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An efficient numerical approach for solving nonlinear Volterra integral equations

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Abstract

This study deals with a numerical solution of a nonlinear Volterra integral equation of the first kind. The method of this research is based on a new kind of orthogonal wavelets, called the Chebyshev cardinal wavelets. These wavelets known as new basis functions contain numerous beneficial features like orthogonality, spectral accuracy, and cardinality. In addition, we assume an expansion of the terms of Chebyshev cardinal wavelets within unknown coefficients as a substitute for an unknown solution. Relatively, considering the mentioned expansion and the cardinality feature within the generated operational matrix of the introduced wavelets, a system of nonlinear algebraic equations is extracted for the stated problem. Finally, by solving the yielded system, the estimated solution results.

Keywords. Chebyshev cardinal wavelets, Operational matrix, Integral equation.
2010 Mathematics Subject Classification. 45D05, 45G10.

1. Introduction

In previous studies, many numerical and theoretical methods for solving the integral equations of the second kind are proposed by different authors. One of the related theoretical studies is brought in [17] and a comprehensive numerical survey which is based on the collocation method is in [13]. Numerical methods based on collocation and implicit Runge-Kutta are presented for the solution of first and second kind Volterra integral equations [3]. Methods using product integrations are proposed in [24] in the case of solving the second kind of Volterra integral equations with singular, nonsingular, and periodic kernels. Other related studies can be found in [18, 29, 33, 38, 40–42, 46].

In comparison with the numerous related research about the numerical analysis of the second kind equations, some computational methods are published for estimating the solutions of the equations of the first kind, especially the nonlinear ones [1, 32, 47]. Volterra integral equations of the first kind exist in various fields and subfields of science and engineering, e.g. electrochemical systems [12], electrostatic [16], heat conduction problems [9], diffusion problems [8], the concrete problem of mechanics or physics [51], etc. Volterra integral equations of the first kind are well-known as ill-posed problems due to their unstable solutions. Relatively, it should be noted that small changes in their solutions can make large errors [4, 15]. Hence, it is complicated to find the exact solutions to these equations in many cases or fields. Moreover, since a small error may end in an unbounded error, it is also complicated to find numerical solutions to these problems. Considering this argument, various research and several regularization methods are proposed to defeat these ill-posed problems [25–27]. In 1977, Tikhonov and Arsenin [50] introduced regularization techniques to overcome the ill-posedness matter of the problems. But it should be highlighted that achieving an appropriate filter to regularize the solution method is too twisted and time-consuming in practice. Generally, many methods are developed to solve these types of equations, but a few numerical methods can be utilized in nonlinear cases. For example, the approaches introduced in [6, 36], solved the Volterra integral equations of the first kind within the usage of the expansion-iterative methods and the operational matrix ones. In [30, 31, 34] presented numerical techniques based on wavelets, modified block pulse functions, and Bernstein's approximation method for solving the Volterra integral equations of the first kind. In [23] defined optimal Homotopy asymptotic method for solving these equations.

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As an effort for overcoming both the ill-posedness and the nonlinearity features of these equations, in [6, 7] introduced a direct method to solve some specific nonlinear Volterra integral equations of the first kind containing the operational matrix with block-pulse functions and operational matrices of piecewise constant orthogonal functions. Singh and Kumar [49] presented the Haar wavelet operational matrix method for a class of nonlinear Volterra integral equations of the first kind. Indeed, it seems that operational matrices play a preconditioner role in the equations of the first kind. For complementary details, see [5, 6, 35, 36, 39, 49].

Here we can mention some other methods that have been used to solve nonlinear integral equations of the first kind. The usage of the homotopy perturbation method for solving nonlinear Volterra integral equations of the first kind is specifically brought in [11]. Moreover, the Adomian method [10] is used for the aim of solving nonlinear Volterra integral equations. One of the methods proposed recently is the Sinc Nyström method that is utilized to solve the equations of the first kind, as mentioned in Ma et al. [28]. Nedaiasl et al. [44] presented the hp-version collocation method for a class of nonlinear Volterra integral equations of the first kind. In the present paper, while using the Chebyshev cardinal wavelets, we keep in mind that the Chebyshev cardinal wavelets include numerous significant belongings, like orthogonality, cardinality, and spectral accuracy. Due to the mentioned properties, this set of basis functions is called a powerful tool in approximation theory. In addition, these wavelets inherit both the properties of wavelets and the Chebyshev cardinal polynomials at the same time. These wavelets have been used in [37, 48] for solving nonlinear constrained optimal control problems. The key supremacy of the wavelets in comparison with other popular ones mentioned above is their cardinality. The cardinality feature rescues us from computing integrals which often arise in obtaining the coefficients of the Chebyshev cardinal wavelets expansion of a function. The aimed coefficients are acquired by computing the values of the considered function at some grid points which are used as well in producing the wavelets [19–22]. According to this feature, the nonlinear terms in the studied problems are calculated efficiently. Relatively, these basis functions are suitable to achieve the estimated solution of the nonlinear integral equations of the first kind. The main goal of the present method in comparison with other methods, on the one hand, it is conceptually simple, accurate, and fast, with minimum computational cost, and on the other hand, it is to overcome the ill-posedness and nonlinearity of integral equations of the first kind.

This research studies the numerical solution of the nonlinear Volterra integral equation of the first kind, given as follows:

$$\mathcal{K}y(x) := \int_0^x K(x,t)F(t,y(t))\mathrm{d}t = f(x), \qquad x \in [x_0, x_f]. \tag{1.1}$$

A general form of Eq. (1.1) can be defined as follows:

$$\mathcal{K}y(x) := \int_0^x K(x,t,y(t))\mathrm{d}t = f(x), \qquad x \in [x_0, x_f]. \tag{1.2}$$

here, $f(x)$ is a known function defined on $[0,1]$ also $K(x,t,y(t))$ is a nonlinear function belonging to $[0,1]\times[0,1]\times\mathbb{R}$, and $y(x)$ is the unknown function to be computed. We also consider the value of x_0 to be zero and $x_f = 1$.

In this study, the main ideas are organized as follows: In section 2, properties of Chebyshev cardinal wavelets are described. Section 3 belongs to the description of the proposed method. Numerical examples of our study are presented in section 4, and at the end, the conclusion is presented in section 5.

2. Properties of Chebyshev cardinal wavelets

The Chebyshev cardinal wavelets are reviewed in summary, and interested properties are presented in this section.

2.1. The Chebyshev cardinal wavelets. By using the process of building polynomial wavelets which has been defined in [19], we can explain the Chebyshev cardinal wavelets over $[0,1]$ as follows:

$$\hat{\varphi}_{nm}(x) = \begin{cases} \sqrt{\frac{2M}{\pi}}2^{\frac{k}{2}}C_m(2^{k+1}x - 2n + 1), & \frac{n-1}{2^k} \le x < \frac{n}{2^k}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

where $k \in \mathbb{N}\cup\{0\}$, x is an independent variable defined on $[0,1]$, $n = 1,2,\ldots,2^k$ and C_m is the Chebyshev cardinal function of order m. Notice that the coefficient $\sqrt{\frac{2M}{\pi}}$ is used for normalizing. The set $\{\hat{\varphi}_{nm}(x) | n = 1,2,\ldots,2^k, m =$

$1, 2, \ldots, M, M \in N\}$ produces an orthonormal basis for $L^2_{w_n}[0,1]$, i.e.

$$\langle \hat{\varphi}_{nm}(x), \hat{\varphi}_{n'm'}(x) \rangle_{w_n} = \int_0^1 \hat{\varphi}_{nm}(x)\hat{\varphi}_{n'm'}(x) w_n(x) \mathrm{d}x = \begin{cases} 1, & (n,m) = (n',m'), \\ 0, & (n,m) \neq (n',m'), \end{cases}$$

where

$$w_n(x) = \begin{cases} \left(1 - (2^{k+1}x - 2n + 1)^2\right)^{-\frac{1}{2}}, & \frac{n-1}{2^k} \leq x < \frac{n}{2^k}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

By the usage of some simplifications, Eq. (2.1) can be rephrased in the following form

$$\hat{\varphi}_{nm}(x) = \begin{cases} \sqrt{\frac{2M}{\pi}} 2^{\frac{k}{2}} \prod\limits_{\substack{l=1 \\ l \neq m}}^{M} \left(\frac{x - \gamma_{nl}}{\gamma_{nm} - \gamma_{nl}} \right), & \frac{n-1}{2^k} \leq x < \frac{n}{2^k}, \\ 0, & \text{otherwise,} \end{cases} \tag{2.3}$$

in Eq. (2.3) $\gamma_{nm} = \frac{1}{2^{k+1}}(\lambda_m + 2n - 1)$ for $n = 1, 2, \ldots, 2^k$ and $m = 1, 2, \ldots, M$, and the values $\lambda_m = -cos\left(\frac{(2m-1)\pi}{2M}\right)$ are the zeros of the Chebyshev polynomial [14] of order M defined on $[-1, 1]$ for $m = 1, 2, \ldots, M$.
For constructing a wavelet basis with the interpolation property, we assume a revised form of Eq. (2.3) as follows:

$$\varphi_{nm}(x) = \begin{cases} \prod\limits_{\substack{l=1 \\ l \neq m}}^{M} \left(\frac{x - \gamma_{nl}}{\gamma_{nm} - \gamma_{nl}} \right), & \frac{n-1}{2^k} \leq x < \frac{n}{2^k}, \\ 0, & \text{otherwise.} \end{cases}$$

We remind that the set $\{\varphi_{nm}(x) | n = 1, 2, \ldots, 2^k, m = 1, 2, \ldots, M, M \in N\}$ constructs an orthogonal basis due to the weight function $w_n(t)$ for $L^2_{w_n}[0,1]$ and

$$\langle \varphi_{nm}(x), \varphi_{n'm'}(x) \rangle_{w_n} = \int_0^1 \varphi_{nm}(x)\varphi_{n'm'}(x) w_n(x) \mathrm{d}x = \begin{cases} \frac{\pi}{M 2^{k+1}}, & (n,m) = (n',m'), \\ 0, & (n,m) \neq (n',m'). \end{cases}$$

2.2. **Function of a variable approximation.** Any function $y(x) \in L^2_{w_n}[0,1]$ can be estimated by the Chebyshev cardinal wavelets as follows:

$$y(x) \simeq \sum_{n=1}^{2^k} \sum_{m=1}^{M} c_{nm} \varphi_{nm}(x) = \mathbf{C}^T \mathbf{\Phi}(x), \tag{2.4}$$

where

$$\mathbf{C} = [c_{11}, c_{12}, \ldots, c_{1M} | c_{21}, c_{22}, \ldots, c_{2M} | \ldots | c_{2^k 1}, c_{2^k 2}, \ldots, c_{2^k M}]^T,$$

$$\mathbf{\Phi}(x) = [\varphi_{11}(x), \varphi_{12}(x), \ldots, \varphi_{1M}(x) | \varphi_{21}(x), \ldots, \varphi_{2M}(x) | \ldots | \varphi_{2^k 1}(x), \ldots, \varphi_{2^k M}(x)]^T, \tag{2.5}$$

and

$$c_{nm} = y(\gamma_{nm}), \qquad n = 1, 2, \ldots, 2^k, \quad m = 1, 2, \ldots, M.$$

It has to be considered that c_{nm} are the items of the vector C.

2.3. **Function of two variables approximation.** Suppose $y(x,t)$ is a function of two variables defined over the interval $t \in [0,1]$ and $s \in [0,1]$, then $y(x,t)$ can be extended as following,

$$y(x,t) \simeq \mathbf{\Phi}^T(x) Y \mathbf{\Phi}(t).$$

The following explanation clarifies the above statement:

Remark 2.1. Eq. (2.4) can be shown in a simpler form as follows

$$y(x) \simeq \sum_{p=1}^{\hat{m}} y_p \varphi_p(x) = \mathbf{Y}^T \mathbf{\Phi}(x),$$

where $\hat{m} = 2^k M$, $y_p = y_{nm}$ and $\varphi_p(x) = \varphi_{nm}$ for the index $p = (n-1)M + m$.

For example, we have shown the function $y(x) = \frac{(1-x)x}{1+x}$ and its approximation $y(x) \approx Y^T \Phi(x)$, for $M = 2$, $k = 2$ in Figure 1.

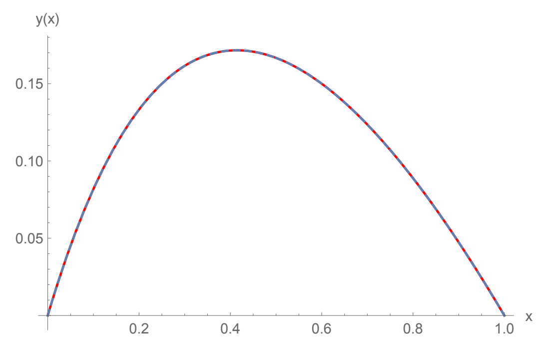

FIGURE 1. $y(x) = \frac{(1-x)x}{1+x}$ and its approximation $y(x) \approx Y^T \Phi(x)$.

Remark 2.2. The Chebyshev cardinal wavelets can be utilized to expand any kind of function $y(x,t) \in L^2_{w_{n.n'}}([0,1] \times [0,1])$

$$y(x,t) \simeq \sum_{p=1}^{\hat{m}} \sum_{q=1}^{\hat{m}} y(x_p, t_q) \varphi_p(x) \varphi_q(t) = \mathbf{\Phi}^T(x) Y \mathbf{\Phi}(t), \tag{2.6}$$

where $\hat{m} = 2^k M$, $Y = [y_{pq}]$ and its elements are computed as $y_{pq} = y(x_p, t_q)$.

For example, we have shown the graph of the function $y(x,t) = t - \sin(10xt)$ and its approximations, for $M = 3$, $k = 1$ and $M = 3$, $k = 2$, in Figures 2-4.

2.4. **Operational matrices.** Operational matrix is a matrix that works on basis like an operator, in other words, if H is an operator, an operational matrix is a matrix like P such that $H(\Phi) \approx P\Phi$.

2.4.1. *The operational matrix of integration.* The operational matrix of integration of the Chebyshev cardinal wavelets has been derived in [20]. The integration of the vector $\mathbf{\Phi}(t)$ defined in Eq. (2.5) can be approximated as

$$\int_0^x \mathbf{\Phi}(\tau) \mathrm{d}\tau \simeq \mathbf{P} \mathbf{\Phi}(x),$$

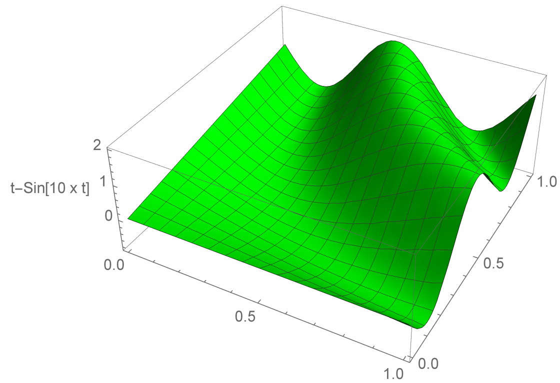

FIGURE 2. $y(x,t) = t - \sin(10xt)$.

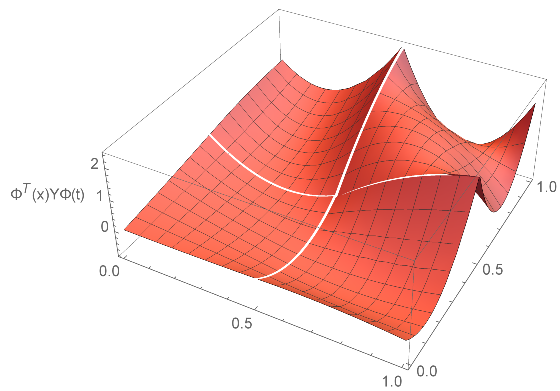

FIGURE 3. $y(x) \simeq \Phi(x)^T Y \Phi(t)$ with $M = 3$ and $k = 1$.

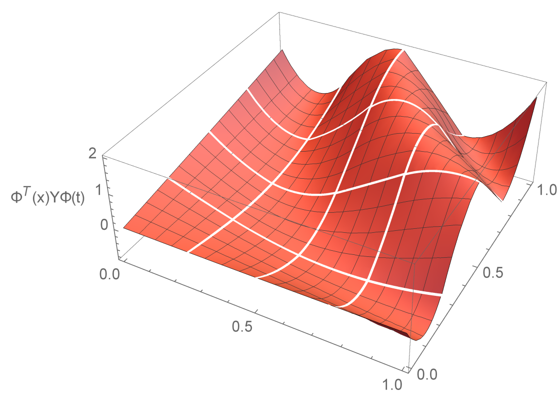

FIGURE 4. $y(x) \simeq \Phi(x)^T Y \Phi(t)$ with $M = 3$ and $k = 2$.

where $\mathbf{P}$ is the $\hat{m} \times \hat{m}$ matrix as follows

$$\mathbf{P} = \begin{pmatrix} A & B & B & B & \dots & B \\ 0 & A & B & B & \dots & B \\ 0 & 0 & \ddots & \ddots & \ddots & B \\ \vdots & \vdots & \ddots & A & B & B \\ 0 & 0 & \dots & 0 & A & B \\ 0 & 0 & 0 & \dots & 0 & A \end{pmatrix}_{\hat{m} \times \hat{m}} . \tag{2.7}$$

In Eq. (2.7), $A = [a_{ij}]$ and $B = [b_{ij}]$ are $M \times M$ matrices, and their components are obtained via using the below relations:

$$a_{ij} = \frac{1}{2^{k+1}} \int_{-1}^{\eta_j} \Big(\prod_{\substack{l=1 \\ l \neq m}}^{M} (\frac{\tau - \eta_l}{\eta_i - \eta_l}) \Big) \mathrm{d}\tau, \qquad b_{ij} = \frac{1}{2^{k+1}} \int_{-1}^{1} \Big(\prod_{\substack{l=1 \\ l \neq m}}^{M} (\frac{\tau - \eta_l}{\eta_i - \eta_l}) \Big) \mathrm{d}\tau.$$

As an illustrative example for $k = 1$, $M = 2$ and $k = 1$, $M = 3$ we have

$$\mathbf{P} = \begin{pmatrix} \frac{1}{8} - \frac{1}{16\sqrt{2}} & \frac{1}{8} + \frac{3}{16\sqrt{2}} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8} - \frac{3}{16\sqrt{2}} & \frac{1}{8} + \frac{1}{16\sqrt{2}} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{8} - \frac{1}{16\sqrt{2}} & \frac{1}{8} + \frac{3}{16\sqrt{2}} \\ 0 & 0 & \frac{1}{8} - \frac{3}{16\sqrt{2}} & \frac{1}{8} + \frac{1}{16\sqrt{2}} \end{pmatrix}_{4\times 4},$$

$$\mathbf{P} = \begin{pmatrix} \frac{1}{18} - \frac{1}{32\sqrt{3}} & \frac{1}{18} + \frac{1}{8\sqrt{3}} & \frac{1}{18} + \frac{\sqrt{3}}{32} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{5}{36} - \frac{1}{4\sqrt{3}} & \frac{5}{36} & \frac{5}{36} + \frac{1}{4\sqrt{3}} & \frac{5}{18} & \frac{5}{18} & \frac{5}{18} \\ \frac{1}{18} - \frac{\sqrt{3}}{32} & \frac{1}{18} - \frac{1}{8\sqrt{3}} & \frac{1}{18} + \frac{1}{32\sqrt{3}} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ 0 & 0 & 0 & \frac{1}{18} - \frac{1}{32\sqrt{3}} & \frac{1}{18} + \frac{1}{8\sqrt{3}} & \frac{1}{18} + \frac{\sqrt{3}}{32} \\ 0 & 0 & 0 & \frac{5}{36} - \frac{1}{4\sqrt{3}} & \frac{5}{36} & \frac{5}{36} + \frac{1}{4\sqrt{3}} \\ 0 & 0 & 0 & \frac{1}{18} - \frac{\sqrt{3}}{32} & \frac{1}{18} - \frac{1}{8\sqrt{3}} & \frac{1}{18} + \frac{1}{32\sqrt{3}} \end{pmatrix}_{6\times 6}.$$

3. The proposed computational scheme

In order to find a solution using the Chebyshev cardinal wavelets, at first, we will explain the following theorem.

Theorem 3.1. *Suppose $\mathcal{T} : [0, 1] \times [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous real-valued operation and $Y^T \mathbf{\Phi}(t)$ is an estimation of the function $y(t)$ using the Chebyshev cardinal wavelets. Then we have*

$$\mathcal{T}(x, t, y(t)) \simeq \mathbf{\Phi}^T(x) Y \mathbf{\Phi}(t),$$

where $Y = [y_{pq}]$ and its elements are calculated as $y_{pq} = \mathcal{T}(x_p, t_q, y_q)$ for $p = 1, 2, \ldots, \hat{s}$ and $q = 1, 2, \ldots, \hat{s}$.

Proof. The evidence is clear via assuming Eq. (2.6). □

The Theorem 3.1 and the outcome achieved for the Chebyshev cardinal wavelets are applied in order to figure out the issue shown in Eq. (1.2). For this aim, the target solution by the Chebyshev cardinal wavelets with unknown coefficients is estimated. The cardinality of the stated basic functions as well as the resulted operational matrix are applied for converting the principal problem to an adaptive algebraic system of equations. The solution is approximately acquired by resolving the resulted system and computing the coefficients of the expansion by an sutiable approch. Due to that, consider this

$$y(x) \simeq Y^T \mathbf{\Phi}(x), \tag{3.1}$$

where Y is an $\hat{m}$-column vector which has the unknown coefficients and the vector $\mathbf{\Phi}(x)$ is supposed as in Eq. (2.5). The Chebyshev cardinal wavelets are applied to state the function $f(x)$ as

$$f(x) \simeq F^T \mathbf{\Phi}(x), \tag{3.2}$$

whereas the coefficients of the Chebyshev cardinal wavelets cover in the $\hat{m}$-column vector F.
Replacing Eqs. (3.1) and (3.2) into Eq. (1.2), we have

$$\int_0^x K(x, t, Y^T \mathbf{\Phi}(t)) \mathrm{d}t \simeq F^T \mathbf{\Phi}(x). \tag{3.3}$$

In addition, using Theorem 3.1, leads to

$$K(x, t, Y^T \mathbf{\Phi}(t)) \simeq \mathbf{\Phi^T}(x) \mathbf{K} \mathbf{\Phi}(t), \tag{3.4}$$

whereas

$$\mathbf{K} = [k_{ij}] = K(x_i, t_j, y_j), \qquad i = 1, 2, \dots, \hat{m}, \quad j = 1, 2, \dots, \hat{m}.$$

Eq. (3.4) is replaced in Eq. (3.3) which results in

$$\int_0^x \mathbf{\Phi^T}(x)\mathbf{K}\mathbf{\Phi}(t)\mathrm{d}t \simeq F^T\mathbf{\Phi}(x), \tag{3.5}$$

or the following equivalent

$$\mathbf{\Phi^T}(x)\mathbf{K}\int_0^x \mathbf{\Phi}(t)\mathrm{d}t \simeq F^T\mathbf{\Phi}(x). \tag{3.6}$$

By means of operational matrix, the mentioned relation is potentially shown as

$$\mathbf{\Phi^T}(x)\mathbf{KP}\mathbf{\Phi}(x) \simeq F^T\mathbf{\Phi}(x). \tag{3.7}$$

Also, using cardinality of the basic function, this equation is achieved as

$$\mathbf{\Phi^T}(x)\mathbf{KP}\mathbf{\Phi}(x) \simeq diag(\mathbf{KP})\mathbf{\Phi}(x) = \Gamma^T\mathbf{\Phi}(x). \tag{3.8}$$

Hence, Eqs. (3.1)-(3.8) give

$$\Gamma^T\mathbf{\Phi}(x) \simeq F^T\mathbf{\Phi}(x),$$

and we attain the below system of nonlinear algebraic equations by employing the orthogonal property of the Chebyshev cardinal wavelets:

$$\Gamma^T - F^T = 0. \tag{3.9}$$

The resulting system can be solved with a suitable method for determining Y. At last, the estimated solution is gained by placing Y in Eq. (1.2).

4. Illustrative examples

In this part, we introduce may instances to estimate a solution for nonlinear Volterra equations by the usage of numerical method illustrated in preceding parts. For showing the function of the function of the method and illustrating the precision and utility of the presented scheme, we separately compare the results of the proposed scheme with the results from other ones, in such way that $y(t)$ and $Y^T\mathbf{\Phi}(t)$ are the exact and calculated solutions using introduced scheme. This study shows that the absolute error values of the mentioned method are less than the absolute error values of other previous methods. Numerical calculations related to the examples mentioned in the software Mathematica 7 have been performed. The simulation was conducted on the Intel core i5-2400 with 4 GB RAM.

Example 4.1. We consider the following Volterra integral equation of the first kind(see [44, 52]):

$$\int_0^x e^{-xt}y(t)\mathrm{d}t = \frac{e^{-x(x+1)}\mathrm{sin}x - (x+1)\mathrm{cos}xe^{-x(x+1)} + x + 1}{1 + (x+1)^2}, \quad 0 \le x \le 1.$$

It can be checked that the exact solution $y(x) = e^{-x}\mathrm{cos}x$. This example is examined in [44] and [52]. The highest result reported in [44] with m = 4, n = 10 has an absolute error of about 10^{-10} and also the best result reported in [52] with m = 4, n = 10 has an absolute error of about 10^{-14}, while the method presented in the absolute error is about 10^{-15} for $k = 2, M = 10$. Graph of the absolute error for values of $k = 2, M = 10$ in Example 4.1 is shown in Figure 5. Table 1 shows the absolute error of the mentioned method for different values of k and M. The matrices P and K in Eqs. (2.7) and (3.4) have large numbers of zero elements, hence this method is very attractive and reduces the CPU time. The CPU time for this example is equal to 0.345 sec.

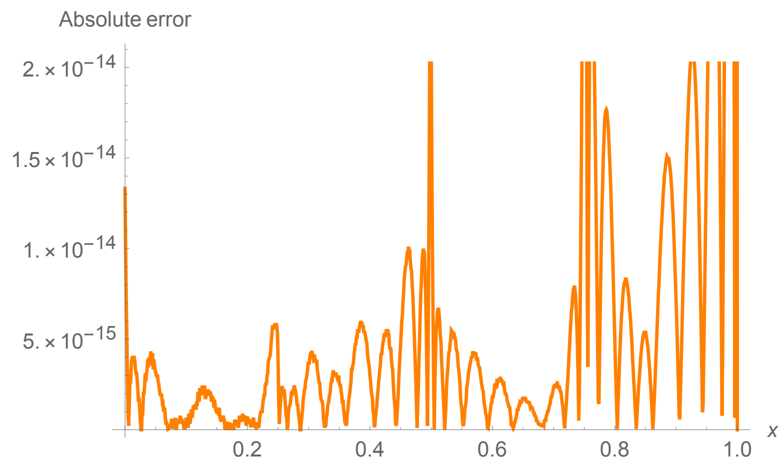

FIGURE 5. Absolute errors with $k = 2$ and $M = 10$ of Example 4.1.

TABLE 1. Absolute errors for Example 4.1.

x	Our method with $k = 1, M = 10$	Our method with $k = 2, M = 10$
0.1	1.63203×10^{-14}	6.66134×10^{-16}
0.2	5.55112×10^{-15}	6.66134×10^{-16}
0.3	1.27676×10^{-14}	3.66374×10^{-15}
0.4	8.61533×10^{-14}	2.99760×10^{-15}
0.5	1.13688×10^{-12}	2.18749×10^{-14}
0.6	1.40686×10^{-14}	1.88738×10^{-15}
0.7	1.67921×10^{-13}	2.16493×10^{-15}
0.8	2.19672×10^{-13}	4.87110×10^{-15}
0.9	2.57155×10^{-13}	7.13318×10^{-15}

Example 4.2. Consider the following nonlinear VIE (see [43]):

$$\int_0^x (x - t + 1)e^{-y(t)}\mathrm{d}t = f(x), \quad 0 \le x \le 1,$$

where $f(x)$ is selected such that the exact solution to be $y(x) = x$.
Table 2 illustrates a comparison between the absolute error values of the proposed method and the method in [43] for different values of k and M. Graph of the absolute error for values of $k = 2$ and $M = 10$ of Example 4.2 is shown in Figure 6. The CPU time for this example is equal to 0.253 sec.

Example 4.3. Consider the following nonlinear VIE (see [28, 49]):

$$\int_0^x e^{x-t}\ln(y(t))\mathrm{d}t = e^x - x - 1, \quad 0 \le x \le 1,$$

whose exact solution is $y(x) = e^x$. Table 3 shows the numerical results. Graph of the absolute error for values of $k = 2, M = 10$ for Example 4.3 is shown in Figure 7. According to the results in Table 3, we see that our method with less basis functions is more accurate than previous methods. For example, the absolute error of our method with $\hat{m} = 10$ is 10^{-11}, while for the methods in [49] and [28], the absolute error is $2.1E-7$, $1E-5$ with $L = 512$, 33 basis functions, respectively. The CPU time for this example is equal to 0.721 sec.

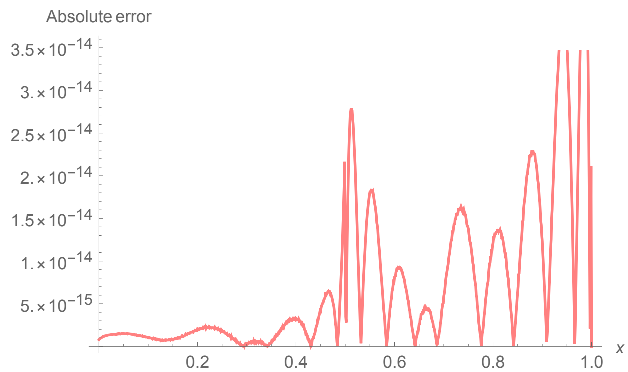

FIGURE 6. Absolute errors with $k = 2$ and $M = 10$ of Example 4.2.

TABLE 2. Absolute errors for Example 4.2.

x	Our method with $k = 1, M = 10$	Our method with $k = 1, M = 11$	Method in [43] with $m = 8$
0	5.81132×10^{-15}	7.77156×10^{-16}	4.6×10^{-9}
0.15	1.76942×10^{-16}	7.77156×10^{-16}	8.3×10^{-9}
0.30	5.66214×10^{-15}	2.77556×10^{-16}	3.2×10^{-9}
0.45	6.32827×10^{-15}	4.16334×10^{-15}	3.1×10^{-9}
0.60	2.30926×10^{-14}	8.11851×10^{-15}	3.8×10^{-9}
0.75	3.32789×10^{-14}	1.37668×10^{-14}	4.5×10^{-9}
0.90	4.34097×10^{-14}	1.23235×10^{-14}	1.4×10^{-7}

TABLE 3. Absolute errors for Example 4.3.

x	Our method with $k = 0, M = 10$	Our method with $k = 1, M = 10$	Our method with $k = 1, M = 12$
0.1	1.18223×10^{-11}	3.33067×10^{-15}	2.67841×10^{-15}
0.2	1.26390×10^{-11}	2.22045×10^{-15}	1.33227×10^{-15}
0.3	5.36227×10^{-12}	2.88658×10^{-15}	1.77636×10^{-15}
0.4	1.19014×10^{-11}	2.22045×10^{-15}	6.21725×10^{-15}
0.5	1.87652×10^{-11}	9.08162×10^{-14}	3.82888×10^{-14}
0.6	1.39910×10^{-12}	4.77396×10^{-15}	9.26342×10^{-15}
0.7	2.83102×10^{-11}	7.02216×10^{-15}	2.30371×10^{-15}
0.8	3.19355×10^{-11}	6.77236×10^{-15}	5.44009×10^{-15}
0.9	2.42706×10^{-11}	1.37668×10^{-14}	8.88178×10^{-16}

Example 4.4. Consider the following nonlinear VIE (see [28, 49]):

$$\int_0^x (\sin(x-t)+1)\cos(y(t))\mathrm{d}t = \frac{x\sin x}{2} + \sin x, \quad 0 \le x \le 1,$$

with the exact solution $y(x) = x$. The results reported in Table 4 show the fact that the absolute error of this method is much less than the absolute error presented in [28] and [49] with fewer basis functions. Graph of the absolute error for values of $k = 3$ and $M = 10$ of Example 4.4 is shown in Figure 8. The CPU time for this example is equal to 0.546 sec.

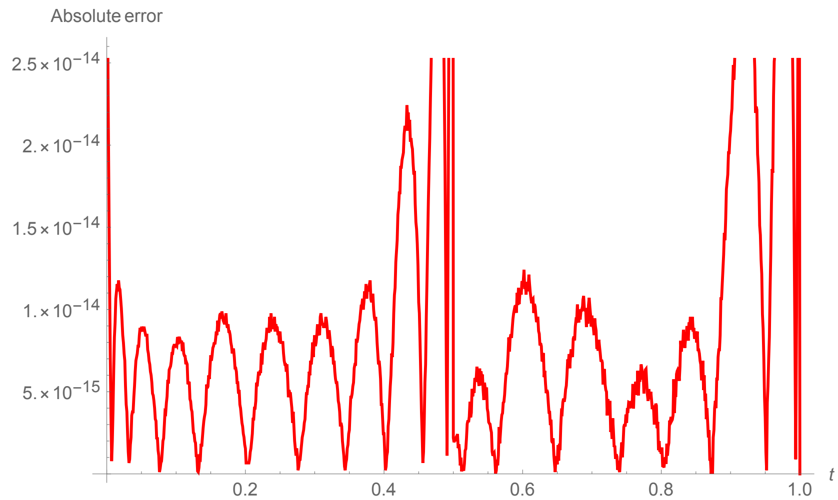

FIGURE 7. Absolute errors with $k = 2$ and $M = 10$ of Example 4.3.

TABLE 4. Absolute errors for Example 4.4.

x	Our method with $k = 0, M = 10$	Our method with $k = 1, M = 10$	Our method with $k = 1, M = 12$
0.1	1.32469×10^{-10}	2.45359×10^{-13}	3.40006×10^{-15}
0.2	1.72721×10^{-10}	4.19942×10^{-13}	2.12330×10^{-14}
0.3	5.86761×10^{-10}	4.90108×10^{-13}	1.39888×10^{-14}
0.4	5.65781×10^{-10}	6.43929×10^{-14}	2.51466×10^{-14}
0.5	4.46749×10^{-11}	3.90039×10^{-12}	6.75293×10^{-14}
0.6	5.80446×10^{-10}	1.89321×10^{-13}	6.82787×10^{-15}
0.7	5.22602×10^{-10}	6.99385×10^{-13}	2.02616×10^{-14}
0.8	6.40366×10^{-11}	5.74624×10^{-13}	7.49401×10^{-15}
0.9	1.94803×10^{-10}	3.48166×10^{-13}	2.93099×10^{-14}

Example 4.5. Consider the following nonlinear VIE (see[7, 49]):

$$\int_0^x e^{x-t} y^2(t)\mathrm{d}t = e^{2x} - e^x, \quad 0 \le x \le 1.$$

The analytical solution of this example is $y(x) = e^x$. In Table 5, the absolute error for the mentioned method is compared with the obtained absolute error in [7] and [49]. Graph of the absolute error for values of $k = 1$ and $M = 10$ of Example 4.5 is shown in Figure 9. The CPU time for this example is equal to 0.612 sec.

Example 4.6. Consider the following first-kind NVIE(see [2]):

$$\int_0^x \cos(x-t) y^2(t)\mathrm{d}t = \sin(x) + \frac{2}{3}\cos(x) - \frac{2}{3}\cos(2x), \quad 0 \le x \le 1.$$

The exact solution is $y(x) = \cos(x) + \sin(x)$. For instance, the absolute error of the proposed method is compared with the absolute error reported in [2], the results of which are given in Table 6, the results indicate that the proposed method is effective. Graph of the absolute error for values of $k = 2$ and $M = 10$ of Example 4.6 is shown in Figure 10. The CPU time for this example is equal to 0.803 sec.

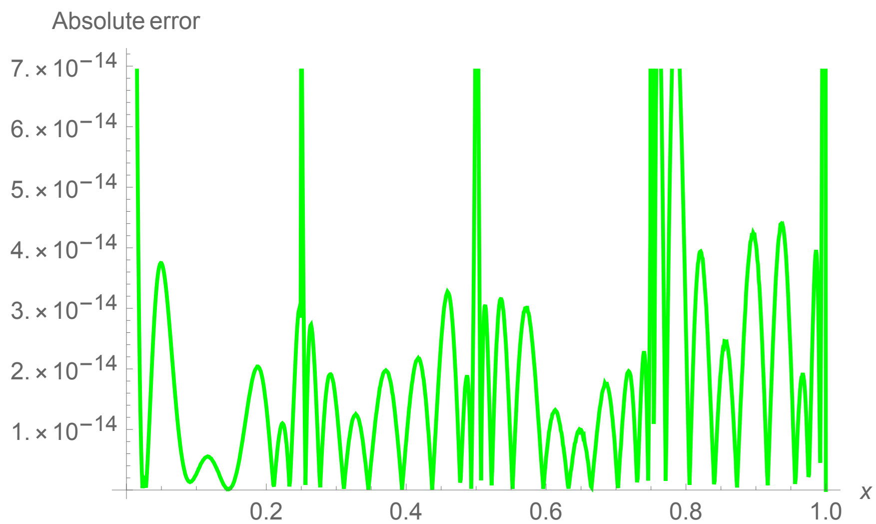

FIGURE 8. Absolute errors with $k = 3$ and $M = 10$ of Example 4.4.

TABLE 5. Absolute errors for Example 4.5.

x	Our method with $k = 1, M = 11$	Method in [49]	Method in [7]
1/16	1.55431×10^{-15}	1.95×10^{-3}	3.28×10^{-2}
3/16	4.44089×10^{-16}	1.98×10^{-3}	3.74×10^{-2}
5/16	1.11022×10^{-15}	2.04×10^{-3}	4.25×10^{-2}
7/16	2.22045×10^{-15}	2.14×10^{-3}	4.85×10^{-2}
9/16	6.66134×10^{-16}	2.26×10^{-3}	5.52×10^{-2}
11/16	6.66134×10^{-16}	2.43×10^{-3}	6.28×10^{-2}
13/16	3.10862×10^{-15}	2.63×10^{-3}	7.16×10^{-2}
15/16	8.88178×10^{-16}	2.87×10^{-3}	8.15×10^{-2}

TABLE 6. Absolute errors for Example 4.6.

t	Our method with $k = 1, M = 10$	Our method with $k = 2, M = 10$	Method in [2] ($\frac{3}{8}$Simpson)
0.1	2.67653×10^{-12}	4.77396×10^{-15}	1.789×10^{-6}
0.2	3.93707×10^{-12}	4.44089×10^{-16}	1.597×10^{-6}
0.3	2.73648×10^{-12}	3.77476×10^{-15}	2.8601×10^{-5}
0.4	2.53753×10^{-12}	9.99201×10^{-15}	1.4331×10^{-5}
0.5	5.34717×10^{-11}	2.59792×10^{-13}	1.3613×10^{-5}
0.6	3.52104×10^{-12}	3.23491×10^{-14}	3.7060×10^{-5}
0.7	1.02465×10^{-11}	1.03251×10^{-14}	2.0975×10^{-5}
0.8	7.79191×10^{-12}	3.19189×10^{-16}	1.8903×10^{-5}
0.9	7.18492×10^{-12}	2.31482×10^{-14}	4.5269×10^{-5}

Example 4.7. Consider the following nonlinear VIE :

$$\int_0^x y^3(t)\mathrm{d}t = f(x), \quad 0 \le x \le 1,$$

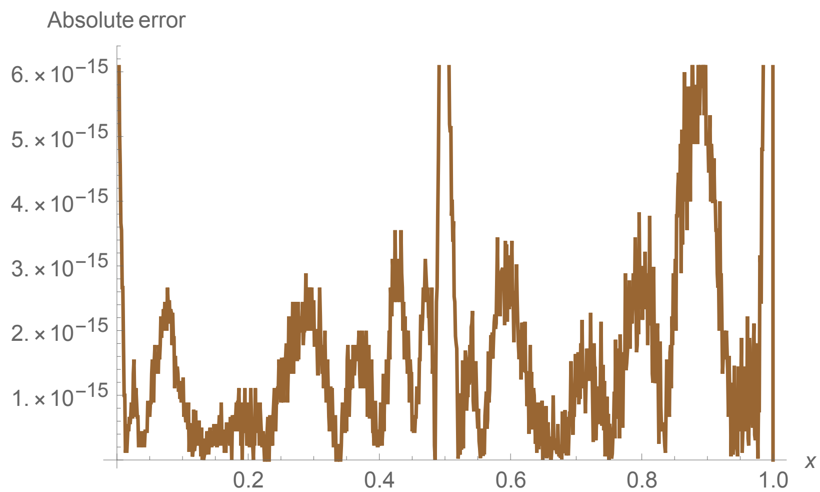

FIGURE 9. Absolute errors with $k = 1$ and $M = 10$ of Example 4.5.

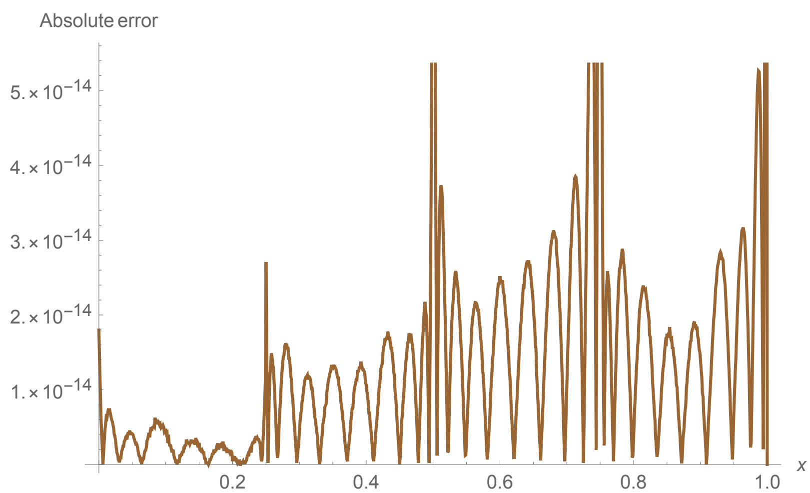

FIGURE 10. Absolute errors with $k = 2$ and $M = 10$ of Example 4.6.

whose $f(x)$ is determined by noting that the non-smooth solution $y(x) = |x - \frac{1}{2}|$, hence $f(x)$ is $\frac{1}{64} + \frac{1}{4}\left(x - \frac{1}{2}\right)^3 |x - \frac{1}{2}|$. The unknown function belongs to $C^0 \backslash C^1$. Table 7 shows the absolute error of the proposed method. Graph of the absolute error for values of $k = 3$ and $M = 5$ of Example 4.7 is shown in Figure 11. The CPU time for this example is equal to 1.003 sec.

5. Conclusion

The proposed method for solving the nonlinear Volterra integral equations is based on the Chebyshev cardinal wavelets. The most significant advantage of the proposed method in comparison with other methods is its simplicity and method comprehensibility. Eventually, some numerical examples were included to demonstrate the applicability of the presented method. It is noteworthy that the results of numerical examples show that our proposed method is an efficient and very simple plan to overcome the ill-posedness and the nonlinearity of these problems. Studies and research to find more applications of this method are one of the goals of our research team.

TABLE 7. Absolute errors for Example 4.7.

x	Our method with $k=2, M=7$	Our method with $k=3, M=7$
0.1	2.22045×10^{-16}	5.55112×10^{-16}
0.2	3.33067×10^{-16}	5.55112×10^{-17}
0.3	3.08628×10^{-13}	9.02056×10^{-16}
0.4	6.32633×10^{-13}	8.4148×10^{-14}
0.5	2.04925×10^{-10}	1.31130×10^{-11}
0.6	6.75515×10^{-12}	2.05987×10^{-13}
0.7	3.28459×10^{-13}	2.34535×10^{-15}
0.8	5.20417×10^{-16}	3.88578×10^{-16}
0.9	1.38778×10^{-15}	2.22045×10^{-16}

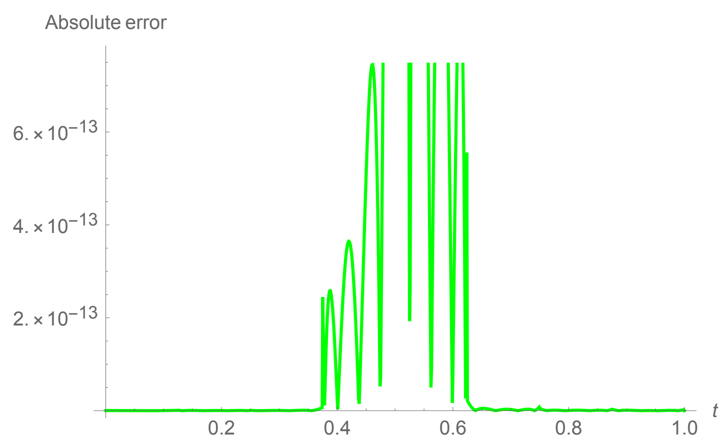

FIGURE 11. Absolute errors with $k = 3$ and $M = 5$ of Example 4.7.

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