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An optimum solution for multi-dimensional distributed-order fractional differential equations

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Abstract

This manuscript investigates a computational method based on fractional-order Fibonacci functions (FFFs) for solving distributed-order (DO) fractional differential equations and DO time-fractional diffusion equations. Extra DO fractional derivative operator and pseudo-operational matrix of fractional integration for FFFs are proposed. To evaluate the unknown coefficients in the FFF expansion, utilizing the matrices, an optimization problem relating to considered equations is formulated. This approach converts the original problems into a system of algebraic equations. The approximation error is proposed. Several problems are proposed to investigate the applicability and computational efficiency of the scheme. The approximations achieved by some existing schemes are also tested conforming to the efficiency of the present method. Also, the model of the motion of the DO fractional oscillator is solved, numerically.

Keywords. Fractional-order Fibonacci functions, Optimization method, Extra distributed-order fractional derivative operator.
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1. Introduction

The idea of DO fractional derivative is raised by Caputo [7]. Then, the concept was extended by himself [8] and some mathematicians. This concept is used in the scientific model of some phenomena. Then, It has been considered by some researchers.

For example, the existence and the uniqueness of a class of DO equations are investigated in [2]. This equation is arisen in DO models of viscoelasticity and system identification theory. The existence of the solution of DO equations is investigated by Torvik and Bagley [3].

The authors in [5] perused the stability of linear DO fractional systems with distributed delays. Also, Saberi Najafi et al. analyzed the stability of DO of fractional differential equations [21].

However, there is a difficult task to solve the problems. So, it is necessary to develop an approximate algorithm is suggested. So far, some procedures dealing numerically with DO fractional equations have been presented.

Improved composite collocation method [24], Legendre wavelets method [31], finite difference/spectral-Galerkin method [15], meshless upwind local radial basis function-finite difference method [1], Müntz-Legendre wavelet method [19], the second kind Chebyshev wavelet method [25], Jacobi wavelet method [11], and Legendre hybrid function [12] are examples of numerical techniques proposed by some researches.

The diffusion equation can appear in the mathematical modeling of many phenomena in the economy, medical science, physics, etc. However, these equations are not effective for modeling abnormal diffusion processes in complex environments, because of the long profile in the spatial or memristive distribution. Also, fractional diffusion equations with the fractional derivative on the spatial derivative are used for studying Markovian processes. A study on a spatial fractional derivative was presented in [9]. In recent years, it has been found that the single discriminant order is not suitable for representing phenomena where the discriminant order varies within a certain range. Distributed order fractional differential equations were first introduced by Caputo [6]. The time-distributed order fractional equation can model processes that lack power-law scaling over the time domain, such as the ultraslow diffusion in which the

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particle cloud propagates with logarithmic velocity. The authors in [13] proposed a computational method based on the second kind Chebyshev wavelets and shifted fractional-order Jacobi polynomials to solve linear and nonlinear distributed order time-fractional diffusion equations.

Spatially distributed fractional-order equations offer more flexibility in representing medium actions and spatial interactions with fluids, such as acceleration of superdiffusion. Another class of these equations is the Riesz space distributed order equation which is solved and obtained second-order accuracy [30]. Also, a finite volume method was proposed for solving Riesz space distributed order diffusion equation [18]. A Crank-Nicolson finite volume approximation was presented for solving the nonlinear distributed order space-fractional diffusion equations in three space dimensions [32].

Now, given the above discussion, an optimization scheme based on fractional-order Fibonacci functions is developed to the solution of DO fractional differential equations (FDEs) and DO time-fractional diffusion equations.

Concerning the advantage of Fibonacci polynomials in [28], we consider a set of fractional functions named FFFs. The functions were introduced in [29] to define fractional order Fibonacci-hybrid functions. In this study, we apply FFFs for developing an efficient numerical technique for solving the following problems.

* DO fractional differential equation:

$$\int_a^b \tau(\gamma) D_t^{\gamma} u(t) d\gamma + \omega u(t) = G(t), \quad t \in [0,1], \tag{1.1}$$

subject to the following constraints

$$u^{(l)}(0) = u_{0,l}, \quad l = 0, 1, \cdots, \lceil b \rceil - 1. \tag{1.2}$$

* DO time-fractional diffusion equations

$$\int_a^b \tau(\gamma) \frac{\partial^{\gamma} \mathfrak{U}}{\partial t^{\gamma}}(x,t) = \frac{\partial^2 \mathfrak{U}}{\partial x^2}(x,t) + \mathfrak{G}(x,t), \quad (x,t) \in [0,1] \times [0,1], \tag{1.3}$$

subject to

$$\mathfrak{U}(x,0) = \varphi(x), \quad \mathfrak{U}(0,t) = \phi_0(t), \quad \mathfrak{U}(1,t) = \phi_1(t),$$

where, ω is constant, and D^{γ} shows the fractional derivative in the Caputo type which is defined in [26]. a and b are positive numbers. $\tau(\gamma)$ is a non-negative smooth weight function and [16]

$$\int_0^1 \tau(\gamma) \mathrm{d}\gamma = \mathfrak{C} > 0.$$

Fibonacci polynomials are some advantages that have been mentioned in different studies. For instance, the advantages of Fibonacci polynomials over shifted Legendre polynomials are listed in [27, 28]

- Fibonacci polynomials have fewer terms than the shifted Legendre polynomials, which reduces CPU time using Fibonacci polynomials.
- The coefficient of individual terms in Fibonacci polynomials is smaller than corresponding ones in shifted Legendre polynomials, which reduces CPU time using Fibonacci polynomials.
- The integration operational matrix of Fibonacci polynomials has less error than the same operational matrix for shifted Legendre polynomials.

Then, the properties of these polynomials are also inherited to the fractional-order Fibonacci functions. In addition, fractional-order polynomials have two degrees of freedom $(M; \alpha)$ but polynomials have one degree of freedom (M).

Also, in the procedure of deriving the Riemann-Liouville pseudo-operational matrix, the factor t^{γ} takes out from the approximation process which causes errors in calculations. This work helps to improve the accuracy of the technique. Then, features of the fractional-order Fibonacci functions and the pseudo-operational matrix create good conditions to get appropriate results.

On the other hand, the main idea of spectral methods is to express the solution of the equation as the sum of the basic functions and then derive the coefficients to minimize the error between the approximate and exact solutions in a suitable sense. Among the common types of spectral methods are the collocation method, Petrov-Galerkin method, and the tau method. Here, we present another appropriate scheme to achieve the coefficients which still guarantee the minimum value of approximation error. The use of the least square approximation method based on the Riemann-Liouville pseudo-operational matrix for solving considered a problem with the initial and boundary conditions. Indeed, converting the considered problems with their conditions into one system is another advantage of our technique because greatly decreases the computational calculation while maintaining a higher level of accuracy.

The rest of this manuscript comes as follows. Section 2 recalls the fundamental feature of Fibonacci polynomials and FFFs. Section 3 defines a pseudo-operational matrix of fractional integration and extra DO fractional operator for these functions. Section 4 suggests an algorithm to find the solution of two classes of problems. Error estimation is presented in section 5. Section 6 analyses the mentioned problems that illustrate the accuracy of the present scheme. Section 7 summarises the conclusions.

2. Functions

Here, we recall some concepts which are used in this study.

2.1. Fibonacci Polynomials. A natural generalization of Fibonacci numbers is the Fibonacci polynomials defined recursively by [4]

$$\begin{cases} F_0(x) = 1, \\ F_1(x) = x, \\ F_m(x) = xF_{m-1}(x) + F_{m-2}(x). \end{cases} \tag{2.1}$$

When $x = 1$, the initial conditions and recurrence simply generate the Fibonacci numbers [4]. Inductively, it is clear that $F_m(x)$ is an mth degree polynomial and therefore has as follows [29]

$$F_m(x) = \sum_{i=0}^{\lfloor m/2 \rfloor} \begin{pmatrix} m-i \\ i \end{pmatrix} x^{m-2i}, \quad m \geq 0. \tag{2.2}$$

2.2. Fractional-order Fibonacci functions. Given the definition Fibonacci polynomials in Eq. (2.2) and on the interval $[0, 1]$, FFFs are defined as follows [29]

$$F_m^{\alpha}(t) = \sum_{i=0}^{\lfloor m/2 \rfloor} \begin{pmatrix} m-i \\ i \end{pmatrix} t^{\alpha(m-2i)}, \quad m \geq 0, \tag{2.3}$$

where the mentioned functions are constructed explicitly utilizing the change of variable $t \to t^{\alpha} (\alpha > 0)$.

3. New achievements

The section is devoted to presenting pseudo-operational matrix of fractional integration and extra distributed-order fractional derivative operator for FFFs.

3.1. Fractional integral pseudo-operational matrix. Here, we extract an explicit pseudo-operational matrix related to fractional integration of FFFs. To this aim, let

$$\Phi_{\alpha}(t) = [F_0^{\alpha}(t), F_1^{\alpha}(t), \cdots, F_{M-1}^{\alpha}(t)]^T. \tag{3.1}$$

The Riemann-Liouville integral of the vector $\Phi_{\alpha}(t)$ in relation (3.1) can be represented as follows

$$\mathfrak{I}^{\gamma} \Phi_{\alpha}(t) \simeq \mathfrak{P}(\gamma, \alpha, t) \Phi_{\alpha}(t), \tag{3.2}$$

where $\mathfrak{P}(\gamma,\alpha,t)=t^{\gamma}\mathfrak{P}(\gamma,\alpha)$ is fractional integral pseudo-operational matrix for FFFs. $\mathfrak{P}(\gamma,\alpha)$ and $\mathfrak{P}(\gamma,\alpha,t)$ are derived in the following manner.

Due to equations (2.3)-(3.1) and properties of Riemann-Liouville fractional integral presented in [26], we get

$$\begin{aligned}\mathfrak{I}^{\gamma}F_m^{\alpha}(t) &= \sum_{i=0}^{\lfloor m/2\rfloor}\begin{pmatrix} m-i\\ i\end{pmatrix}\mathfrak{I}^{\gamma}t^{\alpha(m-2i)} \\ &= t^{\gamma}\sum_{i=0}^{\lfloor m/2\rfloor}\begin{pmatrix} m-i\\ i\end{pmatrix}\frac{\Gamma(\alpha(m-2i)+1)}{\Gamma(\alpha(m-2i)+1+\gamma)}t^{\alpha(m-2i)}, \quad m=0,1,\cdots,M-1.\end{aligned} \tag{3.3}$$

Now, by expanding $t^{\alpha(m-2i)}, i=0,1,\cdots,\lfloor m/2\rfloor,\ m=0,1,\cdots,M-1$, regarding FFFs, we have

$$t^{\alpha(m-2i)}\simeq\sum_{j=0}^{M-1}a_jF_j^{\alpha}(t),$$

so due to relation (3.3), we achieve

$$\begin{aligned}\mathfrak{I}^{\gamma}F_m^{\alpha}(t) &\simeq t^{\gamma}\sum_{i=0}^{\lfloor m/2\rfloor}\begin{pmatrix} m-i\\ i\end{pmatrix}\frac{\Gamma(\alpha(m-2i)+1)}{\Gamma(\alpha(m-2i)+1+\gamma)}\left(\sum_{j=0}^{M-1}a_jF_j^{\alpha}(t)\right) \\ &= t^{\gamma}\sum_{j=0}^{M-1}\left[\sum_{i=0}^{\lfloor m/2\rfloor}\begin{pmatrix} m-i\\ i\end{pmatrix}\frac{\Gamma(\alpha(m-2i)+1)}{\Gamma(\alpha(m-2i)+1+\gamma)}a_j\right]F_j^{\alpha}(t) \\ &= t^{\gamma}\sum_{j=0}^{M-1}\mathfrak{P}_j(\gamma,\alpha)F_j^{\alpha}(t)=t^{\gamma}\tilde{\mathfrak{P}}_m(\gamma,\alpha)\Phi_{\alpha}(t), \quad m=0,1,\cdots,M-1,\end{aligned} \tag{3.4}$$

in which

$$\mathfrak{P}_j(\gamma,\alpha)=\sum_{i=0}^{\lfloor m/2\rfloor}\begin{pmatrix} m-i\\ i\end{pmatrix}\frac{\Gamma(\alpha(m-2i)+1)}{\Gamma(\alpha(m-2i)+1+\gamma)}a_j,$$

and $\mathfrak{P}(\gamma,\alpha)=\left[\tilde{\mathfrak{P}}_m(\gamma,\alpha)\right], m=0,1,\cdots,M-1$. For example, for $M=3,\alpha=\gamma=\frac{1}{2}$, we have

$$\mathfrak{P}(\frac{1}{2},\frac{1}{2})=\begin{pmatrix}\frac{2}{\sqrt{\pi}} & 0 & 0\\ 0 & \frac{\sqrt{\pi}}{2} & 0\\ \frac{2}{3\sqrt{\pi}} & 0 & \frac{4}{3\sqrt{\pi}}\end{pmatrix}.$$

3.2. **Extra DO fractional derivative operator.** In this subsection, we present another necessary tool to propose an efficient numerical algorithm. For this purpose, we obtain an extra DO fractional derivative operator for FFFs in the following form:

$$\int_a^b\tau(\gamma)D_t^{\gamma}(t^{\gamma}\Phi_{\alpha}(t))\simeq\Theta^{\alpha}(\gamma,t), \tag{3.5}$$

where, $\Theta^{\alpha}(\gamma,t)=[\Theta_m^{\alpha}(\gamma,t)]_{M\times 1},\quad m=0,1,\cdots,M-1$.

Considering equation (2.3) and properties of Caputo derivative given in [26], the above integral can be computed each element of $\Theta^{\alpha}(\gamma,t)$ via an $\mathfrak{N}$-point Gauss-Legendre integration which is as follows

$$\begin{aligned}
\int_a^b \tau(\gamma) D_t^\gamma (t^\gamma F_m^\alpha(t)) \mathrm{d}\gamma &= \int_a^b \tau(\gamma) D_t^\gamma \Big(\sum_{i=0}^{\lfloor m/2 \rfloor} \begin{pmatrix} m-i \\ i \end{pmatrix} t^{\alpha(m-2i)+\gamma} \Big) \mathrm{d}\gamma \\
&= \int_a^b \tau(\gamma) \Big(\sum_{i=0}^{\lfloor m/2 \rfloor} \begin{pmatrix} m-i \\ i \end{pmatrix} D_t^\gamma t^{\alpha(m-2i)+\gamma} \Big) \mathrm{d}\gamma \\
&= \int_a^b \tau(\gamma) \Big(\sum_{i=0}^{\lfloor m/2 \rfloor} \begin{pmatrix} m-i \\ i \end{pmatrix} \frac{\Gamma(\alpha(m-2i)+\gamma+1)}{\Gamma(\alpha(m-2i)+1)} t^{\alpha(m-2i)} \Big) \mathrm{d}\gamma \\
&= \sum_{i=0}^{\lfloor m/2 \rfloor} \frac{t^{\alpha(m-2i)}}{\Gamma(\alpha(m-2i)+1)} \begin{pmatrix} m-i \\ i \end{pmatrix} \Big(\int_a^b \tau(\gamma) \Gamma(\alpha(m-2i)+\gamma+1) \mathrm{d}\gamma \Big) \\
&\simeq \sum_{i=0}^{\lfloor m/2 \rfloor} \frac{t^{\alpha(m-2i)}}{\Gamma(\alpha(m-2i)+1)} \begin{pmatrix} m-i \\ i \end{pmatrix} \Big[\sum_{\vartheta=1}^{\mathfrak{N}} \frac{b-a}{2} \omega_\vartheta \tau(\eta_\vartheta) \Gamma(\alpha(m-2i)+\eta_\vartheta+1) \Big] \\
&= \sum_{i=0}^{\lfloor m/2 \rfloor} \Big(\sum_{\vartheta=1}^{\mathfrak{N}} \frac{b-a}{2} \frac{t^{\alpha(m-2i)}}{\Gamma(\alpha(m-2i)+1)} \begin{pmatrix} m-i \\ i \end{pmatrix} \omega_\vartheta \tau(\eta_\vartheta) \Gamma(\alpha(m-2i)+\eta_\vartheta+1) \Big) \\
&= \Theta_m^\alpha(\gamma, t), \quad m = 0, 1, \cdots, M-1,
\end{aligned}$$

where ω_ϑ and η_ϑ are weights and nodes of Gauss-Legendre (for more details, see [10]).

4. Suggested technique

Here, we utilize the FFFs expansion and new achievements in the previous section for manufacturing an efficient computational technique for solving the multi-dimensional DO FDEs in equations (1.1)-(1.3).

- DO FDE

  In fact, assuming $b > \gamma$, for finding the solution of the expressed problem, we consider

$$D_t^b u(t) \simeq U^T \Phi_\alpha(t), \tag{4.1}$$

then using equations (3.2) and (1.3), the following relation is obtained.

$$u(t) \simeq U^T \mathfrak{P}(b, \alpha, t) \Phi_\alpha(t) + \sum_{l=0}^{\lceil b \rceil - 1} \frac{t^l}{l!} u_{0,l}, \tag{4.2}$$

and we get

$$\begin{aligned}
D_t^\gamma u(t) &\simeq U^T \mathfrak{P}(b-\gamma, \alpha, t) \Phi_\alpha(t) + \sum_{l=0}^{\lceil b \rceil - 1} \frac{D_t^\gamma (t^l)}{l!} u_{0,l} \\
&= t^{b-\gamma} U^T \mathfrak{P}(b-\gamma, \alpha) \Phi_\alpha(t) + \sum_{l=0}^{\lceil b \rceil - 1} \frac{D_t^\gamma (t^l)}{l!} u_{0,l},
\end{aligned} \tag{4.3}$$

inserting Eq. (4.3) in equation (1.1) and utilizing Eq. (3.5), we derive

$$\begin{aligned}
& \int_a^b \tau(\gamma)\Big(t^{b-\gamma}U^T\mathfrak{P}(b-\gamma,\alpha)\Phi_\alpha(t)+\sum_{l=0}^{\lceil b\rceil-1}\frac{D_t^\gamma(t^l)}{l!}u_{0,l}\Big)\mathrm{d}\gamma+\omega\Big(U^T\mathfrak{P}(b,\alpha,t)\Phi_\alpha(t)+\sum_{l=0}^{\lceil b\rceil-1}\frac{t^l}{l!}u_{0,l}\Big) \\
- \quad & G(t)\simeq U^T\mathfrak{P}(b-\gamma,\alpha)\int_a^b\tau(\gamma)(t^{b-\gamma}\Phi_\alpha(t))\mathrm{d}\gamma+\int_a^b\Big(\tau(\gamma)\sum_{l=0}^{\lceil b\rceil-1}\frac{D_t^\gamma(t^l)}{l!}u_{0,l}\Big)\mathrm{d}\gamma \\
+ \quad & \omega\Big(U^T\mathfrak{P}(b,\alpha,t)\Phi_\alpha(t)+\sum_{l=0}^{\lceil b\rceil-1}\frac{t^l}{l!}u_{0,l}\Big)-G(t) \\
\simeq \quad & U^T\mathfrak{P}(b-\gamma,\alpha)\Theta^\alpha(b-\gamma,t)-\frac{b-a}{2}\sum_{\vartheta=1}^{\mathfrak{N}}\omega_\vartheta\tau(\eta_\vartheta)\Big(\sum_{l=0}^{\lceil b\rceil-1}\frac{D_t^{\eta_\vartheta}(t^l)}{l!}u_{0,l}\Big) \\
+ \quad & \omega\Big(U^T\mathfrak{P}(b,\alpha,t)\Phi_\alpha(t)+\sum_{l=0}^{\lceil b\rceil-1}\frac{t^l}{l!}u_{0,l}\Big)-G(t) \\
= \quad & R(\alpha,t), \qquad (4.4)
\end{aligned}$$

in which $R(\alpha,t)\simeq 0$. In order to obtain an numerical solution of equations (1.1)-(1.2), we define 2-norm of the residual functions in the following form

$$\mathfrak{J}=\int_0^1 R^2(\alpha,t)\mathrm{d}t. \qquad (4.5)$$

The necessary conditions for the extreme $\mathfrak{J}$ are given as

$$\frac{\partial\mathfrak{J}}{\partial U}=0. \qquad (4.6)$$

The above system can be solved using the "FindRoot" package in *Mathematica* to derive the unknown coefficients vector.

- DO time-fractional diffusion equations

We assume $b=1$. $\frac{\partial^3\mathfrak{U}(x,t)}{\partial x^2\partial t}$ can be expanded by FFFs as

$$\frac{\partial^3\mathfrak{U}(x,t)}{\partial x^2\partial t}\simeq {}_M\Phi_\alpha^T(x)\,\mathcal{U}\,{}_{\tilde M}\Phi_{\tilde\alpha}(t), \qquad (4.7)$$

where, ${}_M\Phi_\alpha(x)$ and ${}_{\tilde M}\Phi_{\tilde\alpha}(t)$ are the vectors of $M\times 1$ and $\tilde M\times 1$-order, respectively. Next, in view of the pseudo-operational matrix of fractional integration, we have

$$\frac{\partial^2\mathfrak{U}(x,t)}{\partial x^2}\simeq {}_M\Phi_\alpha^T(x)\,\mathcal{U}\,\mathfrak{P}(1,\alpha,t)\,{}_{\tilde M}\Phi_{\tilde\alpha}(t)+\varphi''(x), \qquad (4.8)$$

and

$$\frac{\partial\mathfrak{U}(x,t)}{\partial x}\simeq {}_M\Phi_\alpha^T(x)\,\mathfrak{P}^T(1,\alpha,x)\mathcal{U}\,\mathfrak{P}(1,\alpha,t)\,{}_{\tilde M}\Phi_{\tilde\alpha}(t)+\varphi'(x)-\varphi'(0)+\frac{\partial\mathfrak{U}(0,t)}{\partial x}, \qquad (4.9)$$

in which, $\frac{\partial\mathfrak{U}(0,t)}{\partial x}$ is unknown function.

By integrating the aforesaid function respect to x, the following relations are derived.

$$\mathfrak{U}(x,t)\simeq {}_M\Phi_\alpha^T(x)\mathfrak{P}^T(2,\alpha,x)\,\mathcal{U}\,\mathfrak{P}(1,\alpha,t)\,{}_{\tilde M}\Phi_{\tilde\alpha}(t)+\varphi(x)-\varphi(0)-x\varphi'(0)+x\frac{\partial\mathfrak{U}(0,t)}{\partial x}+\phi_0(t), \qquad (4.10)$$

and

$$\frac{\partial \mathfrak{U}(0,t)}{\partial x} \simeq \phi_1(t) - \phi_0(t) - \Big(\Lambda \mathcal{U}\, \mathfrak{P}(1,\alpha,t)\ {}_{\tilde{M}}\Phi_{\tilde{\alpha}}(t) + \varphi(1) - \varphi(0) - \varphi^{'}(0)\Big), \tag{4.11}$$

where, $\Lambda = \int_0^1 \ {}_M\Phi_{\alpha}^T(x)\mathfrak{P}^T(1,\alpha,x)\mathrm{d}x$. Also, we have

$$\frac{\partial \mathfrak{U}(x,t)}{\partial t} \simeq \ {}_M\Phi_{\alpha}^T(x)\mathfrak{P}^T(2,\alpha,x)\mathcal{U}\ {}_{\tilde{M}}\Phi_{\tilde{\alpha}}(t) + \phi_0^{'}(t), \tag{4.12}$$

and

$$\frac{\partial^{\gamma} \mathfrak{U}(x,t)}{\partial t^{\gamma}} \simeq \ {}_M\Phi_{\alpha}^T(x)\mathfrak{P}^T(2,\alpha,x)\mathcal{U}\mathfrak{P}^T(1-\gamma,\alpha,t)\ {}_{\tilde{M}}\Phi_{\tilde{\alpha}}(t) + \mathfrak{I}_t^{1-\gamma}(\phi_0^{'}(t)). \tag{4.13}$$

Regarding to the pervious section and with the help of the aforesaid equations, we obtain

$$\begin{aligned}
\int_a^b \tau(\gamma)\frac{\partial^{\gamma}\mathfrak{U}}{\partial t^{\gamma}}(x,t)\mathrm{d}\gamma \ &\simeq\ \int_a^b \tau(\gamma)D_t^{\gamma}\Big(\ {}_M\Phi_{\alpha}^T(x)\mathfrak{P}^T(2,\alpha,x)\mathcal{U}\mathfrak{P}(1,\alpha,t)_{\tilde{M}}\Phi_{\tilde{\alpha}}(t) + \varphi(x) - \varphi(0) - x\varphi^{'}(0)\\
&+\ x\frac{\partial \mathfrak{U}(0,t)}{\partial x} + \phi_0(t)\Big)\mathrm{d}\gamma\\
&\simeq\ {}_M\Phi_{\alpha}^T(x)\mathfrak{P}^T(2,\alpha,x)\mathcal{U}\mathfrak{P}(1,\alpha)\int_a^b \tau(\gamma)D_t^{\gamma}(t\ {}_{\tilde{M}}\Phi_{\tilde{\alpha}}(t))\mathrm{d}\gamma + \varphi(x) - \varphi(0) - x\varphi^{'}(0)\\
&+\ x\int_a^b \tau(\gamma)D_t^{\gamma}\frac{\partial \mathfrak{U}(0,t)}{\partial x}\mathrm{d}\gamma + \int_a^b \tau(\gamma)D_t^{\gamma}\phi_0(t)\mathrm{d}\gamma\\
&\simeq\ {}_M\Phi_{\alpha}^T(x)\mathfrak{P}^T(2,\alpha,x)\mathcal{U}\mathfrak{P}(1,\alpha)\Theta^{\alpha}(1,t) + \varphi(x) - \varphi(0) - x\varphi^{'}(0)\\
&+\ x\Big[\int_a^b \tau(\gamma)D_t^{\gamma}\phi_1(t)\mathrm{d}\gamma - \int_a^b \tau(\gamma)D_t^{\gamma}\phi_0(t)\mathrm{d}\gamma\\
&-\ \Big(\Lambda\mathcal{U}\int_a^b \tau(\gamma)D_t^{\gamma}\mathfrak{P}(1,\alpha,t)_{\tilde{M}}\Phi_{\tilde{\alpha}}(t)\mathrm{d}\gamma + \varphi(1) - \varphi(0) - \varphi^{'}(0)\Big)\Big]\\
&+\ \int_a^b \tau(\gamma)D_t^{\gamma}\phi_0(t)\mathrm{d}\gamma,
\end{aligned}$$

then, by implementing Gauss-Legendre formula, we get

$$\begin{aligned}
\int_a^b \tau(\gamma)\frac{\partial^{\gamma}\mathfrak{U}}{\partial t^{\gamma}}(x,t)\mathrm{d}\gamma \ &\simeq\ {}_M\Phi_{\alpha}^T(x)\mathfrak{P}^T(2,\alpha,x)\mathcal{U}\mathfrak{P}(1,\alpha)\Theta^{\alpha}(1,t) + \varphi(x) - \varphi(0) - x\varphi^{'}(0)\\
&+\ x\Big[\sum_{\vartheta=1}^{\mathfrak{N}} -\frac{b-a}{2}\omega_{\vartheta}\tau(\eta_{\vartheta})D_t^{\eta_{\vartheta}}\phi_1(t) - \sum_{\vartheta=1}^{\mathfrak{N}} -\frac{b-a}{2}\omega_{\vartheta}\tau(\eta_{\vartheta})D_t^{\eta_{\vartheta}}\phi_0(t)\\
&-\ \Big(\Lambda\mathcal{U}\mathfrak{P}(1,\alpha)\Theta^{\alpha}(1,t)\gamma + \varphi(1) - \varphi(0) - \varphi^{'}(0)\Big)\Big]\\
&+\ \sum_{\vartheta=1}^{\mathfrak{N}} -\frac{b-a}{2}\omega_{\vartheta}\tau(\eta_{\vartheta})D_t^{\eta_{\vartheta}}\phi_0(t)\\
&=\ \Upsilon(\alpha,x,t).
\end{aligned} \tag{4.14}$$

Consequently, we substitute equations (4.7)-(4.14) into equation (1.3). Then, we derive

$$\Upsilon(\alpha,x,t) - \Big(\ {}_M\Phi_{\alpha}^T(x)\mathcal{U}\mathfrak{P}(1,\alpha,t)_{\tilde{M}}\Phi_{\tilde{\alpha}}(t) + \varphi^{''}(x)\Big) - \mathfrak{G}(t) = \mathfrak{J}^*(x,t). \tag{4.15}$$

The 2-norm of the residual functions for the above relation can be expressed as

$$\mathfrak{J}^\star(\mathcal{U}) = \int_0^1 \int_0^1 \mathfrak{J}^{*2}(x,t)\mathrm{d}x\mathrm{d}t. \tag{4.16}$$

The necessary conditions for the extreme $\mathfrak{J}^\star(\mathcal{U})$ are given as

$$\frac{\partial \mathfrak{J}^\star(\mathcal{U})}{\partial \mathcal{U}} = 0. \tag{4.17}$$

Similar to the previous system, we solve this via "FindRoot" package.

5. Error estimation

A function $f \in L^2[0,\ 1]$ can be expanded as

$$f(t) \simeq \sum_{i=0}^{M-1} c_i F_i^\alpha(t) = C^T \Phi_\alpha(t) := f_M(t).$$

Then, we have error function $\widehat{E}(t)$ as follows:

$$\widehat{E}(t) = |f(t) - f_M(t)|, \quad t \in [0,\ 1] \tag{5.1}$$

Theorem 5.1. *Suppose that $D^{m\alpha} f \in C(0,1]$, $m = 0,\ 1,\ \cdots,\ M-1$ and $Y_M^\alpha = \{F_0^\alpha(t),\ F_1^\alpha(t),\ \cdots,\ F_{M-1}^\alpha(t)\}$. If $f_M(x)$ is the best approximation to $f(t)$ out of Y_M^α, then the error bound of the approximate solution $f_M(t)$ by using FFFs series would be obtained as follows:*

$$\|f - f_M\|_2 \leq \frac{\mathfrak{M}_\alpha}{\Gamma(M\alpha+1)\sqrt{2M\alpha+1}}, \tag{5.2}$$

where $\mathfrak{M}_\alpha = \sup_{t\in[0,1]} |D^{M\alpha} f(t)|$.

Proof. We define

$$\tilde{f}(t) = \sum_{i=0}^{M-1} \frac{t^{m\alpha}}{\Gamma(m\alpha+1)} D^{m\alpha} f(0^+). \tag{5.3}$$

From the generalized Taylors formula [22], we have

$$|f(t) - \tilde{f}(t)| \leq \frac{t^{M\alpha}}{\Gamma(M\alpha+1)} \sup_{t\in[0,1]} |D^{M\alpha} f(t)|. \tag{5.4}$$

Since, $\tilde{f}(t)$ is the best approximation of f out of Y_M^α, $\tilde{f}(t) \in Y_M^\alpha$ and from the above relation, we have

$$\begin{aligned}
\|f - f_M\|_2^2 &\leq \|f - \tilde{f}\|_2^2 = \int_0^1 |f(t) - \tilde{f}(t)|^2 dx \\
&\leq \int_0^1 \frac{t^{2M\alpha}}{\Gamma(M\alpha+1)^2} \mathfrak{M}_\alpha^2 \mathrm{d}t \\
&= \frac{\mathfrak{M}_\alpha^2}{\Gamma(M\alpha+1)^2} \int_0^1 t^{2M\alpha} \mathrm{d}t \\
&= \frac{\mathfrak{M}_\alpha^2}{\Gamma(M\alpha+1)^2(2M\alpha+1)},
\end{aligned} \tag{5.5}$$

the theorem is proved by taking the square roots.

□

Recent theorem proves convergence of approximations of FFFs to $f(t)$.

Theorem 5.2. *Let $D^{m\alpha+\gamma}f \in C(0,1], \quad m = 0,\ 1,\ \cdots,\ M-1$. If $(D^{\gamma}f)_M(t)$ is the best approximation to $D^{\gamma}f(t)$ from Y_M^{α}, then*

$$\|D^{\gamma}f - (D^{\gamma}f)_M\|_2 \leq \frac{\mathcal{M}_{\alpha}^{\gamma}}{\Gamma(M\alpha+1)\sqrt{2M\alpha+1}}, \tag{5.6}$$

where $\mathcal{M}_{\alpha}^{\gamma} = \sup_{t\in[0,1]} |D^{M\alpha+\gamma}f(t)|$.

Proof. Due to the concept of the best approximation, $\forall \tilde{f} \in Y_M^{\alpha}$, we have

$$\|D^{\gamma}f - (D^{\gamma}f)_M\|_2 \leq \|D^{\gamma}f - D^{\gamma}\tilde{f}\|_2,$$

considering the generalized Taylor formula $(\tilde{f})$, we get

$$|D^{\gamma}f(t) - D^{\gamma}\tilde{f}(t)| = \left| f(t) - \sum_{i=0}^{M-1} \frac{t^{m\alpha}}{\Gamma(m\alpha+1)} D^{m\alpha}\left(D^{\gamma}f\right)(0^{+}) \right| \leq \mathcal{M}_{\alpha}^{\gamma} \frac{t^{M\alpha}}{\Gamma(M\alpha+1)},$$

where $\mathcal{M}_{\alpha}^{\gamma} = \sup_{t\in[0,1]} |D^{M\alpha+\gamma}f(t)|$. Taking $L^2[0,1]$-norm in both sides of the above inequality leads to

$$\begin{aligned} \|D^{\gamma}f - D^{\gamma}\tilde{f}\|_2^2 &\leq \frac{(\mathcal{M}_{\alpha}^{\gamma})^2}{\Gamma(M\alpha+1)^2} \int_0^1 t^{2M\alpha}\mathrm{d}t \\ &= \frac{(\mathcal{M}_{\alpha}^{\gamma})^2}{\Gamma(M\alpha+1)^2(2M\alpha+1)}. \end{aligned}$$

The theorem is proved by taking the square roots. □

Canuto et al. [10] defined the Sobolev norm of integer order $s \geq 0$ in the domain $(a,b)^d$ in $R^d, d = 2,3$ as

$$\|u\|_{H^s(a,b)} = \left(\sum_{j=0}^{s} \sum_{i=1}^{d} \|D_i^j u\|_{L^2(a,b)}^2 \right)^{\frac{1}{2}}, \tag{5.7}$$

subject to D_i^j shows the jth derivative of u relative to the ith variable.

Theorem 5.3. *Suppose that $u \in H^s(0,1), s \geq 0$. If*

$$P_{\alpha,M}u = \sum_{m=0}^{M-1} u_m F_m^{\alpha}(t),$$

is the best approximation of the function u, so the following estimations are achieved.

$$\|u - P_{\alpha,M}u\|_{L^2(0,1)} \leq c(\alpha(M-1))^{1-s} \left(\sum_{j=\min(s,(M-1)\alpha+1)}^{s} \sum_{i=1}^{d} \|D_i^j u\|_{L^2(0,1)}^2 \right)^{\frac{1}{2}}, \tag{5.8}$$

and for $1 \leq r \leq s$, we have

$$\|u - P_{\alpha,M}u\|_{H^r(0,1)} \leq c(\alpha(M-1))^{\mu(r)-s} \left(\sum_{j=\min(s,(M-1)\alpha+1)}^{s} \sum_{i=1}^{d} \|D_i^j u\|_{L^2(0,1)}^2 \right)^{\frac{1}{2}}, \tag{5.9}$$

where

$$\mu(r) = \begin{cases} 0, & r = 0, \\ 2r - \frac{1}{2}, & r > 0, \end{cases}$$

and c depends on s.

Proof. Concerning the results proposed by Canuto [10] and uniqueness of the best approximation [17], the following relation holds:

$$\|u - P_{\alpha,M}u\|_{L^2(0,1)} = \|u - P_{M\alpha}u\|_{L^2(0,1)} \leq c(\alpha(M-1))^{1-s}\Bigg(\sum_{j=\min(s,(M-1)\alpha+1)}^{s}\sum_{i=1}^{d}\|D_i^j u\|^2_{L^2(0,1)}\Bigg)^{\frac{1}{2}},$$

and for $1 \leq r \leq s$, we get

$$\|u - P_{\alpha,M}u\|_{H^r(0,1)} = \|u - P_{\alpha M}u\|_{H^r(0,1)} \leq c(\alpha(M-1))^{\mu(r)-s}\Bigg(\sum_{j=\min(s,(M-1)\alpha+1)}^{s}\sum_{i=1}^{d}\|D_i^j u\|^2_{L^2(0,1)}\Bigg)^{\frac{1}{2}}.$$

Hence, the mentioned results are established. □

6. Numerical experiments

Here, we include some numerical experiments for various choices of α, γ, and M to demonstrate the effectiveness of the developed scheme. The computations associated with the examples were performed using Mathematica 12.3 on a 2.67GHz Corei5 personal computer with 4GB of RAM.

Example 6.1. Consider the following DO FDE [23]

$$\int_0^1 \frac{\Gamma(\frac{7}{2}-\gamma)}{\Gamma(\frac{7}{2})} D^\gamma u(t)\mathrm{d}\gamma = \frac{t^{\frac{3}{2}}(t-1)}{ln(t)},$$

where $u(0) = 0$. The analytical solution of the example is $u(t) = t^{\frac{5}{2}}$. The proposed technique is utilized to solve this example for some values of M and α. Table 1 lists the maximum absolute errors of the suggested technique for $M = 3$ and $\alpha = 1, \frac{1}{2}$. A comparison is performed in this table between the reported results using the method based on Legendre polynomials [23]. Also, in this table we assess the CPU time (in seconds) of the proposed method. Moreover, Figure 1 shows the absolute error of the scheme for $M = 5$ and $\alpha = \frac{1}{2}$. Besides, to demonstrate the stability of the method, we make a small change in the initial condition ($u(0) = \delta$). By implementing the present scheme for various values of δ, we report the maximum errors in Table 2. From this table, it can be observed that by creating small changes in the problem, the major error has not been created in the results.

The results demonstrate confirm the high accuracy of the present scheme and we can achieve an excellent approximation for the exact solutions by applying a limited number of basis functions.

Example 6.2. Consider the following DO FDE

$$\int_0^2 \Gamma(4-\gamma)\sinh(\gamma)D^\gamma u(t)\mathrm{d}\gamma = \frac{6t(t^2 - \cosh(2) - \sinh(2)ln(t)}{(ln(t))^2 - 1}),$$

with the initial conditions

$$u(0) = u^{'}(0) = 0.$$

The exact solution is $u(t) = t^3$. The comparison of absolute errors of the suggested algorithm with fractional Chelyshkov wavelet method (FChW) [24] are listed in Table 3. Also, the CPU time of the present method are reported in this table. This table shows that we obtain more accurate results than the FChW.

TABLE 1. Comparison of the L_2 errors in Example 1.

Methods	L_2 errors	CPU
[23]		
$m = 4$	2.34×10^{-4}	–
$m = 6$	2.73×10^{-5}	–
$m = 10$	1.84×10^{-6}	–
$m = 12$	7.00×10^{-7}	–
Present method		
$M = 4, \alpha = 1$	6.18088×10^{-5}	10^{-4}
$M = 3, \alpha = \frac{1}{2}$	1.01748×10^{-15}	10^{-3}

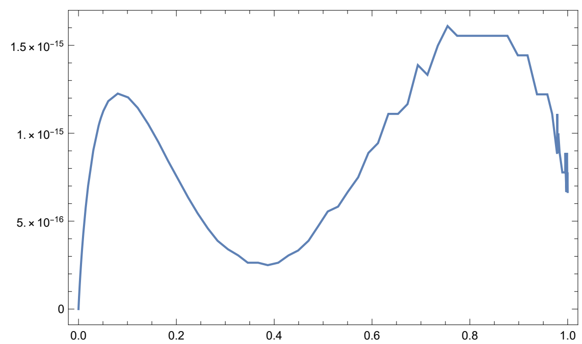

FIGURE 1. Absolute error for $M = 3, \alpha = \frac{1}{2}$ in Example 1.

TABLE 2. Maximum errors with $M = 5, \alpha = \frac{1}{2}$ and different values of δ in Example 1.

$\delta = 10^{-10}$	$\delta = 10^{-12}$	$\delta = 10^{-14}$
1.85866×10^{-10}	1.85651×10^{-12}	1.75415×10^{-14}

TABLE 3. Comparison of the absolute errors for several values of α, M for Example 2.

t	FChW [24] $\lambda = 1, k = 2, M = 3$	FChW [24] $\lambda = 1, k = 2, M = 4$	Present method $\alpha = 1, M = 2$	Present method $\alpha = \frac{1}{2}, M = 3$
0.1	4.84×10^{-9}	4.59×10^{-12}	2.69×10^{-16}	5.99×10^{-16}
0.3	1.60×10^{-8}	9.57×10^{-12}	2.58×10^{-15}	2.19×10^{-15}
0.5	2.09×10^{-8}	1.45×10^{-11}	7.58×10^{-15}	1.08×10^{-15}
0.7	3.46×10^{-8}	4.39×10^{-13}	1.12×10^{-14}	1.16×10^{-15}
0.9	4.19×10^{-8}	5.94×10^{-12}	2.73×10^{-14}	1.59×10^{-14}
CPU	–	–	10^{-2}	0.109

Example 6.3. Consider the following DO FDE which is described the motion of the DO fractional oscillator [31]

$$u^{''}(t) + \sigma(t) + w^2 u(t) = G_0 \sin(\Omega t),$$

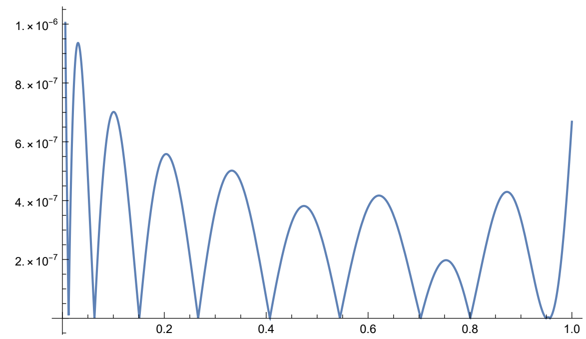

FIGURE 2. The residual error of Example 3 for $a = b = 2$ and $\alpha = 1, M = 4$.

$$\int_0^1 \mathsf{a}^\gamma D^\gamma \sigma(t)\mathrm{d}\gamma = \lambda \int_0^1 \mathsf{b}^\gamma D^\gamma u(t) d\gamma,$$

subject to

$$u(0) = u^{'}(0) = 0.$$

Let $\Omega = 1.2w, w = 3, w_* = \sqrt{10}$, and $G_0 = \lambda = 1$. The residual error of this problem for $a = b = 2$ and $\alpha = 1, M = 12$ are displayed in Figure 2. This figure shows that the approximate solution has a good agreement with the exact solution.

Example 6.4. Consider the following two dimensional DO FDE

$$\int_0^1 (\Gamma(\frac{5}{2} - \gamma)\frac{\partial^\gamma \mathfrak{U}}{\partial t^\gamma}(x,t)\mathrm{d}\gamma = \frac{\partial^2 \mathfrak{U}}{\partial x^2}(x,t) + \mathfrak{G}(x,t),$$

where

$$\mathfrak{G}(x,t) = \frac{\sqrt{t}(x-1)^2\Big(3\sqrt{\pi}(t-1)(x-1)^2x^2 - 8t(5x(3x-2)+1)ln(t)\Big)}{ln(t)},$$

and

$$\mathfrak{U}(x,0) = 0, \quad \mathfrak{U}(0,t) = \mathfrak{U}(1,t) = 0.$$

The analytical solution of the mentioned problem is $\mathfrak{U}(x,t) = t^{\frac{3}{2}}x^2(1-x)^4$. To solve this problem, we select some values of $M, \tilde{M}, \alpha,$ and $\tilde{\alpha}$ using the proposed technique. The absolute errors of this scheme with $M = 5, \tilde{M} = 3, \alpha = \tilde{\alpha} = 1$, $M = 5, \tilde{M} = 2, \alpha = 1, \tilde{\alpha} = \frac{1}{2}$, and $M = \tilde{M} = 5, \alpha = 1, \tilde{\alpha} = \frac{1}{4}$, are displayed in Figure 3. In addition, Table 4 compares the values of L_∞-errors of the proposed scheme with Finite difference method [14], and Chebyshev collocation method [20]. Moreover, CPU times of the proposed technique are reported in this table. Then, the quality of the results achieved utilizing the proposed technique can be seen numerically and graphically from Table 4 and Figure 3, respectively.

Example 6.5. Consider the following two dimensional DO FDE

$$\int_0^1 (\Gamma(\frac{7}{2} - \gamma)\frac{\partial^\gamma \mathfrak{U}}{\partial t^\gamma}(x,t)\mathrm{d}\gamma = \frac{\partial^2 \mathfrak{U}}{\partial x^2}(x,t) + \mathfrak{G}(x,t),$$

where

$$\mathfrak{G}(x,t) = \frac{t^{\frac{3}{2}}\Big(15\sqrt{\pi}(t-1)(x-1)^2x + 16t(2-3x)ln(t)\Big)}{8ln(t)},$$

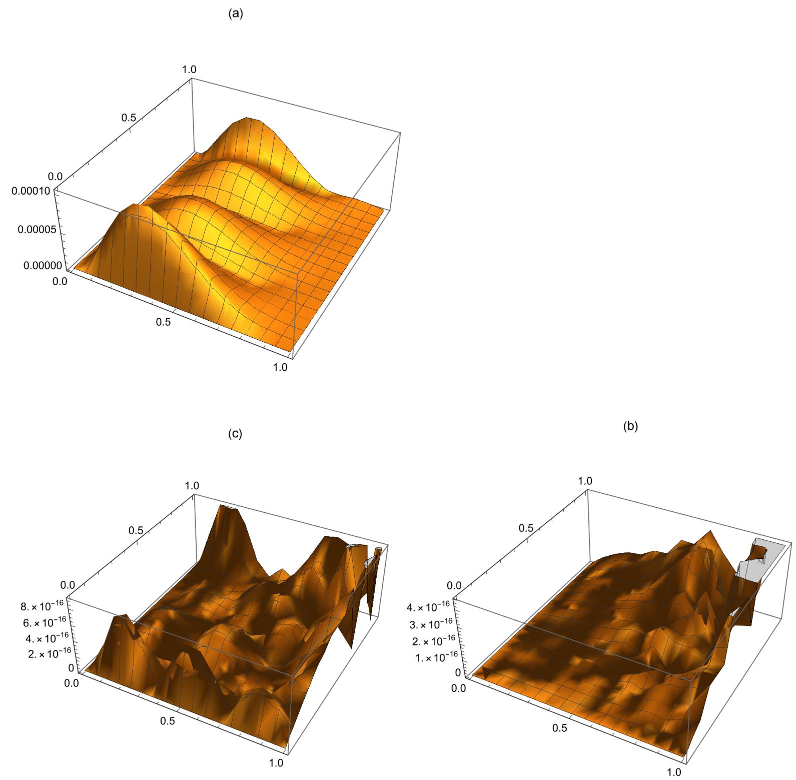

FIGURE 3. Absolute error for (a) $M = 5, \tilde{M} = 3, \alpha = \tilde{\alpha} = 1$, (b) $M = 5, \tilde{M} = 2, \alpha = 1, \tilde{\alpha} = \frac{1}{2}$, and (c) $M = \tilde{M} = 5, \alpha = 1, \tilde{\alpha} = \frac{1}{4}$ in Example 4.

TABLE 4. Comparison of L_∞ errors for several values of α, M with some other methods for Example 4.

	L_∞ errors	CPU
Finite difference method [14]		
$h = 0.5$	8.40×10^{-3}	–
$h = 0.25$	2.45×10^{-3}	–
$h = 0.125$	6.36×10^{-4}	–
$h = 0.0625$	1.62×10^{-4}	–
Chebyshev collocation method [20]		
$n = m = 5$	1.21×10^{-3}	–
$n = m = 7$	1.06×10^{-5}	–
$n = m = 9$	4.47×10^{-6}	–
$n = m = 11$	1.69×10^{-6}	–
Present method		
$M = 5, \tilde{M} = 2, \alpha = 1, \tilde{\alpha} = \frac{1}{2}$	1.50×10^{-15}	0.22
$M = \tilde{M} = 5, \alpha = 1, \tilde{\alpha} = \frac{1}{4}$	8.88×10^{-16}	0.37

TABLE 5. Comparison of the absolute errors for $M = 2, \tilde{M} = 4, \alpha = 1$, and $\tilde{\alpha} = \frac{1}{2}$ with Chelyshkov wavelet method [24] for Example 5.

(x,t)	Chelyshkov wavelet method [24] $k = k^{'} = 1, M = M^{'} = 4, \lambda = \lambda^{'} = \frac{1}{2}$	Chelyshkov wavelet method [24] $k = k^{'} = 1, M = 3, M^{'} = 4, \lambda = \lambda^{'} = \frac{1}{2}$	Present method $M = 2, \tilde{M} = 4, \alpha = 1$, and $\tilde{\alpha} = \frac{1}{2}$
$(0.2, 0.2)$	1.71×10^{-15}	6.51×10^{-18}	3.90×10^{-18}
$(0.4, 0.4)$	1.04×10^{-17}	2.78×10^{-17}	3.47×10^{-18}
$(0.6, 0.6)$	3.33×10^{-16}	1.91×10^{-16}	5.55×10^{-17}
$(0.8, 0.8)$	2.15×10^{-16}	3.54×10^{-16}	5.55×10^{-17}
CPU	–	–	0.64

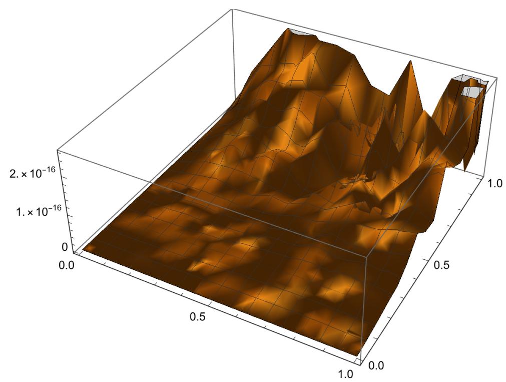

FIGURE 4. Absolute error of the present method for $M = 2, \tilde{M} = 4, \alpha = 1$, and $\tilde{\alpha} = \frac{1}{2}$, in Example 5.

and

$$\mathfrak{U}(x, 0) = 0, \quad \mathfrak{U}(0, t) = \mathfrak{U}(1, t) = 0.$$

The exact solution of this problem is $\mathfrak{U}(x,t) = t^{\frac{5}{2}} x(1-x)^2$. To solve this problem, we select values of $M = 2, \tilde{M} = 4, \alpha = 1$, and $\tilde{\alpha} = \frac{1}{2}$ using the technique. The absolute errors of this method (togethed CPU time of the method) are compared to the values of absolute errors of our method with Chelyshkov wavelet method [24] in Table 5. Figure 4 displays the absolute error obtained by the suggested scheme. Besides, the absolute error of Chebyschev collocation method in $n = m = 10$ is plotted in [20]. From this figure and Figure 4, we see that we can obtain a reasonable approximation with the analytical solution of this problem.

Example 6.6. Consider the following two dimensional DO FDE

$$\int_0^1 (\Gamma(3-\gamma) \frac{\partial^\gamma \mathfrak{U}}{\partial t^\gamma}(x,t) \mathrm{d}\gamma = \frac{\partial^2 \mathfrak{U}}{\partial x^2}(x,t) + \mathfrak{G}(x,t),$$

where

$$\mathfrak{G}(x,t) = \frac{2t(t-1)(1-x)\cos(x)}{ln(t)} - 2t^2 \sin(x) + t^2(1-x)\cos(x),$$

and

$$\mathfrak{U}(x,0) = 0, \quad \mathfrak{U}(0,t) = t^2, \mathfrak{U}(1,t) = 0.$$

The exact solution of this problem is $\mathfrak{U}(x,t) = t^2(1-x)\cos(x)$. We solve this problem using the present scheme. Figure 5 shows the approximate solution and the plot of the absolute error obtained with $M = 2, \tilde{M} = 4, \alpha = 1$, and $\tilde{\alpha} = 1$. This figure shows that we obtained a numerical solution that has good agreement with the analytical solution.

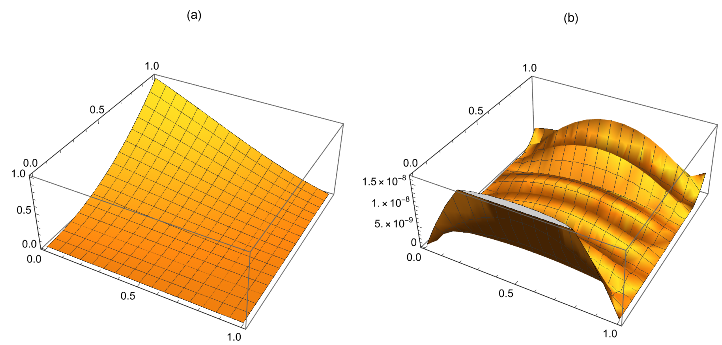

Figure 5. (a) The approximate solution and (b) the absolute error of the present method for $M = 2, \tilde{M} = 7, \alpha = 1$, and $\tilde{\alpha} = 1$, in Example 6.

7. Conclusion

In this manuscript, the fractional-order Fibonacci functions and their features are implemented to derive a numerical solution of DO fractional differential equations and DO time-fractional diffusion equations. An extra DO fractional derivative operator and pseudo-operational matrix of fractional integration for FFFs are proposed. The algorithm converts the considered problem to a system of algebraic equations in order to achieve the unknown coefficients optimally. Moreover, a set of numerical tests has been proposed. The numerical results are compared with analytical solutions and some previous techniques. The examples show the effectiveness and efficiency of the method.

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