$$

and $\lim\limits_{h \to 0} \mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) = 0$.Thus, Euler method converges in this case.

Case 2.

When $x(t)$ is 2-GHD, let suppose $\mathcal{C}_n = \ominus(-1)\frac{h^2}{2} \odot (x_{2-GH}(t_n))''$, so by equation (6.1) and $\mathcal{C}_n$, the exact solution $x(t) = \prec x^i(t) \succ,\ i = 1, 2, \cdots, m$ of (5.1) satisfies

$$x(t_{n+1}) = x(t_n) \ominus (-1)h \odot \mathcal{F}(t_n, x(t_n)) \oplus \mathcal{C}_n.$$

Now

$$\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) = \mathfrak{D}\big(x(t_n) \ominus (-1)h \odot \mathcal{F}(t_n, x(t_n)) \oplus \mathcal{C}_n, x_n \ominus (-1)h \odot \mathcal{F}(t_n, x_n)\big)$$
$$\leq \mathfrak{D}\big(x(t_n), x_n\big) + h\mathfrak{D}\big(\mathcal{F}(t_n, x(t_n) \ominus_{gH} \mathcal{F}(t_n, x_n), 0\big) + \mathfrak{D}(\mathcal{C}_n, 0).$$

Since, $\mathcal{F}$ satisfies Lipschitz condition, so there exists $\mathcal{L}_n > 0$ such that

$$\mathfrak{D}\big(\mathcal{F}(t_n, x(t_n) \ominus_{gH} \mathcal{F}(t_n, x_n), 0\big) = \mathfrak{D}\big(\mathcal{F}(t_n, x(t_n), \mathcal{F}(t_n, x_n)\big)$$
$$\leq \mathcal{L}_n \mathfrak{D}(x(t_n), x_n).$$

Thus

$$\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) \leq (1 - h\mathcal{L}_n)\mathfrak{D}\big(x(t_n), x_n\big) + \mathfrak{D}(\mathcal{C}_n, 0). \tag{7.2}$$

Suppose $\mathcal{L} = \max\limits_{0 \leq n \leq N} \mathcal{L}_n$ and $\mathcal{C} = \max\limits_{0 \leq n \leq \mathcal{N}} \mathfrak{D}_1(\mathcal{C}_n, 0)$, then (7.6) can be written as

$$\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) \leq (1 - h\mathcal{L})\mathfrak{D}\big(x(t_n), x_n\big) + \mathcal{C}.$$

Using iterations, we have

$$\begin{aligned}\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) \leq &(1-h\mathcal{L})^{n+1}\mathfrak{D}\big(x(t_0), x_0\big) + \{1 + (1-h\mathcal{L}) + \cdots + (1+h\mathcal{L})^n\}\mathcal{C}\\ &= (1-h\mathcal{L})^{n+1}\mathfrak{D}\big(x(t_0), x_0\big) + \{\frac{1-(1-h\mathcal{L})^{n+1}}{h\mathcal{L}}\}\mathcal{C}.\end{aligned}$$

Now, $0 \leq (n+1)h \leq T$ for $(n+1) \leq (\mathcal{N}-1)$ and by Lemma 7.7, we have

$$\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) \leq e^{-\mathcal{L}T}\mathfrak{D}\big(x(t_0), x_0\big) + \frac{d}{h\mathcal{L}}[1-e^{-\mathcal{L}T}].$$

Moreover, $\mathcal{C} = \max\limits_{0\leq n\leq N-1}\mathfrak{D}(\mathcal{C}_n, 0) = -\dfrac{h^2}{2}\max\limits_{0\leq n\leq N}\mathfrak{D}_1((x_{2-gH}(t_n))'', 0)$ and $\mathfrak{D}\big(x(t_0), x_0\big) = 0$, so

$$\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) \leq -\frac{h}{2\mathcal{L}}[1-e^{-\mathcal{L}T}]\max_{0\leq t\leq T}\mathfrak{D}(x_{1-GH}(t_n))'', 0).$$

and $\lim\limits_{h\to 0}\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) = 0$.
Thus, the Euler method converges in this case.
The convergence of the Euler method for other cases can be proved in a similar manners. □

Theorem 7.9. Let $x'''_{gH}(t)$ exists and $\mathcal{F}(t, x(t))$ satisfies Lipschitz condition on $\{\mathcal{F}(t, x(t)), t \in [0, \mathcal{P}], x \in \overline{B}(x_0, \mathcal{Q}), \mathcal{P}, \mathcal{Q} > 0\}$, then proposed Modified Euler method converges to the solution of m-polar fuzzy IVPs (5.1).

Proof. **Case 1**.
When $x(t)$ is 1-GHD. We can write the Modified Euler method as

$$x(t_{n+1}) = x(t_n) \oplus h \odot \psi(t_n, x_n; h),$$

where

$$\psi(t_n, x_n; h) = \frac{1}{2}\big[\mathcal{F}(t_n, x_n) \oplus \mathcal{F}\big(t_n + h, x_n \oplus h \odot \mathcal{F}(t_n, x_n)\big)\big].$$

Obviously, ψ is the continuous function. First, we prove that ψ satisfies Lipschitz condition.

$$\mathfrak{D}\big(\psi(t, u; h), \psi(t, v; h)\big) \leq \frac{1}{2}\mathfrak{D}\big(\mathcal{F}(t, u), \mathcal{F}(t, v)\big) + \frac{1}{2}\mathfrak{D}\big(\mathcal{F}(t+h, u \oplus h \odot \mathcal{F}(t, u)), \mathcal{F}(t+h, v \oplus h \odot \mathcal{F}(t, v))\big).$$

Since $\mathcal{F}$ satisfies Lipschitz condition, so there is $\mathcal{L} > 0$ such that

$$\mathfrak{D}(\mathcal{F}(t, u), \mathcal{F}(t, v)) \leq \mathcal{L}\mathfrak{D}(u, v).$$

Thus

$$\begin{aligned}\mathfrak{D}\big(\psi(t, u; h), \psi(t, v; h)\big) &\leq \frac{\mathcal{L}}{2}\mathfrak{D}\big(u, v\big) + \frac{\mathcal{L}}{2}\mathfrak{D}\big(u \oplus h \odot \mathcal{F}(t, u)), v \oplus h \odot \mathcal{F}(t, v))\big)\\ &\leq \frac{\mathcal{L}}{2}\mathfrak{D}\big(u, v\big) + \frac{\mathcal{L}}{2}\mathfrak{D}\big(u, v\big) + \frac{h\mathcal{L}}{2}\mathfrak{D}\big(\mathcal{F}(t, u), \mathcal{F}(t, v)\big)\\ &\leq \mathcal{L}\big(1 + \frac{h\mathcal{L}}{2}\big)\mathfrak{D}(u, v) = \mathcal{L}'\mathfrak{D}(u, v).\end{aligned}$$

i.e. ψ satisfies Lipschitz condition with constant $\mathcal{L}' = \mathcal{L}(1 + \frac{h\mathcal{L}}{2})$.
Let suppose $\mathcal{C}_n = \frac{h^3}{3!}\odot(x_{1-GH}(t_n))'''$, so by equation (6.11) and $\mathcal{C}_n$, the exact solution $x(t) = \prec x^1(t), x^2(t), \cdots, x^m(t) \succ$ of (5.1) satisfies

$$x(t_{n+1}) = x(t_n) \oplus h \odot \psi(t_n, x(t_n)) \oplus \mathcal{C}_n.$$

Now

$$\begin{aligned}\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) &= \mathfrak{D}\big(x(t_n) \oplus h \odot \psi(t_n, x(t_n)) \oplus \mathcal{C}_n, x_n \oplus h \odot \psi(t_n, x_n)\big)\\ &\leq \mathfrak{D}\big(x(t_n), x_n\big) + h\mathfrak{D}\big(\psi(t_n, x(t_n)), \psi(t_n, x_n)\big) + \mathfrak{D}(\mathcal{C}_n, 0).\end{aligned}$$

Since, ψ satisfies Lipschitz condition, so there exists $\mathcal{L}'_n > 0$ such that

$$\mathfrak{D}\big(\psi(t_n, x(t_n), \psi(t_n, x_n)\big) \leq \mathcal{L}'_n \mathfrak{D}(x(t_n), x_n).$$

Thus

$$\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) \leq (1 + h\mathcal{L}'_n)\mathfrak{D}\big(x(t_n), x_n\big) + \mathfrak{D}(\mathcal{C}_n, 0). \tag{7.3}$$

Suppose $\mathcal{L}' = \max_{0\leq n\leq \mathcal{N}} \mathcal{L}'_n$ and $\mathcal{C} = \max_{0\leq n\leq \mathcal{N}} \mathfrak{D}_1(\mathcal{C}_n, 0)$, then (7.3) can be written as

$$\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) \leq (1 + h\mathcal{L}')\mathfrak{D}\big(x(t_n), x_n\big) + \mathcal{C}.$$

Continuing in this way, we have

$$\begin{aligned}\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) \leq& (1 + h\mathcal{L}')^{n+1}\mathfrak{D}\big(x(t_0), x_0\big) + \{1 + (1 + h\mathcal{L}') + \cdots + (1 + h\mathcal{L}')^n\}\mathcal{C} \\ &= (1 + h\mathcal{L}')^{n+1}\mathfrak{D}\big(x(t_0), x_0\big) + \{\frac{(1 + h\mathcal{L}')^{n+1} - 1}{h\mathcal{L}'}\}\mathcal{C}.\end{aligned}$$

Now, $0 \leq (n + 1)h \leq T$ for $(n + 1) \leq (\mathcal{N} - 1)$ and by Lemma 7.7, we have

$$\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) \leq e^{\mathcal{L}'T}\mathfrak{D}\big(x(t_0), x_0\big) + \frac{d}{h\mathcal{L}'}[e^{\mathcal{L}'T} - 1].$$

Moreover, $\mathcal{C} = \max_{0\leq n\leq \mathcal{N}-1} \mathfrak{D}(\mathcal{C}_n, 0) = \max_{0\leq n\leq \mathcal{N}} \mathfrak{D}(\frac{h^3}{3!} \odot (x_{1-GH}(t_n))''', 0)$ and $\mathfrak{D}\big(x(t_0), x_0\big) = 0$, so

$$\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) \leq \frac{h^2}{3!\mathcal{L}}[e^{\mathcal{L}'T} - 1] \max_{0\leq t\leq T} \mathfrak{D}(x_{1-GH}(t_n))''', 0)$$

and $\lim_{h\to 0} \mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) = 0$.

Thus, the Modified Euler method converges in this case.

Case 2.

When $x(t)$ is 2-GHD, we can write the Modified Euler method as

$$x(t_{n+1}) = x(t_n) \ominus (-1)h \odot \psi(t_n, x_n; h),$$

where

$$\psi(t_n, x_n; h) = \frac{1}{2}\big[\mathcal{F}(t_n, x_n) \oplus \mathcal{F}\big(t_n + h, x_n \ominus (-1)h \odot \mathcal{F}(t_n, x_n)\big)\big].$$

Obviously, ψ is the continuous function. First, we prove that ψ satisfies Lipschitz condition.

$$\mathfrak{D}\big(\psi(t, u; h), \psi(t, v; h)\big) \leq \frac{1}{2}\mathfrak{D}\big(\mathcal{F}(t, u), \mathcal{F}(t, v)\big) + \frac{1}{2}\mathfrak{D}\big(\mathcal{F}(t + h, u \ominus (-1)h \odot \mathcal{F}(t, u)), \mathcal{F}(t + h, v \ominus (-1)h \odot \mathcal{F}(t, v))\big).$$

Since $\mathcal{F}$ satisfies Lipschitz condition, so there is $\mathcal{L} > 0$ such that

$$\mathfrak{D}(\mathcal{F}(t, u), \mathcal{F}(t, v)) \leq \mathcal{L}\mathfrak{D}(u, v).$$

Thus

$$\begin{aligned}\mathfrak{D}\big(\psi(t, u; h), \psi(t, v; h)\big) &\leq \frac{\mathcal{L}}{2}\mathfrak{D}\big(u, v\big) + \frac{\mathcal{L}}{2}\mathfrak{D}\big(u \ominus (-1)h \odot \mathcal{F}(t, u)),\ v \ominus (-1)h \odot \mathcal{F}(t, v))\big) \\ &\leq \frac{\mathcal{L}}{2}\mathfrak{D}\big(u, v\big) + \frac{\mathcal{L}}{2}\mathfrak{D}\big(u, v\big) - \frac{h\mathcal{L}}{2}\mathfrak{D}\big(\mathcal{F}(t, u), \mathcal{F}(t, v)\big) \\ &\leq \mathcal{L}\big(1 - \frac{h\mathcal{L}}{2}\big)\mathfrak{D}(u, v) = \mathcal{L}'\mathfrak{D}(u, v),\end{aligned}$$

i.e. ψ satisfies Lipschitz condition with constant $\mathcal{L}' = \mathcal{L}(1 - \frac{h\mathcal{L}}{2})$.
Let suppose $\mathcal{C}_n = \ominus(-1)\frac{h^2}{2}\odot(x_{2-GH}(t_n))'''$, so by equation (6.12) and $\mathcal{C}_n$, the exact solution $x(t) = \prec x^1(t), x^2(t), \cdots, x^m(t) \succ$ of (5.1) satisfies

$$x(t_{n+1}) = x(t_n) \ominus (-1)h \odot \psi(t_n, x(t_n)) \oplus \mathcal{C}_n.$$

Now

$$\begin{aligned}\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) &= \mathfrak{D}\big(x(t_n) \ominus (-1)h \odot \psi(t_n, x(t_n)) \oplus \mathcal{C}_n, x_n \ominus (-1)h \odot \psi(t_n, x_n)\big) \\ &\leq \mathfrak{D}\big(x(t_n), x_n\big) + h\mathfrak{D}\big(\psi(t_n, x(t_n) \ominus_{gH} \psi(t_n, x_n), 0\big) + \mathfrak{D}(\mathcal{C}_n, 0).\end{aligned}$$

Thus

$$\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) \leq (1 - h{\mathcal{L}'}_n)\mathfrak{D}\big(x(t_n), x_n\big) + \mathfrak{D}(\mathcal{C}_n, 0). \tag{7.4}$$

Suppose $\mathcal{L} = \max_{0\leq n\leq \mathcal{N}} \mathcal{L}_n$ and $\mathcal{C} = \max_{0\leq n\leq \mathcal{N}} \mathfrak{D}(\mathcal{C}_n, 0)$, then (7.4) can be written as

$$\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) \leq (1 - h\mathcal{L}')\mathfrak{D}\big(x(t_n), x_n\big) + \mathcal{C}.$$

Using above relation, we have

$$\begin{aligned}\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) &\leq (1 - h\mathcal{L}')^{n+1}\mathfrak{D}\big(x(t_0), x_0\big) + \{1 + (1 - h\mathcal{L}) + \cdots + (1 + h\mathcal{L}')^n\}\mathcal{C} \\ &= (1 - h\mathcal{L}')^{n+1}\mathfrak{D}\big(x(t_0), x_0\big) + \{\frac{1 - (1 - h\mathcal{L}')^{n+1}}{h\mathcal{L}'}\}\mathcal{C}.\end{aligned}$$

Now, $0 \leq (n + 1)h \leq T$ for $(n + 1) \leq (\mathcal{N} - 1)$ and by Lemma 7.7, we have

$$\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) \leq e^{-\mathcal{L}'T}\mathfrak{D}\big(x(t_0), x_0\big) + \frac{d}{h\mathcal{L}'}[1 - e^{-\mathcal{L}'T}].$$

Moreover, $\mathcal{C} = \max_{0\leq n\leq N-1} \mathfrak{D}(\mathcal{C}_n, 0) = -\frac{h^3}{3!} \max_{0\leq n\leq N} \mathfrak{D}((x_{2-GH}(t_n))''', 0)$ and
$\mathfrak{D}\big(x(t_0), x_0\big) = 0$, so

$$\mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) \leq -\frac{h^2}{3!\mathcal{L}}[1 - e^{-\mathcal{L}'T}] \max_{0\leq t\leq T} \mathfrak{D}(x_{1-GH}(t_n))''', 0)$$

and $\lim_{h\to 0} \mathfrak{D}\big(x(t_{n+1}), x_{n+1}\big) = 0$.
Thus, the Modified Euler method converges in this case.
The convergence of Modified Euler method for other cases can be proved in similar manners. □

7.3. **Stability.**

Definition 7.10. Let $x_{n+1},\ n + 1 \geq 0$ be the solution of m-polar fuzzy IVP $x'(t) = \mathcal{F}(t, x(t))$ with initial condition $x_0 \in \mathfrak{X}$ and let z_{n+1} be the solution obtained by same numerical method with perturbed initial condition $z_0 = x_0 \oplus \delta_0 \in \mathfrak{X}$, then the method is stable if there exists $h_1, \mathcal{L} > 0$ such that

$$\mathfrak{D}(x_{n+1}, z_{n+1}) \leq \mathcal{L}\delta,\ \forall\ (n + 1)h < T,\ n \leq \mathcal{N} - 1,\ h \in (0, h_1), \quad \text{whenever } \mathfrak{D}(\delta_0, 0) \leq \delta.$$

Theorem 7.11. The Euler method is stable.

Proof. When $x(t)$ is 1-GHD, then by equation (6.1), we have

$$z_{n+1} = z_n \oplus h \odot \mathcal{F}(t_n, z_n),\ z_0 = x_0 \oplus \delta_0. \tag{7.5}$$

Using (6.1) and (7.6), we have

$$\mathfrak{D}(z_{n+1}, x_{n+1}) \leq \mathfrak{D}(z_n, x_n) + h\mathfrak{D}\big(\mathcal{F}(t_n, z_n), \mathcal{F}(t_n, x_n)\big),$$

Since, $\mathcal{F}$ satisfies Lipscitz condition, there exists $\mathcal{L}_n > 0$ such that
$\mathfrak{D}\big(\mathcal{F}(t_n, z_n), \mathcal{F}(t_n, x_n)\big) \leq \mathcal{L}_n\mathfrak{D}(z_n, x_n)$, so

$$\mathfrak{D}(z_{n+1}, x_{n+1}) \leq \mathfrak{D}(z_n, x_n) + h\mathcal{L}_n\mathfrak{D}(z_n, x_n).$$

Let $\mathcal{L} = \max_{0 \le n \le N} \mathcal{L}_n$, then

$$\mathfrak{D}(z_{n+1}, x_{n+1}) \le (1 + h\mathcal{L})\mathfrak{D}(z_n, x_n).$$

Continuing, we get

$$\mathfrak{D}(z_{n+1}, x_{n+1}) \le (1 + h\mathcal{L})^{(n+1)}\mathfrak{D}(z_0, x_0).$$

By Lemma 7.7, we have

$$\begin{aligned}\mathfrak{D}(z_{n+1}, x_{n+1}) &\le e^{h\mathcal{L}(n+1)}\mathfrak{D}(z_0, x_0)\\ &\le e^{h\mathcal{L}(n+1)}\mathfrak{D}(z_0 \ominus_{gH} x_0, 0)\\ &\le e^{\mathcal{L}T}\mathfrak{D}(\delta_0, 0) \le \mathcal{K}\delta,\end{aligned}$$

where $e^{\mathcal{L}T} = \mathcal{K} > 0$ and $\mathfrak{D}(\delta_0, 0) \le \delta$. Hence, the Euler method is stable in this case.
When $x(t)$ is 2-GHD, then by equation (6.1), we have

$$z_{n+1} = z_n \ominus (-1)h \odot \mathcal{F}(t_n, z_n),\ \ z_0 = x_0 \oplus \delta_0. \tag{7.6}$$

Using (6.1) and (7.6), we have

$$\mathfrak{D}(z_{n+1}, x_{n+1}) \le \mathfrak{D}(z_n, x_n) - h\mathfrak{D}\big(\mathcal{F}(t_n, z_n), \mathcal{F}(t_n, x_n)\big),$$

Since, $\mathcal{F}$ satisfies Lipscitz condition, there exists $\mathcal{L}_n > 0$ such that
$\mathfrak{D}\big(\mathcal{F}(t_n, z_n), \mathcal{F}(t_n, x_n)\big) \le \mathcal{L}_n\mathfrak{D}(z_n, x_n)$, so

$$\mathfrak{D}(z_{n+1}, x_{n+1}) \le \mathfrak{D}(z_n, x_n) - h\mathcal{L}_n\mathfrak{D}(z_n, x_n).$$

Let $\mathcal{L} = \max_{0 \le n \le N} \mathcal{L}_n$, then

$$\mathfrak{D}(z_{n+1}, x_{n+1}) \le (1 - h\mathcal{L})\mathfrak{D}(z_n, x_n).$$

Continuing, we have

$$\mathfrak{D}(z_{n+1}, x_{n+1}) \le (1 - h\mathcal{L})^{(n+1)}\mathfrak{D}(z_0, x_0).$$

By Lemma 7.7, we have

$$\begin{aligned}\mathfrak{D}(z_{n+1}, x_{n+1}) &\le e^{-h\mathcal{L}(n+1)}\mathfrak{D}(z_0, x_0)\\ &\le e^{-h\mathcal{L}(n+1)}\mathfrak{D}(z_0 \ominus_{gH} x_0, 0)\\ &\le e^{-\mathcal{L}T}\mathfrak{D}(\delta_0, 0) \le \mathcal{K}\delta,\end{aligned}$$

where $e^{-\mathcal{L}T} = \mathcal{K} > 0$ and $\mathfrak{D}(\delta_0, 0) \le \delta$. Hence, the Euler method is stable in this case. For the other cases, we can prove easily that the Euler method is stable. □

Theorem 7.12. The Modified Euler method is stable.

Proof. When $x(t)$ is 1-GHD, then by Equation (6.11), we have

$$z_{n+1} = z_n \oplus \frac{h}{2} \odot \big[\mathcal{F}(t_n, z_n) \oplus \mathcal{F}(t_n + h, z_n \oplus h \odot \mathcal{F}(t_n, z_n))\big], \tag{7.7}$$

$$z_0 = x_0 \oplus \delta_0. \tag{7.8}$$

Since, $\mathcal{F}$ satisfies Lipschitz condition, there exists $\mathcal{L}_n > 0$ such that
$\mathfrak{D}\big(\mathcal{F}(t_n, z_n), \mathcal{F}(t_n, x_n)\big) \le \mathcal{L}_n\mathfrak{D}(z_n, x_n)$, using (6.11) and (7.7), we have

$$\mathfrak{D}(z_{n+1}, x_{n+1}) \le \big(1 + h\mathcal{L}_n + \frac{(h\mathcal{L}_n)^2}{2}\big)\mathfrak{D}(z_n, x_n).$$

Let $\mathcal{L} = \max_{0 \le n \le N} \mathcal{L}_n$, then

$$\mathfrak{D}(z_{n+1}, x_{n+1}) \le (1 + h\mathcal{L} + \frac{(h\mathcal{L})^2}{2})\mathfrak{D}(z_n, x_n).$$

Using above inequality, we have

$$\mathfrak{D}(z_{n+1}, x_{n+1}) \le (1 + h\mathcal{L} + \frac{(h\mathcal{L})^2}{2})^{(n+1)}\mathfrak{D}(z_0, x_0).$$

By Lemma 7.7, we have

$$\begin{aligned}\mathfrak{D}(z_{n+1}, x_{n+1}) &\le e^{[h\mathcal{L} + \frac{(h\mathcal{L})^2}{2}](n+1)}\mathfrak{D}(z_0, x_0)\\ &\le e^{[h\mathcal{L} + \frac{(h\mathcal{L})^2}{2}](n+1)}\mathfrak{D}(z_0 \ominus_{gH} x_0, 0)\\ &\le e^{(\mathcal{L} + \frac{h\mathcal{L}^2}{2})T}\mathfrak{D}(\delta_0, 0) \le \mathcal{K}\delta,\end{aligned}$$

where $e^{(\mathcal{L} + \frac{h\mathcal{L}^2}{2})T} = \mathcal{K} > 0$ and $\mathfrak{D}(\delta_0, 0) \le \delta$. Hence, the Modified Euler method is stable in this case.
When $x(t)$ is 2-GHD, then by equation (6.14), we have

$$z_{n+1} = z_n \ominus (-1)\frac{h}{2} \odot \big[\mathcal{F}(t_n, z_n) \oplus \mathcal{F}(t_n + h, z_n \ominus (-1)h \odot \mathcal{F}(t_n, z_n))\big], \quad z_0 = x_0 \oplus \delta_0. \tag{7.9}$$

Since, $\mathcal{F}$ satisfies Lipschitz condition, there exists $\mathcal{L}_n > 0$ such that
$\mathfrak{D}\big(\mathcal{F}(t_n, z_n), \mathcal{F}(t_n, x_n)\big) \le \mathcal{L}_n\mathfrak{D}(z_n, x_n)$, using (6.14) and (7.9), we have

$$\mathfrak{D}(z_{n+1}, x_{n+1}) \le \big(1 - h\mathcal{L}_n + \frac{(h\mathcal{L}_n)^2}{2}\big)\mathfrak{D}(z_n, x_n).$$

Let $\mathcal{L} = \max_{0 \le n \le N} \mathcal{L}_n$, then

$$\mathfrak{D}(z_{n+1}, x_{n+1}) \le (1 - h\mathcal{L} + \frac{(h\mathcal{L})^2}{2})\mathfrak{D}(z_n, x_n).$$

Using above inequality, we have

$$\mathfrak{D}(z_{n+1}, x_{n+1}) \le (1 - h\mathcal{L} + \frac{(h\mathcal{L})^2}{2})^{(n+1)}\mathfrak{D}(z_0, x_0).$$

By Lemma 7.7, we have

$$\begin{aligned}\mathfrak{D}(z_{n+1}, y_{n+1}) &\le e^{[h\mathcal{L} + \frac{(-h\mathcal{L})^2}{2}](n+1)}\mathfrak{D}(z_0, y_0)\\ &\le e^{[-h\mathcal{L} + \frac{(h\mathcal{L})^2}{2}](n+1)}\mathfrak{D}(z_0 \ominus_{gH} y_0, 0)\\ &\le e^{(-\mathcal{L} + \frac{h\mathcal{L}^2}{2})T}\mathfrak{D}(\delta_0, 0) \le \mathcal{K}\delta,\end{aligned}$$

where $e^{(-\mathcal{L} + \frac{h\mathcal{L}^2}{2})T} = \mathcal{K} > 0$ and $\mathfrak{D}(\delta_0, 0) \le \delta$. Hence, the Modified Euler method is stable in this case. For the other cases, we can easily prove that the Modified Euler method is stable. □

8. Numerical Example

In this section, we solve an m-polar fuzzy IVPs by the Euler method(EM) and modified Euler method(MEM). We give the comparison of these methods by calculating global truncation errors. All computations are performed on Maple 13 software.

Example 8.1. Consider the 3-polar fuzzy IVP

$$\begin{aligned}&\tilde{x}'_{gH}(t) = (t-1)\tilde{x}(t),\\ &\tilde{x}(0) = \prec [1 + 2\delta_1, 4 - \delta_1], [1.5 + 0.5\delta_2, 2.5 - 0.5\delta_2], [2 + \delta_3, 5 - 2\delta_3] \succ, \quad t \in [0, 2], \ \delta_1, \delta_2, \delta_3 \in [0, 1].\end{aligned} \tag{8.1}$$

Obviously, the IVP is 2-GHD on [0,1] and at $t = 1$, the problem is switched to 1-GHD. Thus $t = 1$ is a switching point. The solution is 2-GHD on [0,1] and 1-GHD on (1,2]. The exact 2-GHD solution can be obtained by solving the following system

$$\begin{cases} (\underline{x}^1)'(t;\delta_1) = \begin{cases} (t-1)(\underline{x}^1)(t;\delta_1), & t \in [0,1], \\ (t-1)(\overline{x}^1)'(t;\delta_1), & t \in (1,2], \end{cases} \\ (\overline{x}^1)'(t;\delta_1) = \begin{cases} (t-1)(\overline{x}^1)(t;\delta_1), & t \in [0,1], \\ (t-1)(\underline{x}^1)(t;\delta_1), & t \in (1,2], \end{cases} \\ (\underline{x}^2)'(t;\delta_2) = \begin{cases} (t-1)(\underline{x}^2)(t;\delta_2), & t \in [0,1], \\ (t-1)(\overline{x}^2)(t;\delta_2), & t \in (1,2], \end{cases} \\ (\overline{x}^2)'(t;\delta_2) = \begin{cases} (t-1)(\overline{x}^2)(t;\delta_2), & t \in [0,1], \\ (t-1)(\underline{x}^2)(t;\delta_2), & t \in (1,2], \end{cases} \\ (\underline{x}^3)'(t;\delta_3) = \begin{cases} (t-1)(\underline{x}^3)(t;\delta_3), & t \in [0,1], \\ (t-1)(\overline{x}^3)(t;\delta_3), & t \in (1,2], \end{cases} \\ (\overline{x}^3)'(t;\delta_3) = \begin{cases} (t-1)(\overline{x}^3)(t;\delta_3), & t \in [0,1], \\ (t-1)(\underline{x}^3)(t;\delta_3), & t \in (1,2], \end{cases} \\ \tilde{x}(0) = \prec [1+2\delta_1, 4-\delta_1], [1.5+0.5\delta_2, 2.5-0.5\delta_2], [2+\delta_3, 5-2\delta_3] \succ . \end{cases}$$

The exact 1-GHD solution can be obtained by solving the following system

$$\begin{cases} (\underline{x}^1)'(t;\delta_1) = \begin{cases} (t-1)(\overline{x}^1)(t;\delta_1), & t \in [0,1], \\ (t-1)(\underline{x}^1)(t;\delta_1), & t \in (1,2], \end{cases} \\ (\overline{x}^1)'(t;\delta_1) = \begin{cases} (t-1)(\underline{x}^1)(t;\delta_1), & t \in [0,1], \\ (t-1)(\overline{x}^1)(t;\delta_1), & t \in (1,2], \end{cases} \\ (\underline{x}^2)'(t;\delta_2) = \begin{cases} (t-1)(\overline{x}^2)(t;\delta_2), & t \in [0,1], \\ (t-1)(\underline{x}^2)(t;\delta_2), & t \in (1,2], \end{cases} \\ (\overline{x}^2)'(t;\delta_2) = \begin{cases} (t-1)(\underline{x}^2)(t;\delta_2), & t \in [0,1], \\ (t-1)(\overline{x}^2)(t;\delta_2), & t \in (1,2], \end{cases} \\ (\underline{x}^3)'(t;\delta_3) = \begin{cases} (t-1)(\overline{x}^3)(t;\delta_3), & t \in [0,1], \\ (t-1)(\underline{x}^3)(t;\delta_3), & t \in (1,2], \end{cases} \\ (\overline{x}^3)'(t;\delta_3) = \begin{cases} (t-1)(\underline{x}^3)(t;\delta_3), & t \in [0,1], \\ (t-1)(\overline{x}^3)(t;\delta_3), & t \in (1,2], \end{cases} \\ \tilde{x}(0) = \prec [1+2\delta_1, 4-\delta_1], [1.5+0.5\delta_2, 2.5-0.5\delta_2], [2+\delta_3, 5-2\delta_3] \succ . \end{cases}$$

For $0 \leq t \leq 1$, the exact 2-GHD solution is

$$\begin{aligned} [x(t)]^\delta = \prec & [(1+2\delta_1)e^{\frac{t^2-2t}{2}}, (4-\delta_1)e^{\frac{t^2-2t}{2}}], [(1.5+0.5\delta_2)e^{\frac{t^2-2t}{2}}, \\ & (2.5-0.5\delta_2)e^{\frac{t^2-2t}{2}}], [(2+\delta_3)e^{\frac{t^2-2t}{2}}, (5-2\delta_3)e^{\frac{t^2-2t}{2}}] \succ, \ \delta_1, \delta_2, \delta_3 \in [0,1]. \end{aligned}$$

and for $1 < t \leq 2$, the exact 1-GHD solution is

$$\begin{aligned} [X(t)]^\delta = \prec & [(1+2\delta_1)e^{\frac{t^2-2t}{2}}, (4-\delta_1)e^{\frac{t^2-2t}{2}}], [(1.5+0.5\delta_2)e^{\frac{t^2-2t}{2}}, (2.5-0.5\delta_2)e^{\frac{t^2-2t}{2}}], \\ & [(2+\delta_3)e^{\frac{t^2-2t}{2}}, (5-2\delta_3)e^{\frac{t^2-2t}{2}}] \succ, \ \delta_1, \delta_2, \delta_3 \in [0,1]. \end{aligned}$$

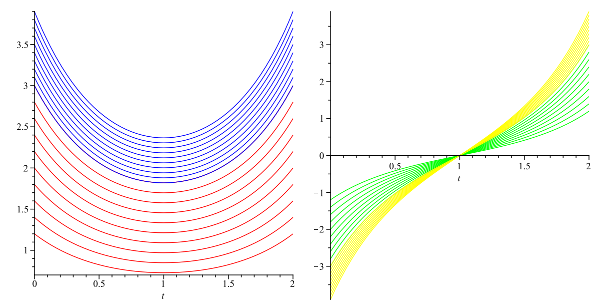
.

TABLE 1. Global truncation errors for $h = 0.01$ of Example 8.1.

t	EM	MEM
0	0	0
0.2	$7.62e^{-03}$	$1.02e^{-05}$
0.4	$1.20e^{-02}$	$1.31e^{-05}$
0.6	$1.50e^{-02}$	$1.32e^{-05}$
0.8	$1.75e^{-02}$	$1.26e^{-05}$
1	$2.02e^{-02}$	$1.23e^{-05}$
1.2	$2.37e^{-02}$	$1.250e^{-05}$
1.4	$2.88e^{-02}$	$1.28e^{-05}$
1.6	$3.63e^{-02}$	$1.25e^{-05}$
1.8	$4.79e^{-02}$	$9.47e^{-06}$
2	$6.66e^{-02}$	$9.47e^{-06}$

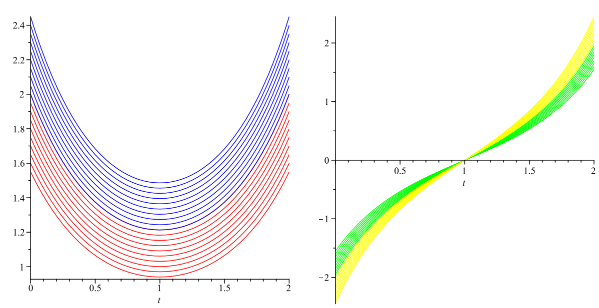
TABLE 2. Global truncation errors for $h = 0.001$ of Example 8.1.

t	EM	MEM
0	0	0
0.2	$7.57e^{-04}$	$1.05e^{-07}$
0.4	$1.20e^{-03}$	$1.31e^{-07}$
0.6	$1.49e^{-03}$	$1.36e^{-07}$
0.8	$1.75e^{-03}$	$1.26e^{-07}$
1	$2.02e^{-03}$	$1.25e^{-07}$
1.2	$2.37e^{-03}$	$1.26e^{-07}$
1.4	$2.88e^{-03}$	$1.35e^{-07}$
1.6	$3.64e^{-03}$	$1.30e^{-07}$
1.8	$4.81e^{-03}$	$1.03e^{-07}$
2	$6.66e^{-03}$	$2.0e^{-09}$



(a) (b)

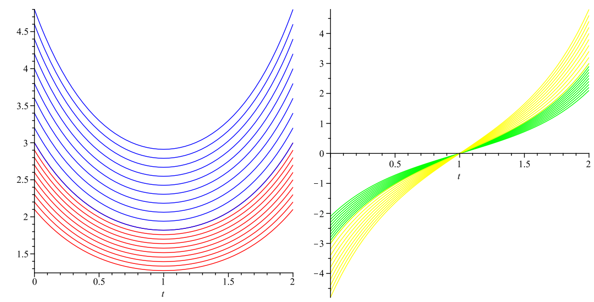
FIGURE 1. (a) Level sets of first component of the exact solution defined in Example 8.1. (b) Level sets of gH-derivative of first component of the solution defined in Example 8.1.



(a) (b)

FIGURE 2. (a) Level sets of the second component of exact solution defined in Example 8.1. (b) Level sets of gH-derivative of the second component of the solution defined in Example 8.1.

In Figure 1(a), red lines represent $\underline{x}^1(t;\delta_1)$ and blue lines represent $\overline{x}^1(t;\delta_1)$. In Figure 1(b), green lines represent $(\underline{x}^1)'(t;\delta_1)$ and yellow lines represent $(\overline{x}^1)'(t;\delta_1)$.



(a) (b)

FIGURE 3. (a) Level sets of the third component of exact solution defined in Example 8.1. (b) Level sets of gH-derivative of the third component of the solution defined in Example 8.1.

In Figure 2(a), red lines represent $\underline{x}^2(t;\delta_2)$ and blue lines represent $\overline{x}^2(t;\delta_2)$. In Figure 2(b), green lines represent $(\underline{x}^2)'(t;\delta_2)$ and yellow lines represent $(\overline{x}^2)'(t;\delta_2)$.

In Figure 3(a), red lines represent $\underline{x}^3(t;\delta_3)$ and blue lines represent $\overline{x}^3(t;\delta_3)$. In Figure 3(b), green lines represent $(\underline{x}^3)'(t;\delta_3)$ and yellow lines represent $(\overline{x}^3)'(t;\delta_3)$.

9. Conclusion

In many dynamical systems, we must deal with uncertainty and imprecision. An m-polar fuzzy set model is an extension of the fuzzy set model, and it is a powerful tool to deal with fuzzy and ambiguous problems in multidimensional problems. Compared with the fuzzy framework, this framework is more attractive to researchers. We have considered differential equations in in m-polar fuzzy environment. The fuzzy initial value problem has been extended to m-polar fuzzy initial value problem, where we have m-polar information about the unknown function and initial conditions. Different types of gH-differentiability of m-polar FVF are discussed. We have presented some results on gH-differentiability of m polar FVF. The Taylor expansion of the m-polar FVF is obtained by considering different types of differentiability. We have demonstrated the Euler and Modified Euler methods for m-polar fuzzy IVP. The consistency, stability, and convergence analysis of these methods are discussed. We have given some numerical examples to demonstrate the performance and efficiency of these methods. We calculated the global truncation error. We have seen that by decreasing the step size, the approximate solution converges to the exact solution. Furthermore, it is clear from the numerical results that the Modified Euler method gives better results compared to the Euler method. In the future, we plan to apply the predictor-corrector method to the m-polar fuzzy IVP based on Taylor expansion.

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