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A novel analytical approximation approach for strongly nonlinear oscillation systems based on the energy balance method and He's Frequency-Amplitude formulation

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Abstract

Nonlinear oscillations are an essential fact in physical science, mechanical structures, and other engineering problems. Some of the popular analytical solutions to analyze nonlinear differential equations governing the behavior of strongly nonlinear oscillators are the Energy Balance Method (EBM), and He's Amplitude Frequency Formulation (HAFF). The lack of precision and accuracy despite needing several computational steps to resolve the system frequency is the main demerit of these methods. This research creates a novel analytical approximation approach with a very efficient algorithm that can perform the calculation procedure much easier and with much higher accuracy than classic EBM and HAFF. The presented method's steps rely on Hamiltonian relations described in EBM and the defined relationship between frequency and amplitude in HAFF. This paper demonstrates the substantial precision of the presented method compared to common EBM and HAFF applied in different and well-known engineering phenomena. For instance, the approximate solutions of the equations govern some strongly nonlinear oscillators, including the two-massspring systems, buckling of a column, and duffing relativistic oscillators are presented here. Subsequently, their results are compared with the Runge-Kutta method and exact solutions obtained from the previous research. The proposed novel approach resultant error percentages show an excellent agreement with the numerical solutions and illustrate a very quickly convergent and more precise than mentioned typical methods.

Keywords. Efficient algorithm, Nonlinear oscillators, EBM, HAFF, Analytical solution.
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1. Introduction

There has been a growing interest in analytical solutions for nonlinear oscillator equations in the last few years. The key issue in studying nonlinear oscillation systems is finding the exact solutions. On the other hand, computing the exact results can be challenging, especially in oscillation systems with a high degree of nonlinearity, mainly using traditional analytical techniques. In this view, a wide variety of approximate methods and numerical techniques such as D. Perturbation theory [6, 35], Iteration Perturbation Method (IPM) [1, 11], Hamiltonian Approach (HA) [21, 45], Energy Balance Method (EBM) [22, 23], Harmonic Balance Method (HBM) [3, 8, 14, 33, 44], Homotopy Perturbation Method (HPM) [24–26], and Adomian Decomposition Method [7, 36] have been proposed. The Variational Iteration Method (VIM) [15, 18, 20, 27, 38, 46, 47] is another technique for solving strongly nonlinear systems. The VIM was first proposed by He [28], and its applications to nonlinear oscillators can be found in Refs. [4, 39].

What is more, one of the precise methods to find the frequency of an oscillator is He's amplitude frequency formulation (HAFF). He [29] proposed an amplitude-frequency formulation for nonlinear oscillators to solve nonlinear problems, deduced from an ancient Chinese mathematics method [40], and is now widely used by many authors. [5, 30, 31, 48, 49, 51].

It is evident that frequency is one of the most important physical parameters in oscillatory systems. For several years a fundamental limitation of these studies is that they do not address the accurate determination of the system's frequency, especially in severe nonlinear oscillatory systems. Recently, several authors have tried to improve classic

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methods from different points of view [10, 17, 19, 41, 43]. However, studies on more optimum methodologies to improve traditional methods' performance in computing system frequency are still lacking.

For this reason, a most interesting approach based on two robust methods (EBM&HAFF) is proposed in this study. Firstly, a variational principle for the nonlinear oscillation is established, and then a Hamiltonian formulation is constructed according to EBM. Two trial functions are assumed based on initial conditions and substituted in a defined Hamiltonian. Then by applying a part of He's amplitude frequency formulation, the system frequency can be set.

The author's findings reveal that, because frequency ω is one of the most significant physical parameters in oscillatory systems, and the provided technique demonstrates its capacity to predict the frequency with exceptional precision, it may be easily applied in severely nonlinear oscillation systems.

In this paper, the described novel method is extended and applied to determine approximate solutions for different strongly nonlinear oscillators represented in two famous examples.

These examples are formulated based on a general oscillator differential equation of the below form:

$$\ddot{u} + f(u) = 0, \qquad u(0) = A, \qquad \dot{u}(0) = 0, \tag{1.1}$$

where A is the initial amplitude., and the dot denotes differentiation with respect to time.

The first example involves three various cases of mass-spring oscillatory systems. The second example describes the Duffing-relativistic oscillation system. (See figure 1).

With attention to the error percentages of the results, the presented method profoundly succeeded in compensating for the drawbacks of conventional EBM and HAFF in significant applications. The main advantages of the described approach are less computation burden using fewer terms in expanded series, which leads to less procedure time of solution and more accuracy than its correspondent methods.

Based on the introduced novel approach, this paper evaluates the corresponding relations, and their error percentages and tables are presented for different parameter values. In all the studied cases, it is demonstrated that the solutions found with the proposed novel method show more reliability, accuracy, faster, and convergence than the other referred methods. Moreover, in most cases, the novel method capability is demonstrated in both Single Degree of Freedom (SDOF) and Two Degrees of Freedom (TDOF) oscillation systems and in both high and low amplitude values. It provides a practical and convenient mathematical tool for oscillator equations with high power of nonlinearity.

The presented steps and defined process of this paper are shown in the illustration below:

2. Methods

2.1. **Basic concept of Energy Balance Method (EBM).** In He's energy balance method [22] and [23], a variational formulation for the nonlinear oscillation is established, then a Hamiltonian is constructed, from which the angular frequency can be readily obtained by the collocation method.

The variational principle of Eq. (1.1) can be easily obtained as follows [10]:

$$J(u) = \int_0^t \left(-\frac{1}{2}\dot{u}^2 + F(u) \right) \mathrm{d}t. \tag{2.1}$$

The $F(u) = \int f(u)\ \mathrm{d}u$, which $f(u)$ is the second term of the general equation of a nonlinear oscillator in the form of Eq.(1.1).

The total energy of the oscillator corresponds to its Hamiltonian:

$$H = \frac{1}{2}\dot{u}^2 + F(u). \tag{2.2}$$

In Eq. (2.2), the *kinetic energy* (E) can be expressed as $E = \frac{1}{2}\dot{u}^2$ and $P = F(u) = \int f(u)\ \mathrm{d}u$, representing the *potential energy* (P) through the motion. Hence,the Hamiltonian can be rewritten as below [10]:

$$H = E + P. \tag{2.3}$$

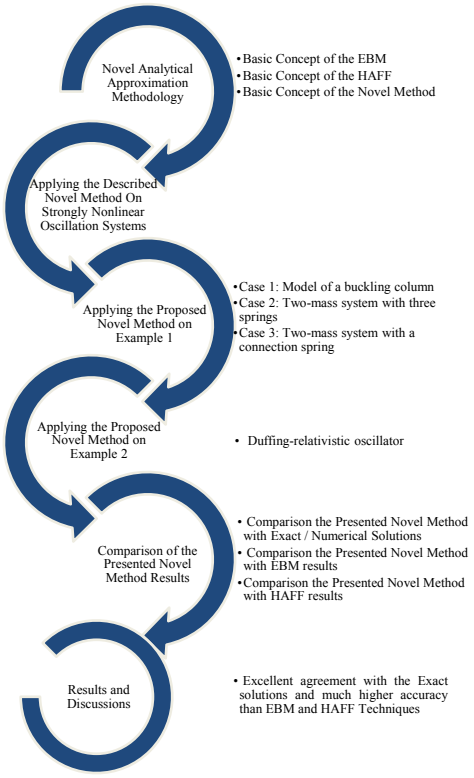

FIGURE 1. The article's steps and process.

Throughout the oscillation, since the system is conservative, the total energy remains unchanged during the motion; the Hamiltonian of the oscillator becomes a constant value:

$$\frac{1}{2}\dot{u}^2 + F(u) = F(A), \tag{2.4}$$

Or

$$\frac{1}{2}\dot{u}^2 + F(u) - F(A) = 0. \tag{2.5}$$

For the first-order approximation, and based on the initial conditions mentioned in Eq.(1.1), the following trial function can be assumed:

$$u(t) = A\cos\omega t, \tag{2.6}$$

with ω being the unknown angular frequency of the motion. Substituting Eq. (2.6) into Eq. (2.5) yields the following residual equation $R(t)$:

$$R(t) = \frac{1}{2}\omega^2 A^2 \sin^2\omega t + F(A\cos\omega t) - F(A). \tag{2.7}$$

It is clear that the residual varies over time. Relation (2.7) gives us the difference between the calculated energy based on the approximate solution for every special value of time and the total energy of the oscillator [34].

Since Eq. (2.6) is only an approximation to the exact solution, $R(t)$ cannot be made zero everywhere. The frequency (ω) is determined by using collocation at $\omega t = \pi/4$, that means by imposing the condition $R(\omega t = \pi/4) = 0$, This

yields:

$$\omega = \frac{2}{A}\sqrt{F(A) - F\left(A/\sqrt{2}\right)}. \tag{2.8}$$

If $T = 2\pi/\omega$ is the period of the nonlinear oscillator; it can be written in the form of:

$$T = \frac{2\pi}{2/A\sqrt{F(A) - F\left(A/\sqrt{2}\right)}}. \tag{2.9}$$

2.2. **Basic concept of He's Amplitude Frequency Formulation (HAFF).** Let us consider the general nonlinear oscillators as follows:

$$\ddot{u} + f(u, \dot{u}, \ddot{u}, t) = 0, \qquad u(0) = A, \qquad \dot{u}(0) = 0, \tag{2.10}$$

where $f(u,\ \dot{u},\ \ddot{u},\ t)$, is a function with the nonlinear term.

For a generalized nonlinear oscillator in Eq. (2.10), we use two trial functions:

$$u_1 = A\cos t, \tag{2.11}$$

$$u_2 = A\cos \omega t. \tag{2.12}$$

Substituting u_1 and u_2 into Eq. (2.10):

$$R_1 = \ddot{u}_1 + f(u_1,\ \dot{u}_1,\ \ddot{u}_1,\ t), \tag{2.13}$$

and

$$R_2 = \ddot{u}_2 + f(u_2, \dot{u}_2, \ddot{u}_2, t). \tag{2.14}$$

To use He's amplitude frequency formulation, we set the:

$$R_{11} = \frac{4}{T_1}\int_0^{\frac{T_1}{4}} R_1 \cos(t)\, \mathrm{d}t, \qquad T_1 = 2\pi, \tag{2.15}$$

and

$$R_{22} = \frac{4}{T_2}\int_0^{\frac{T_2}{4}} R_2 \cos(\omega t)\, \mathrm{d}t, \qquad T_2 = \frac{2\pi}{\omega}. \tag{2.16}$$

Applying He's amplitude frequency formulation, we have [30, 31, 51]:

$$\omega^2 = \frac{\omega_1^2 R_{22} - \omega_2^2 R_{11}}{R_{22} - R_{11}}, \tag{2.17}$$

where

$$\omega_1 = 1, \qquad \omega_2 = \omega. \tag{2.18}$$

2.3. **Basic concept of Novel Method.** In this paper, we consider a general nonlinear oscillator in the form of Eq.(1.1). Considering its initial conditions and based on He's amplitude frequency formulation we employ two trial functions $u_1(t) = A\cos t$ and $u_2(t) = A\cos(\omega t)$. Thanks to the Energy balance method, by constructing the oscillator's Hamiltonian, substituting u_1 and u_2 into Eq.(2.5) yields the following residuals relations:

$$R_1(t) = \frac{1}{2}\dot{u}_1^2 + F(u_1) - F(A) = 0, \tag{2.19}$$

and

$$R_2(t) = \frac{1}{2}\dot{u}_2^2 + F(u_2) - F(A) = 0. \tag{2.20}$$

Through He's amplitude frequency formulation, we set R_{11} and R_{22} according to Eq.(2.15) and Eq.(2.16) respectively. Finally, using Eq.(2.17) and Eq.(2.18), the frequency of the nonlinear oscillator (ω) would be determined.

3. Examples

We consider the following examples to demonstrate the advantages of described novel technique and extend this method's application.

Example 3.1. The Models of Nonlinear Oscillation Systems

This section analyzes a practical case of a nonlinear oscillation system of Single Degree of Freedom (SDOF) and two cases of Two Degrees of Freedom (TDOF) systems.

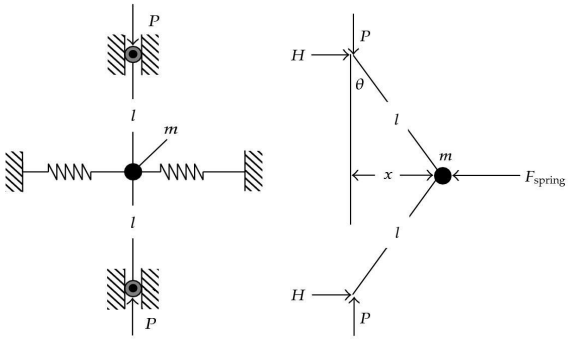

Figure 2. Model for the buckling of a column [13]

Case 1: Model of a buckling column

This section considers a column, as shown in Figure 2. the mass m moves in the horizontal direction only. Based on [12], the column can be modeled by two identical light rigid links with length l, supported by two nonlinear springs with the stiffness of k_1 and k_3, attached at their pivoted joints as shown in Figure 2. The system is then loaded by two axial compressive forces P at both ends.

Using this model representing a column, we demonstrate how one can study its static stability by determining the nature of the singular point at $u = 0$ of the dynamic equations. Neglecting the weight of springs and columns shows that the governing equation for the motion of m is [13]:

$$m\ddot{u} + \left(k_1 - \frac{2P}{l}\right) u + \left(k_3 - \frac{2P}{l^3}\right) u^3 + \cdots = 0, \tag{3.1}$$

where $u(0) = A$, $\dot{u}\,(0) = 0$. The spring force is given by:

$$F_{\text{spring}} = k_1 u + k_3 u^3 + \cdots . \tag{3.2}$$

Case 2: Two-mass system with three springs

A two-mass system with three springs is modeled in Figure 3. In this figure, two equal masses m are linked with the fixed supports using spring k_1. The connection between two masses makes a compact item: a spring with nonlinear properties. The linear coefficient of spring elasticity is k_2 and the cubic nonlinearity is k_3, thus, the system shows two degrees of freedom. The generalized coordinates are x and y.

The mathematical model of the system is presented here [13]:

$$\begin{aligned} &\ddot{u} + \left[\frac{k_1 + 2k_2}{m}\right] u + \left[\frac{2k_3}{m}\right] u^3 = 0, \\ &u\,(0) = y\,(0) - x\,(0) = Y_0 - X_0 = A, \quad \dot{u}\,(0) = 0. \end{aligned} \tag{3.3}$$

Note that in the case of k_3>0 corresponds to a hardening spring while k_3<0 indicates a softening one.

Case 3: Two-mass system with a connection spring

Similarly, the system with one spring is modeled in Figure 4. Two masses, m_1 and m_2, are linked with a spring with a linear coefficient of rigidity k_1 and the nonlinear one k_2. The system has two degrees of freedom.

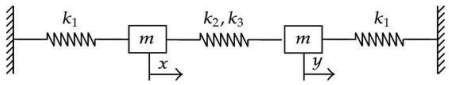

FIGURE 3. The two-mass system with three springs [13]

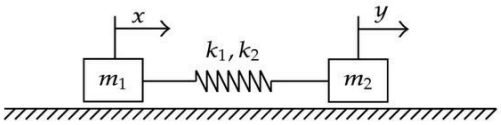

FIGURE 4. The two-mass system with spring [13]

The generalized coordinates of the system are x and y. The motion of the system is described by [13]:

$$\ddot{u} + \left[\frac{k_1 (m_1 + m_2)}{m_1 m_2}\right] u + \left[\frac{k_2 (m_1 + m_2)}{m_1 m_2}\right] u^3 = 0,$$
$$u(0) = y(0) - x(0) = Y_0 - X_0 = A, \quad \dot{u}(0) = 0. \tag{3.4}$$

3.1. Analysis of Mechanical Models and Applying Presented Novel Approach. Applying Novel Approach to Cases 1, 2, 3

We will apply this method to the discussed system to assess the presented novel method capability. Its formulation can be quickly established as below:

These mentioned models can be transformed into a cubic nonlinear differential equation in a general form with different values α and β. The general form of the cubic nonlinear differential is described below:

$$\ddot{u} + \alpha u + \beta u^3 = 0, \qquad \mathrm{u}(0) = A, \qquad \dot{u}(0) = 0. \tag{3.5}$$

The residual relation can be written in the following form based on the oscillator's Hamiltonian:

$$R(t) = \frac{1}{2}\dot{u}^2 + \frac{1}{2}\alpha\, u^2 + \frac{1}{4}\beta\, u^4 - \frac{1}{2}\alpha\, A^2 - \frac{1}{4}\beta\, A^4 = 0, \tag{3.6}$$

To determine the angular frequency ω, i.e., We will use the two trial functions $u_1(t) = Acost$ and $u_2(t) = A\cos(\omega t)$. If we substitute the assumed trial functions into Eq. (3.6), it leads to the following residual equations:

$$R_1(t) = \frac{1}{2}A^2\sin^2(t) + \frac{1}{2}\alpha A^2\cos^2(t) + \frac{1}{4}\beta A^4\cos^4(t) - \frac{1}{2}\alpha A^2 - \frac{1}{4}\beta A^4 = 0, \tag{3.7}$$

$$R_2(t) = \frac{1}{2}A^2\omega^2\sin^2(\omega t) + \frac{1}{2}\alpha A^2\cos^2(\omega\, t) + \frac{1}{4}\beta A^4\cos^4(\omega t) - \frac{1}{2}\alpha A^2 - \frac{1}{4}\beta A^4 = 0. \tag{3.8}$$

If we define R_{11} and R_{22} according to Eq.(2.15) and Eq.(2.16) respectively, we can set the Eq.(2.17) and Eq.(2.18), which leads to:

$$\omega_{\text{Novel Approach}} = 0.1\sqrt{30\beta A^2\cos^2(0.5\pi)\ + 70\beta A^2 + 100\alpha}. \tag{3.9}$$

The analytical results in terms of frequency values as well as outcomes associated with the period of the system are considered for different values of α and β. Subtitling $\alpha = \frac{k_1 - \frac{2P}{l}}{m}$ and $\beta = \frac{k_3 - \frac{2P}{l^3}}{m}$ in Eq. (3.9) yields the following results for the buckling of a column as a nonlinear SDOF system presented in section 3.1:

$$\omega_{\text{Novel Approach}} = 0.1\sqrt{\frac{30\left(k_3 - \frac{2P}{l^3}\right)\ A^2\cos(1.570796327)^2}{m} + \frac{70\left(k_3 - \frac{2P}{l^3}\right)A^2}{m} + \frac{100\left(k_1 - \frac{2P}{l}\right)}{m}}. \tag{3.10}$$

Consequently, we obtain the following period:

$$T_{\text{Novel Approach}} = \frac{2\pi}{0.1\sqrt{\frac{30\left(k_3-\frac{2P}{l^3}\right)A^2\cos(1.570796327)^2}{m}+\frac{70\left(k_3-\frac{2P}{l^3}\right)A^2}{m}+\frac{100\left(k_1-\frac{2P}{l}\right)}{m}}}. \tag{3.11}$$

The approximate frequency formulation of EBM and HAFF in general form, in terms of α and β, is obtained previously as [2]

$$\omega_{\text{EBM}} = \omega_{\text{HAFF}} = \sqrt{a+\frac{3\beta A^2}{4}}. \tag{3.12}$$

TABLE 1. Comparison of the novel method's periods and approximate solutions, with exact- case 1.

A	k_3	k_1	P	l	m	$T\_$ ex	$T\_$ Presented Method	$T\_$ EBM, $T\_\_$ HAFF	$Er(\%)$ Presented Method	Er(%) EBM, HAFF
1	5	10	1	1	1	1.96451	1.977056959	1.962537372	0.638681351	0.100413233
3	6	5	5	1.5	5	3.32368	3.361706968	3.237436612	1.144122419	2.594816228
10	50	10	10	10	10	0.33143	0.33553417	0.324181459	1.238321787	2.187050508
20	100	30	40	25	50	0.26208	0.265392852	0.256402026	1.264061203	2.166504006
10	100	50	-30	20	70	0.61809	0.625930023	0.604857367	1.26842743	2.140890938
100	20	70	150	50	100	0.1658	0.167896891	0.162206277	1.264710856	2.167504644
0.5	500	120	220	150	500	9.71672	9.82307501	9.676369073	1.094556702	0.415273127

Note: $T\_$ **EBM** represents the approximate period obtained by Energy Balance Method, which resulted previously in [2]. $T\_$**HAFF** indicates the approximate period of He's Amplitude Frequency Formulation, obtained previously in [2]. $T\_$ ex represents the exact period stated in [16]. $Er(\%)$ defines the percentage error which been calculated by the relation $\frac{|T-T_{ex}|}{T_{ex}}$ (%)

3.2. **Results and Discussion About Example 1.** Moreover, by substituting $\alpha = \frac{k_1+2k_2}{m}$ and $\beta = \frac{2k_3}{m}$ into Eq. (3.9), we can obtain the approximate solution of the second case according to Eq. (3.13):

$$\omega_{\text{Novel Approach}} = 0.1\sqrt{\frac{60k_3A^2\cos(1.570796327)^2}{m}+\frac{140\ k_3A^2}{m}+\frac{100\ (k_1+2k_2)}{m}}. \tag{3.13}$$

TABLE 2. Comparison of the novel method's frequencies and approximate solutions, with exact - case 2.

Y_0	X_0	A	k_3	k_2	k_1	m	ω_ex	ω _Presented Method	$\omega\_$ EBM, $\omega\_$ HAFF	$Er(\%)$ Presented Method	$Er(\%)$ EBM, HAFF
1	5	-4	1	1	1	1	5.1078	5.039841267	5.196152423	1.33048931	1.729754943
10	8	2	5	3	1	2	4.2406	4.183300133	4.301162634	1.351220747	1.428161911
10	-10	20	30	20	10	5	58.7856	58.05170109	60.08327555	1.248433137	2.207471813
-40	20	-60	90	70	50	10	215.7113	213.0234729	220.4972	1.246029809	2.218659848
10	-10	20	0.5	20	25	10	5.9541	5.873670062	6.041522985	1.350832838	1.468282108
50	-50	100	400	300	200	100	239.6455	236.6600938	244.9653036	1.24575934	2.219863757

Note: $\omega\_$ **EBM** represents the approximate natural frequency obtained by Energy Balance Method, obtained previously in [2]. $\omega\_$ **HAFF** indicates the approximate natural frequency obtained by He's Amplitude Frequency Formulation, obtained previously in [2]. $\omega\_$ ex represents the exact natural frequency stated in [16]. $Er(\%)$ defines the percentage error which been calculated by the relation $|\omega-\omega_{ex}|/\omega_{ex}$ (%)

Similarly, by choosing the $\alpha = \frac{k_1(m_1+m_2)}{m_1 m_2}$ and $\beta = \frac{k_2(m_1+m_2)}{m_1 m_2}$, into Eq. (3.9), the following frequency value is obtained for case 3:

$$\omega_{\text{Novel Approach}} = 0.1\sqrt{\frac{30_2\left(m_l+m_2\right)\mathrm{A}^2\cos(1.570796327)^2}{m_1 m_2}+\frac{70k_2\left(m_1+m_2\right)A^2}{m_1 m_2}+\frac{100k_1\left(m_1+m_2\right)}{m_1 m_2}}. \quad (3.14)$$

TABLE 3. Comparison of the novel method's frequencies and approximate solutions, with exact - case 3.

Y_0	X_0	k_2	k_1	m_2	m_1	A	ω_ex	ω _Presented Method	$\omega\_$ EBM, $\omega\_$ HAFF	$Er(\%)$ _ Presented Method	$Er(\%)$ EBM, HAFF
1	-4	1	5	2	1	5	5.8892	5.809475019	5.968668192	1.353748913	1.349388576
-5	5	5	2	5	3	-10	13.8752	13.70158142	14.17979783	1.251287045	2.195268032
-5	5	1	5	5	1	-10	9.6119	9.486832981	9.79795897	1.301168541	1.935714791
30	20	10	10	5	10	10	14.7806	14.59451952	15.09966887	1.258950787	2.158700391
25	20	5	10	1	100	5	10.0564	9.923457059	10.23657658	1.32197348	1.791660833
25	100	100	50	100	50	-75	110.0633	108.692226	112.5066664	1.245714057	2.219964693
200	400	300	200	100	1000	-200	307.8115	303.9773018	314.6461505	1.245631888	2.220401285

Note: $\boldsymbol{\omega\_}$ **EBM** represents the approximate natural frequency obtained by Energy Balance Method, obtained previously in [2]. $\boldsymbol{\omega\_}$ **HAFF** indicates the approximate natural frequency obtained by He's Amplitude Frequency Formulation, got previously in [2]. $\boldsymbol{\omega\_ ex}$ represents the exact natural frequency which is stated in [16]. $\boldsymbol{Er(\%)}$ defines the percentage error which been calculated by the relation $\left|\boldsymbol{\omega}-\boldsymbol{\omega_{ex}}\right|/\boldsymbol{\omega_{ex}}$ **(%)**

To illustrate and verify the accuracy of the described new approach, the results of HAFF and EBM, compared to the exact solutions, are given in tables 1-3. The exact frequency of the nonlinear differential equation in the cubic form is [13]:

$$\omega_{\text{ex}} = \frac{\pi\sqrt{\alpha+\beta A^2}}{2}\left(\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}t}{1-\delta\sin^2 t}\right)^{-1}, \qquad \delta = \frac{\beta A^2}{2\left(\alpha+\beta A^2\right)}. \quad (3.15)$$

Substituting presented α and β values into Eq. (3.15) give the exact frequencies for cases 1, 2, and 3 in the form of Eqs. (3.16)-(3.18) respectively:

$$\omega_{\text{ex}}\left(A\right) = \frac{\pi}{2}\sqrt{\frac{k_1 l^3-2Pl^2+A^2k_3l^3-2A^2P}{ml^3}}\left(\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}t}{1-\delta\sin^2 t}\right)^{-1}, \quad \delta = \frac{\left(l^3k_3-2P\right)A^2}{2\left(k_1l^3-2Pl^2+A^2k_3l^3-2A^2P\right)}. \quad (3.16)$$

$$\omega_{\text{ex}}\left(A\right) = \frac{\pi}{2}\sqrt{\frac{\left(k_1+2k_2\right)+2A^2k_3}{m}}\left(\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}t}{1-\delta\sin^2 t}\right)^{-1}, \quad \delta = \frac{2k_3A^2}{2\left(k_1+2k_2\right)+2k_3A^2}. \quad (3.17)$$

$$\omega_{\text{ex}}\left(A\right) = \frac{\pi}{2}\sqrt{\frac{\left(m_1+m_2\right)}{m_1m_2}\left(k_1+k_2A^2\right)}\left(\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}t}{1-\delta\sin^2 t}\right)^{-1}, \delta = \frac{k_2\left(m_1+m_2\right)A^2}{2\left(k_1\left(m_1+m_2\right)+k_2\left(m_1+m_2\right)A^2\right)}. \quad (3.18)$$

In this part, the approximate solutions of the introduced approach, HBM and HAFF, and their obtained relative errors for the three cases of Example 1 are presented according to Tables 1,2, and 3. The results of the EBM and HAFF as two powerful and efficient methods are considered two serious competitors of our represented technique.

As shown in Tables 1,2, and 3, the error percentage of approximate solutions of the new approach is dramatically less than the two other methods for most of the different parameter values in all categories of Example 1.

It is evident that the maximum errors percentage of the Novel technic in all range of vibration amplitudes are very negligible, and there is only about a 1% deviation towards exact solutions, while the maximum errors of two other

methods (EBM and HAFF) are about 2% for the same parameters of the described new technique. In other words, the presented novel method's accuracy is twice as the conventional ones. Therefore, this proposed method not only provides a simpler and more efficient solving technique but also provides a highly accurate solution.

Example 3.2. Duffing-relativistic oscillator

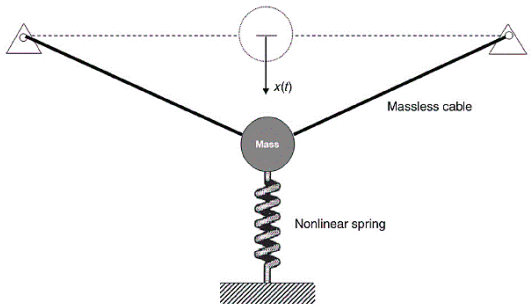

FIGURE 5. Schematic of a Duffing-relativistic oscillator. [50]

In this section, we will study the application of the introduced novel solution procedure for a Duffing-relative oscillator with the following equation [50]:

$$\ddot{u} + u + u^3 - \frac{\lambda u}{\sqrt{u^2+1}} = 0, \tag{3.19}$$

This differential equation can physically govern a motion of a dynamic system of an elastic cable with an attached mass connected to a nonlinear spring, as shown in Figure 5.

With the initial condition is $u(0) = A,\ \dot{u}(0) = 0$, where, the double dot denotes differentiation with respect to time t and the coefficient of the fourth term is introduced with a new parameter λ, which characterizes the system of values between 0 and 1. $(0 < \lambda \leq 1)$. The parameter of λ can be expressed based on initial cable tension and the axial rigidity of the elastic cable according to [9]. For $\lambda = 0$, the equation governs as a type of the well knows Duffing oscillator which represents free undamped vibration of an orthotropic clamped triangular plate [32].

The exact natural frequency can be numerically obtained via the following relation including elliptic integral [50]:

$$\omega_{\text{exact}} = \frac{2\pi}{4\int_0^A \frac{\mathrm{d}u}{\sqrt{(A^2-u^2)+\frac{1}{2}(A^4-u^4)-2\lambda\left(\sqrt{A^2+1}-\sqrt{u^2+1}\right)}}}. \tag{3.20}$$

Basically, elliptic integrals appeared as early as the seventeenth century in the calculation of arc lengths of certain curves, primarily *ellipses*. This is precisely the reason why the integral gets the name *elliptic*. There are several standard forms of elliptic integrals, but in general, they involve radicals of polynomials of degree 3 or 4, that can be evaluated by elliptic integrals [37]. Familiarity with elliptic integrals allows us to solve interesting problems in mathematics and physics that we have heretofore avoided.

It is possible to show that Eq.(3.20) is derived by integrating the differential equation and imposing the corresponding initial conditions, and then equating two relations as follows [42]:

$$\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + u^2 + \frac{1}{2}u^4 - 2\lambda\sqrt{1+u^2} = A^2 + \frac{1}{2}A^4 - 2\lambda\sqrt{1+A^2}. \tag{3.21}$$

Solving Eq.(3.21) for $\mathrm{d}t$ gives :

$$\mathrm{d}t = \frac{\mathrm{d}u}{\sqrt{(A^2-u^2)+\frac{1}{2}(A^4-u^4)-2\lambda\left(\sqrt{A^2+1}-\sqrt{u^2+1}\right)}}. \tag{3.22}$$

The exact period of the oscillation is four times the time taken by the mass to move from $u=0$ to $u=A$ [42]. So

$$T_{\text{exact}} = 4\int_0^A \frac{\mathrm{d}u}{\sqrt{(A^2-u^2)+\frac{1}{2}(A^4-u^4)-2\lambda\left(\sqrt{A^2+1}-\sqrt{u^2+1}\right)}}. \tag{3.23}$$

Then, if $\omega = 2\pi/T$, the exact natural frequency will be obtained in the form of Eq.(3.20).

3.3. **Analysis of Mechanical Models and Applying Presented Novel Approach.** The general form of a Duffing-relativistic oscillator is described as Eq.(3.19). And it's residual relation, therefore, can be written in the form:

$$R(t) = \frac{1}{2}\dot{u}^2 + \frac{1}{2}u^2 + \frac{u^4}{4} - \sqrt{u^2+1} - \frac{A^4}{4} - 0.5\,A^2 + \sqrt{A^2+1} = 0. \tag{3.24}$$

By employing the described new technique, and by assuming the two trial functions $u_1(t) = A\cos t$ and $u_2(t) = A\cos(\omega t)$ and substitute them into Eq. (3.24), the following residual equations can derive:

$$\begin{aligned} R_1(t) =& 0.5A^2\sin(t)^2 + 0.5A^2\cos(t)^2 + \frac{A^4\cos(t)^4}{4} - \lambda\sqrt{A^2\cos(t)^2+1} - \frac{A^4}{4} - 0.5\,A^2 \\ &+ \lambda\sqrt{A^2+1} = 0, \end{aligned} \tag{3.25}$$

$$\begin{aligned} R_2(t) =& 0.5A^2\omega^2\sin(\omega t)^2 + 0.5\,A^2\cos(\omega t)^2 + \frac{A^4\cos(\omega t)^4}{4} - \lambda\sqrt{A^2\cos(\omega t)^2+1} - \frac{A^4}{4} - 0.5\,A^2 \\ &+ \lambda\sqrt{A^2+1} = 0. \end{aligned} \tag{3.26}$$

We set R_{11} and R_{22} according to Eq.(2.15) and Eq.(2.16) respectively. Then, using Eq.(2.17) and Eq.(2.18), the angular frequency (ω) will determine as below:

$$\begin{aligned} \omega_{\text{Novel Technique}} =& \frac{5.9\times10^{-11}}{A^2\sin(0.5\pi)^2}\Bigg\{ -1.6\times10^{10}A\sin(0.5\pi)\Bigg[5\times10^9A^5\sin(0.5\pi)^5 - 1.6\times10^{10}A^5\sin(0.5\pi)^3 \\ &- 1.67\times10^{10}A^3\sin(0.5\pi)^3 - 1.25\times10^{10}IA^2\lambda\ln\left(\frac{-I}{A}\right) + 1.25\times10^{10}IA^2\lambda\ln\left(\frac{I}{A}\right) \\ &+ 2.5\times10^{10}A^2\arctan\left(\frac{0.5\left(2\sin^2(0.5\pi)\,A^2 - 1.\,A^2 - 1\right)}{A\sqrt{-1.0A^2\sin^4(0.5\pi) + \sin^2(0.5\pi)\,A^2 + \sin^2(0.5\pi)}}\right)\lambda \\ &+ 1.0\times10^{11}\sin(0.5\pi)\,\lambda\sqrt{A^2+1.0}\,A \\ &+ 5.0\times10^{10}\lambda\sqrt{-1.0\sin(0.5\pi)^2\left(\sin(0.5\pi)^2\,A^2 - 1.0A^2 - 1.0\right)}\,A \\ &- 1.25\times10^{10}I\lambda\ln\left(\frac{-I}{A}\right) + 1.25\times10^{10}I\lambda\ln\left(\frac{I}{A}\right) \\ &+ 2.5\times10^{10}\arctan\left(\frac{0.5\left(2\sin^2(0.5\pi)\,A^2 - 1.0A^2 - 1\right)}{A\sqrt{-1.0A^2\sin^4(0.5\pi) + \sin^2(0.5\pi)\,A^2 + \sin^2(0.5\pi)}}\right)\lambda\Bigg]\Bigg\}^{\frac{1}{2}}. \end{aligned} \tag{3.27}$$

TABLE 4. Comparison of the New approach frequencies and approximate frequencies with the exact ones when $\lambda = 0.1$.

A	ω_ HAFF	ω_ EBM	ω_ex	ω _Presented Method	Er(%) EBM	Er(%) Presented Method	Er(%) HAFF
10	7.140447968	8.717130618	8.532144867	8.422572101	2.168103733	1.284234707	18.08717477
100	70.71773881	86.60830693	84.68920534	83.67194273	2.266052179	1.201171517	18.34762586

Note: ω_ **EBM** represents the approximate natural frequency obtained by Energy Balance Method studied in [50]. ω_ **HAFF** indicates the approximate natural frequency obtained by He's Amplitude Frequency Formulation. ω_ ex represents the exact natural frequency which is stated in [50]. **Er(%)** defines the percentage error which been calculated by the relation $|\omega - \omega_{ex}|/\omega_{ex}$ (%)

TABLE 5. Comparison of the New approach frequencies and approximate frequencies with the exact ones when $\lambda = 0.5$.

A	ω_ HAFF	ω_ EBM	ω_ex	ω _Presented Method	Er(%) EBM	Er(%) Presented Method	Er(%) HAFF
10	7.136524782	8.714461031	8.528855664	8.408246165	2.176204802	1.414134603	18.10710087
100	70.71769882	86.60827987	84.6891715	83.67194273	2.266061104	1.201132034	18.34764653

Note: ω_ **EBM** represents the approximate natural frequency obtained by Energy Balance Method studied in [50].ω_ **HAFF** indicates the approximate natural frequency obtained by He's Amplitude Frequency Formulation. ω_ ex represents the exact natural frequency which is stated in [50]. **Er(%)** defines the percentage error which been calculated by the relation $|\omega - \omega_{ex}|/\omega_{ex}$ (%)

3.4. **Results and Discussion About Example 2.** Tables 4 and 5 compare the presented novel technique's precision, EBM, and HAFF for different oscillation amplitudes and λ values. It can be clearly seen that the lowest error percentage belongs to the introduced method for all (A, λ) values, and the following two approximation methods have a higher error percentage rate. For instance, for the $A = 100$ and $\lambda = 0.5$, the relative errors of natural frequency are found at 2.26% and 18.34% for EBM and HAFF, respectively, which are much higher than those found using the introduced novel method. It is highly remarkable that an excellent accuracy of the approximate natural frequency has been found, which is valid for the whole range of large and small values of oscillation amplitude compared with the exact ones. The straightforward solution procedure and high-accuracy results reveal the novelty and reliability of the prescribed method.

The presented method's importance is that although the EBM and HAFF are represented as two robust and constructive techniques without the disadvantages of traditional methods, their obtained results are still not as precise and reliable as those described in novel method outcomes. This kind of agreement and convergence is invisible in the other competing methods in all amplitude ranges, especially in an oscillatory system with large vibration amplitude.

4. CONCLUSION

The nonlinear vibration of oscillation systems is modeled by nonlinear differential equations. It is almost difficult to get an exact solution for such nonlinear differential equations. Hence our introduced method can be a constructive tool to address this issue. The presented novel process is defined as a simple and high-accuracy technique based on the obtained results. It can be considered a powerful, efficient alternative to typical methods.

The obtained result emphasizes that the introduced advanced approach shows an excellent agreement with the numerical solutions and is quickly convergent and valid for a wide variety of vibration amplitudes. The paper's main purpose is to establish a modern method with a capable formula, reveal its application through renowned examples in engineering models, and draw attention to the accuracy and ability of this potent technique by showing their results in tables.

It can be assumed that the importance and novelty of this study is this point concerning the conducted research and studies regarding the application of conventional and standard techniques in nonlinear oscillation problems widely

so far, the reconstruction and creation of an efficient method with an optimized computational algorithm have been rarely explored. In addition, the assessment of the introduced technique compared with its related standard methods increases the importance of this research.

From the research carried out, it is possible to conclude that the described formula has been very successful compared to other methods. Moreover, its speed and simplification to attain final results can be potentially used to analyze strongly nonlinear oscillation problems in engineering and applied sciences. The main advantages of this method include its simplicity and computational efficiency, and the ability to find better consistency in different vibration amplitude and time steps of approximate solutions.

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