Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 11, No. 4, 2023, pp. 676-695
DOI:10.22034/cmde.2022.51249.2132

An explicit split-step truncated Milstein method for stochastic differential equations

Amir Haghighi*
Department of Mathematics, Faculty of Science, Razi University, Kermanshah 67149, Iran.

Abstract

In this paper, we propose an explicit split-step truncated Milstein method for stochastic differential equations (SDEs) with commutative noise. We discuss the mean-square convergence properties of the new method for numerical solutions of a class of highly nonlinear SDEs in a finite time interval. As a result, we show that the strong convergence rate of the new method can be arbitrarily close to one under some additional conditions. Finally, we use an illustrative example to highlight the advantages of our new findings in terms of both stability and accuracy compared to the results in Guo et al. (2018).

Keywords. Stochastic differential equations, Non-globally Lipschitz conditions, strong convergence rate, Truncated Milstein method, Split-step methods.

2010 Mathematics Subject Classification. 65H10, 65C30, 60H35.

1. Introduction

Stochastic differential equations are the subject of numerous investigations by scientists [15, 20, 23, 25]. The interest in these equations arises mainly from their applications to many models in physics, economics, chemistry, biology, etc. [1, 5, 7, 15, 31]. However, for the majority of nonlinear SDEs, exact solutions are not known. Therefore, numerical methods become important tools for computing approximate solutions for SDEs [15]. Researchers have proposed several numerical methods to solve such equations and have well studied the convergence properties of these methods under the classical global Lipschitz condition (see, e.g., [3, 4, 6, 10, 19, 24, 26, 27]). However, in many applications, the global Lipschitz and linear growth conditions are perturbed, so most of the proposed methods, such as the Euler-Maruyama (EM) and Milstein methods, face violated convergence properties [12]. Higham, Mao, and Stuart [14] first addressed this issue in their influential 2002 paper. They proved that the uniform boundedness of the moments of both the solution of the SDE and its approximation is sufficient for strong convergence. Subsequently, other numerical techniques have been proposed to solve the divergence caused by the nonlinearity of the coefficients of the original system. We can divide these techniques into implicit [5, 14, 21] and modified versions of explicit techniques, and each of them has its advantages and disadvantages. Implicit methods are characterized by strong convergence and have extended stability regions, which are well suited for solving stiff problems [2, 11, 29, 30]. However, the implementation of implicit methods requires the solution of an additional algebraic equation at each time step, which can drastically increase the computational cost. Therefore, some explicit numerical methods based on changes in drift and diffusion coefficients have been proposed. These numerical methods include the tamed Euler-Maruyama method [13, 28], the tamed Milstein method [32], the stopped EM method [18], the truncated EM method [22] and, the partially truncated Euler–Maruyama method [8]. More recently, Guo et al. [9] introduced a truncated Milstein method for SDEs with commutative noise. This method was further developed in [16] and, the authors present a new truncated Milstein method with order one convergence similar to the Milstein method for SDEs with global Lipschitz coefficients. Liao et al [17] extended the truncated Milstein method to the nonautonomous SDEs with the superlinear state variable

Received: 22 April 2022 ; Accepted: 29 November 2022.
∗ Corresponding author. Email: a.haghighi@razi.ac.ir.

and the continuous Hölder time variable. Moreover, Zhan et al. in [33] proposed a truncated Milstein method with convergence of order one in the mean square sense for superlinear SDEs modulated by a Markov chain.

Despite the performance of the truncated Milstein method [8, 16] in the convergence order, there is still a drawback. The use of explicit methods is often expensive because of the step size reduction due to stability issues. Therefore, for large step sizes, the ability to preserve the qualitative behavior of the solution of the original system is low [2]. In this paper, we propose as a fully explicit method a split-step truncated Milstein method for solving Itô SDEs. We study the convergence properties of the new method under the non-global Lipschitz condition. Following the ideas in [8], we prove that the new method has a strong convergence rate arbitrarily close to one. Finally, we use an illustrative example to show the efficiency of the proposed method in terms of both stability and accuracy.

The rest of the paper is organized as follows. In the next section, we discussed some basic definitions and preliminary results. Then, in section 3, we propose the split-step truncated method for Itô-SDEs with multidimensional noise and obtain the uniform boundedness of the p-th moments. In section 4, we analyze the convergence properties of this method under the non-global Lipschitz conditions. Finally, in section 5, we implement the new method with an example that confirms the theoretical results.

2. Some definitions and preliminary results

Consider the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). If $x \in \mathbb{R}^d$, let $|x| = (x_1^2 + \cdots + x_d^2)^{1/2}$ be the Euclidean norm here and throughout the paper. If $A \in \mathbb{R}^{d\times m}$, then $|A|$ denotes the trace norm for the matrix A, i.e., $|A| = \sqrt{trace(A^TA)}$. Moreover, $a \vee b$ and $a \wedge b$ denote the maximum and minimum, respectively, of the numbers $a, b \in \mathbb{R}$. Finally, the indicator function for a set G is denoted by I_G.

In this paper, we study the numerical solution of the Itô stochastic differential equation

$$dx(t) = f(x(t))dt + \sum_{j=1}^{m} g_j(x(t))dB^j(t), \quad 0 \leq t \leq T, \quad x(0) = x_0 \in \mathbb{R}^d, \tag{2.1}$$

where $B(t) = (B_1(t), \ldots, B_m(t))^T$ is an m-dimensional Brownian motion defined on the probability space and is $\mathcal{F}_t$-adapted. Here $f : \mathbb{R}^d \to \mathbb{R}^d$ is drift and $g : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ with $g = (g_1, \ldots, g_m)^T$ is the diffusion. In the following, we consider numerical methods on a uniform mesh $t_n = n\Delta$ for $n = 1, \ldots, N$, with step size $\Delta = T/N$ and $N \in \mathbb{N}$.

One well-known method for approximating the SDE (2.1) is the Milstein method

$$Y_{k+1}^M = Y_k^M + f(Y_k^M)\Delta + \sum_{j=1}^{m} g_j(Y_k^M)\Delta B_j^k + \sum_{j_1=1}^{m}\sum_{j_2=1}^{m} L^{j_1}g_{j_2}(Y_k^M)I_{(j_1,j_2)}, \tag{2.2}$$

where

$$L^{j_1}g_{j_2}(x) = \sum_{l=1}^{d} g_{l,j_1}(x)G_{j_2}^l(x), \quad I_{(j_2,j_1)} = \int_{t_k}^{t_{k+1}} \Big(B^{j_2}(s) - B^{j_2}(t_k)\Big)dB^{j_1}(s).$$

In the above relation for $l = 1, \ldots, d$ and $j_2 = 1, \ldots, m$, the function $G_{j_2}^l(x)$ is defined as follows:

$$G_{j_2}^l(x) := \frac{\partial}{\partial x^l}g_{j_2}(x) = \Big(\frac{\partial g_{l,j_2}(x)}{\partial x^l}, \ldots, \frac{\partial g_{d,j_2}(x)}{\partial x^l}\Big)^T. \tag{2.3}$$

We consider the case of the SDE (2.1) with commutative noise, i.e., when the diffusion satisfies the commutativity condition

$$L^{j_1}g_{j_2}(x) = L^{j_2}g_{j_1}(x), \quad \forall x \in \mathbb{R}^d,$$

for all $j_1, j_2 = 1, \ldots, m$. With the help of the well-known property $I_{(j_1,j_2)} + I_{(j_2,j_1)} = \Delta B_k^{j_1}\Delta B_k^{j_2}$ for $j_1 \neq j_2$, the Milstein method (2.2) reduces to the following:

$$Y_{k+1}^M = Y_k^M + f(Y_k^M)\Delta + \sum_{j=1}^{m} g_j(Y_k^M)\Delta B_k^j + \frac{1}{2}\sum_{j_1=1}^{m}\sum_{j_2=1}^{m}\sum_{l=1}^{d} g_{l,j_1}(Y_k^M)G_{j_2}^l(Y_k^M)\Big(\Delta B_k^{j_1}\Delta B_k^{j_2} - \delta_{j_1,j_2}\Delta\Big). \tag{2.4}$$

The Milstein method (2.2) is convergent with order one in the mean square sense under the global Lipschitz condition and the linear growth condition [24]. However, when this condition is perturbed, the Milstein method is shown to be no longer convergent [12]. In the next section, we propose a truncated split-step method suitable for numerical solutions of a class of highly nonlinear SDEs in a finite time interval. To construct this method, we require $f, g \in \mathcal{C}^2(\mathbb{R}^d)$. We also estimate the growth rate of the coefficients f and g under the following assumptions.

Assumption 1. There exist real positive constants K_1 and r such that

$$|f(x)-f(y)| \vee |g_j(x)-g_j(y)| \vee |L^{j_1}g_{j_2}(x)-L^{j_1}g_{j_2}(y)| \leq K_1(1+|x|^r+|y|^r)|x-y|,$$

for all $x, y \in \mathbb{R}^d$ and $j, j_1, j_2 = 1, \dots, m$.

From Assumption 1, we can choose a strictly increasing continuous function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mu(u) \to \infty$ as $u \to \infty$ and

$$\sup_{0<|x|\vee|y|<u} \frac{|f(x)-f(y)|}{|x-y|} \vee \frac{|g_j(x)-g_j(y)|}{|x-y|} \leq \mu(u), \tag{2.5}$$

for any $u \geq 2$ and $j = 1, \dots, m$. Besides, Assumption 1 implies that

$$|f(x)| \vee |g_j(x)| \leq \lambda_1(1+|x|^{r+1}), \ \ \forall j = 1, \dots, m, \ \ \forall x \in \mathbb{R}^d, \tag{2.6}$$

where $\lambda_1 \in \mathbb{R}_+$.

Assumption 2. Suppose for all $p \geq 1$, there exists a positive constant K_2, dependent on p, such that

$$\big\langle x-y, f(x)-f(y)\big\rangle + (2p-1)\sum_{j=1}^{m} |g_j(x)-g_j(y)| \leq K_2|x-y|^2. \tag{2.7}$$

If f and g satisfy in Assumption 2, then for all $p \geq 1$, we can prove

$$\big\langle x, f(x)\big\rangle + (2p-1)\sum_{j=1}^{m} |g_j(x)| \leq \lambda_2(1+|x|^2), \ \ \forall x \in \mathbb{R}^d, \tag{2.8}$$

for some constant $\lambda_2 > 0$ depending on p [9].

Theorem 2.1. *[20, pp. 59, Thoprem 4.1] Let Assumptions 1 and 2 hold. Then, the SDE (2.1) with the initial value $x(0) = x_0 \in \mathbb{R}^d$ has a unique global solution $x(t)$. Moreover, for any $t \in [0, T]$, there is a positive constant C, that depends on T, p, and x_0, so that*

$$\mathbb{E}|x(t)|^{2p} \leq C\Big(1+|x(0)|^{2p}\Big). \tag{2.9}$$

The following lemma is a natural result of Theorem 2.1, Assumptions 1 and 2, see, e.g., [9, 20].

Lemma 2.2. *Let x be a solution of (2.1). If the coefficients of the SDE (2.1) fulfill the Assumptions 1 and 2, then for all $p \geq 1$ and $j = 1, \dots, m$,*

$$\sup_{0\leq t\leq T}\Big(\mathbb{E}|x(t)|^p \vee \mathbb{E}|f(x(t))|^p \vee \mathbb{E}|g_j(x(t))|^p\Big) < \infty. \tag{2.10}$$

Remark 2.3. For any real number $R > |x_0|$, consider the stopping time

$$\tau_R := \inf\{t \geq 0, |x(t)| \geq R\}. \tag{2.11}$$

Based on Theorem 2.1, Guo et al. in [9] indicated that there exists a positive constant K independent of R such that

$$\mathbb{P}(\tau_R \leq T) \leq \frac{K}{R^{2p}}. \tag{2.12}$$

We will use the fundamental inequality (2.11) to prove the main theorem in Section 4.

Assumption 3. We assume there is a positive constant K_3 and $r \in \mathbb{R}_+$ such that

$$|\nabla_x f_i(z)| \vee |H_x f_i(z)| \vee |\nabla_x g_{i,j}(z)| \vee |H_x g_{i,j}(z)| \le K_3\big(1+|z|^{r+1}\big), \quad \forall z \in \mathbb{R}^d,$$

for $i = 1,\dots,d$ and $j = 1,\dots,m$, where $\nabla_x f_i$ and $H_x f_i$ are the gradient vector and the Hessian matrix of f_i with respect to the variable x, respectively.

3. The split-step truncated explicit Milstein method

In what follows, C stands for generic positive real constants that can vary from one place to another and depend on p, T, and x_0 but are independent of the step size Δ. To construct the new method, we assume that the function μ satisfies the following condition in addition to the properties (2.5)

$$\sup_{|x|<u}\Big(|f(x)| \vee |g(x)| \vee |G^l_{j_2}|\Big) \le \mu(u), \tag{3.1}$$

for any $u \ge 2$, $j = 1,\dots,m$ and $l = 1,\dots,d$. Due to (2.6) and Assumption 3, the function μ is well-defined. Clearly, $\mu^{-1} : [\mu(0),+\infty) \to (0,+\infty)$ is a strictly increasing continuous function. Assume there is a number $\Delta^* \in (0,1]$ and a strictly decreasing function $h : (0,\Delta^*] \to (0,+\infty)$ such that $h(\Delta) \to \infty$ as $\Delta \to 0$ and for any $\Delta \in (0,\Delta^*]$, we have

$$\Delta^{1/4}h(\Delta) \le 1, \quad \mu(1) \le h(\Delta^*). \tag{3.2}$$

For a given step size $\Delta \in (0,\Delta^*]$, let us define the truncation mapping $\pi_\Delta : \mathbb{R}^d \to \mathbb{R}^d$ by $\pi_\Delta(x) := \big(\mu^{-1}(h(\Delta)) \wedge |x|\big)\frac{x}{|x|}$, where we set $\frac{x}{|x|} = 0$ if $x = 0$.

Remark 3.1. For given $x, y \in \mathbb{R}^d$ and a fixed $\Delta \in (0,\Delta^*]$, if $|x| > \mu^{-1}(h(\Delta))$ and $|y| > \mu^{-1}(h(\Delta))$, then

$$\begin{aligned}\big|\pi_\Delta(x) - \pi_\Delta(y)\big| &= \Big|\frac{\mu^{-1}(h(\Delta))}{|x|}(x-y) + \mu^{-1}(h(\Delta))y\big(\frac{1}{|x|} - \frac{1}{|y|}\big)\Big| \\ &\le \big|x-y\big| + \mu^{-1}(h(\Delta))|y|\frac{\big||y|-|x|\big|}{|x||y|} \le 2\big|x-y\big|.\end{aligned}$$

For $x, y \in \mathbb{R}^d$ with $|y| < \mu^{-1}(h(\Delta)) < |x|$, set $\alpha := \frac{\mu^{-1}(h(\Delta))}{|x|} < 1$. In this case

$$\begin{aligned}\big|\pi_\Delta(x) - \pi_\Delta(y)\big| &= \big|\alpha x - y\big| = \big|\alpha(x-y) + (\alpha-1)y\big| \\ &\le \alpha|x-y| + |y|(1-\alpha) < |x-y| + |y|\frac{|x| - \mu^{-1}(h(\Delta))}{|x|} \\ &< |x-y| + |x| - \mu^{-1}(h(\Delta)) < |x-y| + |x| - |y| < 2|x-y|.\end{aligned}$$

So, from the above relations, we can conclude

$$\big|\pi_\Delta(x) - \pi_\Delta(y)\big| \le 2|x-y|, \tag{3.3}$$

for all $x, y \in \mathbb{R}^d$.

In the following, for a given step size $\Delta \in (0,\Delta^*]$, we define the truncation functions by

$$\tilde{f}(x) := f(\pi_\Delta(x)), \quad \tilde{g}_j(x) := g_j(\pi_\Delta(x)), \quad \tilde{G}^l_j(x) := G^l_j(\pi_\Delta(x)), \tag{3.4}$$

for $l = 1,\dots,d$ and $j = 1,\dots,m$. It is obvious from (3.1) that

$$|\tilde{f}(x)| \vee |\tilde{g}_j(x)| \vee |\tilde{G}^l_j(x)| \le h(\Delta), \quad \forall x \in \mathbb{R}^d. \tag{3.5}$$

Lemma 3.2. *([16, 22]) Let (2.8) hold. Then, for all $\Delta \in (0,\Delta^*]$ and $x \in \mathbb{R}^d$*

$$\big\langle x, \tilde{f}(x)\big\rangle + (2p-1)\sum_{j=1}^{m}|\tilde{g}_j(x)| \le \lambda_3(1+|x|^2), \tag{3.6}$$

for some constat λ_3 independent of Δ.

Now, according to the definition of the truncation functions (3.4), we introduce a split-step numerical method for the SDE (2.1) with $Y_0 = x_0$ and

$$\overline{Y}_k = Y_k + \Delta\tilde{f}(Y_k), \tag{3.7}$$

$$Y_{k+1} = \overline{Y}_k + \sum_{j=1}^{m} \tilde{g}_j(\overline{Y}_k)\Delta B_k^j + \frac{1}{2}\sum_{j_1,j_2=1}^{m}\sum_{l=1}^{d} \tilde{g}_{l,j_1}(\overline{Y}_k)\tilde{G}^l_{j_2}(\overline{Y}_k)\Delta B_k^{j_1}\Delta B_k^{j_2} - \frac{1}{2}\sum_{j=1}^{m}\sum_{l=1}^{d}\tilde{g}_{l,j}(\overline{Y}_k)\tilde{G}^l_j(\overline{Y}_k)\Delta. \tag{3.8}$$

From now on, to simplify the notation, we set

$$L^{j_1}\tilde{g}_{j_2}(x) := \sum_{l=1}^{d}\tilde{g}_{l,j_1}(x)\tilde{G}^l_{j_2}(x). \tag{3.9}$$

Therefore, the proposed method (3.8) reduces to the simple version as below:

$$\overline{Y}_k = Y_k + \Delta\tilde{f}(Y_k),$$

$$Y_{k+1} = \overline{Y}_k + \sum_{j=1}^{m}\tilde{g}_j(\overline{Y}_k)\Delta B_k^j + \frac{1}{2}\sum_{j_1,j_2=1}^{m} L^{j_1}\tilde{g}_{j_2}(\overline{Y}_k)\Big(\Delta B_k^{j_1}\Delta B_k^{j_2} - \delta_{j_1,j_2}\Delta\Big).$$

Now, we form a continuous-time version of the truncated split-step Milstein method (3.8). In this regard, first, for any step size $\Delta \in (0, \Delta^*]$ satisfying in (3.2), we define $Y(t) = Y_k$ and $\overline{Y}(t) = \overline{Y}_k$ for $t_k \le t < t_{k+1}$. Then, continuous-time version of the new method (3.8) is defined by

$$y(t) = Y(t) + \int_{t_k}^{t}\tilde{f}(Y(s))ds + \sum_{j=1}^{m}\int_{t_k}^{t}\tilde{g}_j(\overline{Y}(s))dB^j(s) + \sum_{j_1=1}^{m}\int_{t_k}^{t}\sum_{j_2=1}^{m} L^{j_1}\tilde{g}_{j_2}(\overline{Y}(s))\Delta B^{j_2}(s)dB^{j_1}(s), \tag{3.10}$$

in which

$$\Delta B^{j_2}(s) = \sum_{k=0}^{\infty} I_{[t_k,t_{k+1})}(B^{j_2}(s) - B^{j_2}(t_k)). \tag{3.11}$$

Since $\overline{Y}(t) = Y(t) + \int_{t_k}^{t_{k+1}}\tilde{f}(Y(s))ds$ for all $t_k \le t < t_{k+1}$, so we can rewrite (3.10) as follows

$$y(t) - \overline{Y}(t) = -\int_{t}^{t_{k+1}}\tilde{f}(Y(s))ds + \sum_{j=1}^{m}\int_{t_k}^{t}\tilde{g}_j(\overline{Y}(s))dB^j(s) + \sum_{j_1=1}^{m}\int_{t_k}^{t}\sum_{j_2=1}^{m} L^{j_1}\tilde{g}_{j_2}(\overline{Y}(s))\Delta B^{j_2}(s)dB^{j_1}(s). \tag{3.12}$$

Lemma 3.3. *Fix the step size $\Delta \in (0, \Delta^*]$. Then for all $t \ge 0$ and $p \ge 1$*

$$\mathbb{E}(|y(t) - Y(t)|^{2p}) \le C\Delta^p(h(\Delta))^{2p}, \tag{3.13}$$

where C is a positive constant independent of Δ.

Proof. Obviously, for all $t \ge 0$, there is a unique integer $k \ge 0$ such that $t_k \le t < t_{k+1}$. Using Hölder inequality and (3.5) we can write

$$\begin{aligned}\mathbb{E}\Big|\overline{Y}(t) - Y(t)\Big|^{2p} &= \mathbb{E}\Big|\int_{t_k}^{t}\tilde{f}(Y(s))ds\Big|^{2p} \le \mathbb{E}\left(\int_{t_k}^{t}\Big|\tilde{f}(Y(s))\Big|^{2p}ds\Big|\int_{t_k}^{t}ds\Big|^{2p-1}\right)\\ &\le \Delta^{2p-1}\mathbb{E}\int_{t_k}^{t}\Big|\tilde{f}(Y(s))\Big|^{2p}ds \le \Delta^{2p}(h(\Delta))^{2p}.\end{aligned} \tag{3.14}$$

Now, because of the inequality $|\sum_{i=1}^{n}\alpha_i|^p \le n^{p-1}\sum_{i=1}^{n}|\alpha_i|^p$ from (3.10), we have

$$\mathbb{E}\Big|y(t) - Y(t)\Big|^{2p} \le C\mathbb{E}\left(\Big|\int_{t_k}^{t}\tilde{f}(Y(s))ds\Big|^{2p} + \Big|\sum_{j=1}^{m}\int_{t_k}^{t}\tilde{g}_j(\overline{Y}(s))dB^j(s)\Big|^{2p} + \Big|\sum_{j_1=1}^{m}\int_{t_k}^{t}\sum_{j_2=1}^{m}L^{j_1}\tilde{g}_{j_2}(\overline{Y}(s))\Delta B^{j_2}(s)dB^{j_1}(s)\Big|^{2p}\right). \tag{3.15}$$

Then, by Theorem 7.1 in [20] (Page 39), we have

$$\mathbb{E}\big|y(t)-Y(t)\big|^{2p} \leq C\Bigg(\Delta^{2p-1}\mathbb{E}\int_{t_k}^{t}\Big|\tilde{f}(Y(s))\Big|^{2p}ds + \Delta^{\frac{2p-2}{2}}\sum_{j=1}^{m}\mathbb{E}\int_{t_k}^{t}\Big|\tilde{g}_j(\overline{Y}(s))\Big|^{2p}ds$$
$$+\Delta^{\frac{2p-2}{2}}\sum_{j_1=1}^{m}\sum_{j_2=1}^{m}\mathbb{E}\int_{t_k}^{t}\Big|L^{j_1}\tilde{g}_{j_2}(\overline{Y}(s))\Big|^{2p}\Big|\Delta B^{j_2}(s)\Big|^{2p}ds\Bigg).$$

Applying the fact that $\mathbb{E}|\Delta B^{j_2}(s)|^{2p} \leq C\Delta^p$ for $s \in [t_k, t_{k+1})$ and the relation (3.5), we can conclude

$$\mathbb{E}\big|y(t)-Y(t)\big|^{2p} \leq C\big(\Delta^{2p}(h(\Delta))^{2p} + \Delta^{p}(h(\Delta))^{2p} + \Delta^{2p}(h(\Delta))^{4p}\big). \tag{3.16}$$

□

Lemma 3.4. *If the relation (2.8) holds, then for any $T > 0$ there is a positive constant C independent of $\Delta \in (0, \Delta^*]$ such that*

$$\sup_{0<\Delta\leq\Delta^*}\big(\sup_{0\leq t\leq T}\mathbb{E}|y(t)|^{2p}\big) \leq C\big(1+\mathbb{E}|y(0)|^{2p}\big), \quad \forall p \geq 1. \tag{3.17}$$

Proof. For $t \in [t_k, t_{k+1})$, we have

$$y(t) = Y_k + \int_{t_k}^{t}\tilde{f}(Y(s))ds + \sum_{j=1}^{m}\int_{t_k}^{t}\tilde{g}_j(\overline{Y}(s))dB^j(s) + \sum_{j_1=1}^{m}\int_{t_k}^{t}\sum_{j_2=1}^{m}L^{j_1}\tilde{g}_{j_2}(\overline{Y}(s))\Delta B^{j_2}(s)dB^{j_1}(s).$$

On the other hand, according to the definition Y_k and $\overline{Y}_k$, we can write

$$Y_k = \overline{Y}_{k-1} + \sum_{j=1}^{m}\int_{t_{k-1}}^{t_k}\tilde{g}_j(\overline{Y}_{k-1})dB^j(s) + \sum_{j_1,j_2=1}^{m}L^{j_1}\tilde{g}_{j_2}(\overline{Y}_{k-1})\Big(\int_{t_{k-1}}^{t_k}\big(B^{j_2}(s)-B^{j_2}(t_{k-1})\big)dB^{j_1}(s)\Big), \tag{3.18}$$

in which

$$\overline{Y}_{k-1} = Y_{k-1} + \Delta\tilde{f}(Y_{k-1}) = Y_{k-1} + \int_{t_{k-1}}^{t_k}\tilde{f}(Y(s))ds. \tag{3.19}$$

So, with definition $\Delta B^j(s)$ in (3.11), we obtain

$$Y_k = \overline{Y}_{k-1} + \sum_{j=1}^{m}\int_{t_{k-1}}^{t_k}\tilde{g}_j(\overline{Y}(s))dB^j(s) + \sum_{j_1=1}^{m}\Big(\int_{t_{k-1}}^{t_k}\sum_{j_2=1}^{m}L^{j_1}\tilde{g}_{j_2}(\overline{Y}(s))\Delta B^{j_2}(s)dB^{j_1}(s)\Big). \tag{3.20}$$

Therefore,

$$y(t) = Y_{k-1} + \int_{t_{k-1}}^{t}\tilde{f}(Y(s))ds + \sum_{j=1}^{m}\int_{t_{k-1}}^{t}\tilde{g}_j(\overline{Y}(s))dB^j(s) + \sum_{j_1=1}^{m}\int_{t_{k-1}}^{t}\sum_{j_2=1}^{m}L^{j_1}\tilde{g}_{j_2}(\overline{Y}(s))\Delta B^{j_2}(s)dB^{j_1}(s). \tag{3.21}$$

Continuing this process, we can consider $\{y(t)\}_{t\in[0,t)}$ as an Itô process with the integral form

$$y(t) = y(0) + \int_0^t U(s)ds + \sum_{j=1}^{m}\int_0^t V_j(s)dB^j(s),$$

in which

$$U(t) := \tilde{f}(Y(t)),$$
$$V_j(t) := \tilde{g}_j(\overline{Y}(t)) + \sum_{j_2=1}^{m}L^j\tilde{g}_{j_2}(\overline{Y}(t))\Delta B^{j_2}(t). \tag{3.22}$$

Let $f(t,x)=|x|^{2p}$ and $V(t):=[V_1(t),\dots,V_m(t)]^T$, then by the Itô formula

$$|y(t)|^{2p}=|y(0)|^{2p}+\int_0^t(\nabla_x f)^T V(s)dB(s)+\int_0^t\Big\{\frac{\partial f}{\partial t}+(\nabla_x f)^T U(s)+\frac{1}{2}trace\big(V(s)^T(H_x f)V(s)\big)\Big\}ds, \quad (3.23)$$

hold, in which

$$H_x f(t,x)=2p|x|^{2p-2}I+4p(p-1)|x|^{2p-4}xx^T. \quad (3.24)$$

Applying some matrix calculation, we obtain

$$\begin{aligned}\frac{1}{2}trace\Big(V(s)^T(H_x f)V(s)\Big)&=p|y(s)|^{2p-2}trace\Big(V(s)^TV(s)\Big)+2p(p-1)|y(s)|^{2p-4}trace\Big(\big(V(s)^Ty(s)\big)\big(V(s)^Ty(s)\big)^T\Big)\\&\le 2p\frac{2p-1}{2}|y(s)|^{2p-2}\sum_{j_1=1}^m|V_{j_1}(s)|^2. \quad (3.25)\end{aligned}$$

By inserting (3.25) in (3.23) and applying expectation, we can write

$$\mathbb{E}|y(t)|^{2p}\le\mathbb{E}|y(0)|^{2p}+2p\mathbb{E}\int_0^t|y(s)|^{2p-2}\Big\langle y(s),\tilde f(Y(s))\Big\rangle ds+2p\mathbb{E}\int_0^t\frac{2p-1}{2}|y(s)|^{2p-2}\sum_{j_1=1}^m|V_{j_1}(s)|^2ds.$$

So, we can rewrite the inequality as bellows

$$\begin{aligned}\mathbb{E}|y(t)|^{2p}&\le\mathbb{E}|y(0)|^{2p}+2p\mathbb{E}\int_0^t|y(s)|^{2p-2}\Bigg(\Big\langle\overline{Y}(s),\tilde f(\overline{Y}(s))\Big\rangle+(2p-1)\sum_{j_1=1}^m\big|\tilde g_{j_1}(\overline{Y}(s))\big|^2\Bigg)ds\\&+2p(2p-1)m\mathbb{E}\int_0^t|y(s)|^{2p-2}\sum_{j_1=1}^m\sum_{j_2=1}^m\Big|L^{j_1}\tilde g_{j_2}(\overline{Y}(s))\Delta B^{j_2}(s)\Big|^2ds\\&+2p\mathbb{E}\int_0^t|y(s)|^{2p-2}\Big\langle y(s)-Y(s),\tilde f(Y(s))\Big\rangle ds\\&+2p\mathbb{E}\int_0^t|y(s)|^{2p-2}\Big\langle Y(s)-\overline{Y}(s),\tilde f(Y(s))\Big\rangle ds\\&+2p\mathbb{E}\int_0^t|y(s)|^{2p-2}\Big\langle\overline{Y}(s)-Y(s),\tilde f(Y(s))-\tilde f(\overline{Y})(s)\Big\rangle ds\\&+2p\mathbb{E}\int_0^t|y(s)|^{2p-2}\Big\langle Y(s),\tilde f(Y(s))-\tilde f(\overline{Y}(s))\Big\rangle ds.\end{aligned}$$

Therefore, from (3.1) and (3.6), we can conclude that

$$\begin{aligned}\mathbb{E}|y(t)|^{2p} &\leq \mathbb{E}|y(0)|^{2p} + 2p\lambda_3\mathbb{E}\int_0^t |y(s)|^{2p-2}\big(1+|\overline{Y}(s)|^2\big)ds \\ &+ 2m^3p(2p-1)\mathbb{E}\int_0^t |y(s)|^{2p-2}\big|h(\Delta)\big|^4\Delta ds \\ &+ 2p\mathbb{E}\int_0^t |y(s)|^{2p-2}\Big\langle y(s)-Y(s),\tilde{f}(Y(s))\Big\rangle ds \\ &+ 2p\mathbb{E}\int_0^t |y(s)|^{2p-2}\Big\langle Y(s)-\overline{Y}(s),\tilde{f}(Y(s))\Big\rangle ds \\ &+ 2p\mathbb{E}\int_0^t |y(s)|^{2p-2}\Big\langle \overline{Y}(s)-Y(s),\tilde{f}(Y(s))-\tilde{f}(\overline{Y}(s))\Big\rangle ds \\ &+ 2p\mathbb{E}\int_0^t |y(s)|^{2p-2}\Big\langle Y(s),\tilde{f}(Y(s))-\tilde{f}(\overline{Y}(s))\Big\rangle ds \\ &\leq \mathbb{E}|y(0)|^{2p} + 2p\Big(\lambda_3\Pi_1 + m^3(2p-1)\Pi_2 + \Pi_3 \\ &+ \Pi_4 + \Pi_5 + \Pi_6\Big). \end{aligned} \tag{3.26}$$

Now, by applying the Young inequality that is

$$x^{2p-2}y \leq \frac{2p-2}{2p}x^{2p} + \frac{1}{p}y^p, \quad \forall p \geq 1, \quad \forall x, y \in \mathbb{R}^+, \tag{3.27}$$

we try to estimate the values Π_i, for $i = 1, \dots, 6$. Concerning Π_1, we use (3.27) to arrive at

$$\Pi_1 \leq \frac{2p-2}{2p}\int_0^t \mathbb{E}|y(s)|^{2p}ds + \frac{2^{3p-2}}{p}\int_0^t \mathbb{E}|Y(s)|^{2p}ds + \frac{2^{p-1}}{p}(1+2^{2p-1}\Delta^{\frac{3p}{2}})t. \tag{3.28}$$

For Π_2, (3.27) and (3.2) gives

$$\Pi_2 \leq \frac{2p-2}{2p}\int_0^t \mathbb{E}|y(s)|^{2p}ds + \frac{1}{p}t. \tag{3.29}$$

By the relations (3.2), (3.5), (3.27), and Lemma 3.3, we obtain

$$\begin{aligned}\Pi_3 &\leq \frac{2p-2}{2p}\int_0^t \mathbb{E}|y(s)|^{2p}ds + \frac{1}{p}\mathbb{E}\int_0^t |y(s)-Y(s)|^p|\tilde{f}(Y(s))|^p ds \\ &\leq \frac{2p-2}{2p}\int_0^t \mathbb{E}|y(s)|^{2p}ds + \frac{C}{p}t.\end{aligned} \tag{3.30}$$

Concerning Π_4, similarly as above, Lemma 3.3 gives

$$\begin{aligned}\Pi_4 &\leq \frac{2p-2}{2p}\int_0^t \mathbb{E}|y(s)|^{2p}ds + \frac{1}{p}\mathbb{E}\int_0^t |Y(s)-\overline{Y}(s)|^p|\tilde{f}(Y(s))|^p ds \\ &\leq \frac{2p-2}{2p}\int_0^t \mathbb{E}|y(s)|^{2p}ds + \frac{\Delta^{p/2}}{p}t.\end{aligned} \tag{3.31}$$

From (3.2), (3.5), (3.27) and Lemma 3.3, we have

$$\begin{aligned}\Pi_5 &\leq \frac{2p-2}{2p}\int_0^t \mathbb{E}|y(s)|^{2p}ds + \frac{1}{p}\mathbb{E}\int_0^t |Y(s)-\overline{Y}(s)|^p|\tilde{f}(Y(s))-\tilde{f}(\overline{Y}(s))|^p ds \\ &\leq \frac{2p-2}{2p}\int_0^t \mathbb{E}|y(s)|^{2p}ds + \frac{2^{p-1}}{p}\Delta^{p/2}t.\end{aligned} \tag{3.32}$$

By the relations (3.2), (3.5), (3.27), Lemma 3.3, the inequality (2.5), and Remark 3.1, we can approximate Π_6 as below

$$\begin{aligned} \Pi_6 &\leq \frac{2p-2}{2p}\int_0^t \mathbb{E}|y(s)|^{2p}ds + \frac{1}{2p}\int_0^t \mathbb{E}|Y(s)|^{2p}ds \\ &+ \frac{1}{2p}\mathbb{E}\int_0^t |\tilde{f}(Y(s)) - \tilde{f}(\overline{Y}(s))|^{2p}ds \\ &\leq \frac{2p-2}{2p}\int_0^t \mathbb{E}|y(s)|^{2p}ds + \frac{1}{2p}\int_0^t \mathbb{E}|Y(s)|^{2p}ds + \frac{1}{2p}\Delta^p t. \end{aligned} \tag{3.33}$$

Inserting (3.28)-(3.33) into (3.26) leads to

$$\mathbb{E}|y(t)|^{2p} \leq \mathbb{E}|y(0)|^{2p} + A_1 t + A_2\int_0^t \mathbb{E}|y(s)|^{2p}ds + A_3\int_0^t \mathbb{E}|Y(s)|^{2p}ds,$$

in which

$$\begin{aligned} &A_1 := \max\{\lambda_3 2^p(1+2^{2p-1}), 2m^3(2p-1), 2C, 2^p\} + 1, \\ &A_2 := \max\{(2p-2)\lambda_3, (2p-2)(2p-1)m^3\}, \\ &A_3 := \lambda_3 2^{3p-1} + 1. \end{aligned}$$

So, we can conclude

$$\mathbb{E}|y(t)|^{2p} \leq \mathbb{E}|y(0)|^{2p} + B_1 t + B_2\int_0^t \Big(\sup_{0\leq u\leq s}\mathbb{E}|y(s)|^{2p}ds\Big),$$

for some positive real constants B_1 and B_2. As the sum of the right-hand-side terms in the above inequality is an increasing function of t, we have

$$\sup_{0\leq s\leq t}\mathbb{E}|y(t)|^{2p} \leq \mathbb{E}|y(0)|^{2p} + B_1 t + B_2\int_0^t \Big(\sup_{0\leq u\leq s}\mathbb{E}|y(s)|^{2p}ds\Big).$$

By the Gronwall inequality, we obtain

$$\sup_{0\leq s\leq t}\mathbb{E}|y(t)|^{2p} \leq C(1+\mathbb{E}|y(0)|^{2p}),$$

which complete the proof. □

Remark 3.5. Let (2.8) hold and Δ in $(0,\Delta^*]$ be fixed. For any $t\in[0,T]$, there is integer number k such that $t\in[t_k,t_{k+1})$ and $\overline{Y}(t)=\overline{Y}_k$. From (3.2), (3.5) and (3.7), we can write

$$|\overline{Y}(t)| \leq |Y(t)| + \Delta|\tilde{f}(Y(t))| \leq |Y(t)| + \Delta^{3/4}.$$

Therefore, by elementary calculation and from Lemma 3.4 we write

$$\mathbb{E}|\overline{Y}(t)|^{2p} \leq 2^{2p-1}\Big(\sup_{0\leq\Delta\leq\Delta^*}\big(\sup_{0\leq t\leq T}\mathbb{E}|Y(t)|^{2p}\big)+1\Big),$$

which implies

$$\sup_{0\leq\Delta\leq\Delta^*}\big(\sup_{0\leq t\leq T}\mathbb{E}|\overline{Y}(t)|^{2p}\big) < \infty. \tag{3.34}$$

Below we present two useful lemmas that are a natural consequence of 3.4 and Remark 3.5, see [9] for more details.

Lemma 3.6. *Let $y(t)$ be the numerical approximation generated by (3.10), if Assumptions 1, 2, and 3 hold, then for all $p\geq 1$ and $j_1, j_2 = 1,\dots,m$,*

$$\sup_{0\leq\Delta\leq\Delta^*}\sup_{0\leq t\leq T}\Big(\mathbb{E}|f(y(t))|^p \vee \mathbb{E}|f'(y(t))|^p \vee \mathbb{E}|g(y(t))|^p \vee \mathbb{E}|L^{j_1}(g_{j_2}(y(t))|^p\Big) < \infty.$$

Proof. From Lemma 3.4 and Remark 3.5, the result is obvious. It is worth mentioning, because of the relation (3.34) with Assumptions 1, 2, and 3, one can similarly arrive at

$$\sup_{\Delta\in[0,\Delta^*]}\sup_{t\in[0,T]}\Big(\mathbb{E}|f(\overline{Y}(t))|^p\vee\mathbb{E}|f'(\overline{Y}(t))|^p\vee\mathbb{E}|g(\overline{Y}(t))|^p\vee\mathbb{E}|L^{j_1}(g_{j_2}(\overline{Y}(t))|^p\Big)<\infty.$$

□

Lemma 3.7. *For any real number $R>|x_0|+\Delta^*|f(x_0)|$, consider two stopping times*

$$\gamma_R:=\inf\{t\geq 0,|\overline{Y}(t)|\geq R\},\ \textit{and}\ \rho_R:=\inf\{t\geq 0,|y(t)|\geq R\}. \tag{3.35}$$

Let the condition (2.8) hold and $R>|x_0|+\Delta^*|f(x_0)|$ be fixed. Then for any sufficiently small step size $\Delta\in(0,\Delta^*]$, there exists positive constant K' independent of R and Δ such that

$$\mathbb{P}(\gamma_R\leq T)\vee\mathbb{P}(\rho_R\leq T)\leq\frac{K}{R^{2p}}. \tag{3.36}$$

Proof. By (2.8) and Lemma 3.4, we can derive that

$$\mathbb{E}(|y(\rho_R\wedge T)|^{2p})\leq K_1',$$

for a positive real constant K_1'. Therefore,

$$R^{2p}\mathbb{P}(\rho_R\leq T)=\mathbb{E}\Big(|y(\rho_R)|^{2p}I_{\{\rho_R\leq T\}}\Big)\leq\mathbb{E}|y(\rho_R\wedge T)|^{2p}\leq K_1'.$$

On the other hand, by (2.8) and Remark 3.5, one can similarly arrive at

$$\mathbb{P}(\gamma_R\leq T)\leq\frac{K_2'}{R^{2p}},$$

for some constant K_2' which complete the proof. □

4. Convergence analysis

Let $\psi:\mathbb{R}^d\to\mathbb{R}^d$ be a twice differentiable function. Then, by Taylor's formula, for each $z_1,z_2\in\mathbb{R}$, we have

$$\psi(z_2)-\psi(z_1)=\psi'(z_1)(z_2-z_1)+R(\psi), \tag{4.1}$$

in which

$$R(\psi):=\int_0^1(1-\lambda)\psi''\big(z_1+\lambda(z_2-z_1)\big)\big(z_2-z_1,z_2-z_1\big)d\lambda.$$

Here, for arbitrary z,h_1 and h_2 in $\mathbb{R}^d$, the derivatives have the following expression

$$\psi'(z)(h_1)=\sum_{i=1}^d\frac{\partial\psi}{\partial x^i}h_1^i,\quad \psi''(z)(h_1,h_2)=\sum_{i=1}^d\sum_{j=1}^d\frac{\partial^2\psi}{\partial x^i\partial x^j}h_1^ih_2^j.$$

For any $t\in[t_k,t_{k+1})$ from (3.10) and (3.12), by replacing ψ by g_i in (4.1), we can write

$$g_i(y(t))-g_i(Y(t))=g_i'(Y(t))\Big(\sum_{j=1}^m\int_{t_k}^t\tilde{g}_j(\overline{Y}(s)dB^j(s)\Big)+\tilde{R}_1(g_i), \tag{4.2}$$

in which

$$\tilde{R}_1(g_i)=g_i'(Y(t))\Big(\int_{t_k}^t\tilde{f}(Y(s))ds+\sum_{j_1=1}^m\int_{t_k}^t\sum_{j_2=1}^m L^{j_1}\tilde{g}_{j_2}(\overline{Y}(s))\Delta B^{j_2}(s)dB^{j_1}(s)\Big)$$
$$+\int_0^1(1-\lambda)g_i''\big(Y(s)+\lambda(y(s)-Y(s))\big)\big(y(s)-Y(s),y(s)-Y(s)\big)d\lambda.$$

Similarly, we have

$$g_i(y(t)) - g_i(\overline{Y}(t)) = g_i'(\overline{Y}(t))\Big(\sum_{j=1}^{m}\int_{t_k}^{t}\tilde{g}_j(\overline{Y}(s)dB^j(s)\Big) + \tilde{R}_2(g_i). \tag{4.3}$$

where

$$\tilde{R}_2(g_i) = g_i'(\overline{Y}(t))\Big(-\int_t^{t_{k+1}}\tilde{f}(Y(s))ds + \sum_{j_1=1}^{m}\int_{t_k}^{t}\sum_{j_2=1}^{m}L^{j_1}\tilde{g}_{j_2}(\overline{Y}(s))\Delta B^{j_2}(s)dB^{j_1}(s)\Big)$$
$$+\int_0^1(1-\lambda)g_i''\big(\overline{Y}(s)+\lambda(y(s)-\overline{Y}(s))\big)\big(y(s)-\overline{Y}(s), y(s)-\overline{Y}(s)\big)d\lambda.$$

In the following, with the help of Assumptions 1 and 2, we prove the next useful lemma.

Lemma 4.1. *Let $y(t)$ be the numerical approximation generated by (3.10), if Assumption 1, 2 and 3 hold, then for all $p \geq 1$ and $j = 1, \dots, m$,*

$$\mathbb{E}|\tilde{R}_1(f)|^p \vee \mathbb{E}|\tilde{R}_1(g_j)|^p \vee \mathbb{E}|\tilde{R}_2(f)|^p \vee \mathbb{E}|\tilde{R}_2(g_j)|^p \leq C\Delta^p\big(h(\Delta)\big)^{2p}, \tag{4.4}$$

where C is a positive constant independent of Δ.

Proof. Let $j \in \{1, \dots, m\}$ be fixed. To estimate $\mathbb{E}|\tilde{R}_2(g_i)|^p$ from the Hölder inequality, we can write

$$\mathbb{E}|\tilde{R}_2(g_j)|^p \leq C\Bigg[\Delta^p\mathbb{E}\Big|g_j'(\overline{Y}(t))(\tilde{f}(Y(t))\Big|^p$$
$$+\frac{1}{2^p}\sum_{j_1,j_2=1}^{m}\mathbb{E}\Big|g_j'(\overline{Y}(t))\Big(L^{j_1}\tilde{g}_{j_2}(\overline{Y}(t))(\Delta B_k^{j_1}\Delta B_k^{j_2} - \delta_{j_1,j_2}\Delta)\Big)\Big|^p$$
$$+\mathbb{E}\Big|\int_0^1(1-\lambda)g_j''\big(\overline{Y}(s)+\lambda(y(s)-\overline{Y}(s))\big)\big(y(s)-\overline{Y}(s), y(s)-\overline{Y}(s)\big)d\lambda\Big|^p\Bigg]$$
$$\leq C\Big[\Sigma_1+\Sigma_2+\Sigma_3\Big]. \tag{4.5}$$

Concerning Σ_1, for $t \in [0, T]$ by the Hölder inequality, Lemma 3.6 and (3.5), we have

$$\Sigma_1 \leq \Delta^p\Big(\mathbb{E}|g_j'(\overline{Y}(t))|^{2p}\Big)^{1/2}\Big(\mathbb{E}|\tilde{f}(Y(t))|^{2p}\Big)^{1/2} \leq C\Delta^p(h(\Delta))^p. \tag{4.6}$$

For Σ_2, from the Hölder inequality and the Burkholder-Davis-Gundy inequality, by independence of $\Delta B_k^{j_1}$, $\Delta B_k^{j_2}$ and $Y(t)$, we can write

$$\Sigma_2 \leq \Delta^p\sum_{j_1,j_2=1}^{m}\mathbb{E}\Big|g_j'(\overline{Y}(t))\Big(L^{j_1}\tilde{g}_{j_2}(\overline{Y}(t))\Big)\Big|^p \tag{4.7}$$
$$\leq \Delta^p\sum_{j_1,j_2=1}^{m}\big[\mathbb{E}|g_j'(\overline{Y}(t))|^{2p}\mathbb{E}|L^{j_1}\tilde{g}_{j_2}(\overline{Y}(t)|^{2p}\big]^{1/2} \leq C\Delta^p(h(\Delta))^{2p}.$$

Finally, from Assumption 3, the Jensen inequality, the Hölder inequality, and Lemma 3.4, one can derive

$$\begin{aligned}\Sigma_3 &\leq \int_0^1 (1-\lambda)^p \mathbb{E}\Big| g_j''\big(\overline{Y}(t)+\lambda(y(t)-\overline{Y}(t))\big)\big(y(t)-\overline{Y}(t), y(t)-\overline{Y}(s)\big)\Big|^p d\lambda \\ &\leq \int_0^1 \left[\mathbb{E}\Big| g_j''\big(\overline{Y}(t)+\lambda(y(s)-\overline{Y}(t))\big)\Big|^{2p}\mathbb{E}\big|y(t)-\overline{Y}(t)\big|^{4p}\right]^{1/2} d\lambda \\ &\leq C\big(1+\mathbb{E}|y(t)|^{2p(r+1)}+\mathbb{E}|\overline{Y}(t)|^{2p(r+1)}\big)^{1/2}\big(\mathbb{E}\big|y(t)-\overline{Y}(t)\big|^{4p}\big)^{1/2} \\ &\leq \Delta^p (h(\Delta))^{2p}. \end{aligned} \tag{4.8}$$

Substituting, (4.1)-(4.8) into (4.5), we obtain the desired assertion for $\mathbb{E}|\tilde{R}_2(g_i)|^p$. For to the same reason, the other terms in (4.4) are proven. □

Here, following two theorems, we show that if Assumptions 1, 2, and 3 hold, the associated split-step truncated Milstein method, defined as (3.10), has a strong rate of convergence close to one in the mean square sense. In this context, we define $e(t) := x(t) - y(t)$ for $0 \leq t \leq T$ as the global error. Moreover, for each real number $R > |x_0| + \Delta^*|f(x_0)|$ we consider the stopping times τ_R, ρ_R and γ_R defined in (2.11) and (3.35).

Theorem 4.2. *Consider any given real number $R > |x_0| + \Delta^*|f(x_0)|$. If the coefficients of SED (2.1) satisfy Assumptions 1-3, then for any step size $\Delta \in (0, \Delta^*]$ with property $\mu^{-1}(h(\Delta)) \geq R$ and for any $p \geq 1$*

$$\mathbb{E}(|e(t\wedge\theta)|^{2p}) \leq C\max\{\Delta^{p+1}, \Delta^{2p}(h(\Delta))^{4p}\}, \tag{4.9}$$

where $\theta := \tau_R \wedge \rho_R \wedge \gamma_R$.

Proof. We try to estimate $e(t\wedge\theta)$ for the approximation solution $y(t)$. In this regard, from relations (2.1) and (3.10), we can write

$$e(t\wedge\theta) = \int_0^{t\wedge\theta}\big(f(x(s)-\tilde{f}(Y(s))\big)ds + \sum_{i=1}^m \int_0^{t\wedge\theta}\Big(g_i(x(s)) - \tilde{g}_j(\overline{Y}(s)) - \sum_{j=1}^m L^j\tilde{g}_i(\overline{Y}(s))\Delta B^j(s)\Big)dB^i(s). \tag{4.10}$$

Following the similar approach presented in Theorem 3.4 in [9], we can use the Itô formula to write for any $0 \leq t \leq T$

$$\begin{aligned}\mathbb{E}(|e(t\wedge\theta|^{2p}) &= 2p\mathbb{E}\int_0^{t\wedge\theta}|e(s)|^{2p-2}\Big\langle x(s)-y(s), f(x(s))-\tilde{f}(Y(s))\Big\rangle ds \\ &+ 2p\sum_{i=1}^m \mathbb{E}\int_0^{t\wedge\theta}|e(s)|^{2p-2}\frac{2p-1}{2}\Bigg| g_i(x(s))-\tilde{g}_i(\overline{Y}(s)) \\ &- \sum_{j=1}^m L^j\tilde{g}_i(\overline{Y}(s))\Delta B^j(s)\Bigg|^2 ds. \end{aligned} \tag{4.11}$$

In (4.11), the real values s have been placed at the interval $[0, t\wedge\theta]$, so $|Y(s)| \vee |\overline{Y}(s)| < R$. Given that $\mu^{-1}(h(\Delta)) \geq R$, we have $|Y(s)| \vee |\overline{Y}(s)| < \mu^{-1}(h(\Delta))$. Therefore, from (4.11) and (4.3), we can write

$$\begin{aligned}\mathbb{E}(|e(t\wedge\theta|^{2p}) &\leq 2p\mathbb{E}\int_0^{t\wedge\theta}|e(s)|^{2p-2}\Big(\big\langle x(s)-y(s), f(x(s))-f(y(s))\big\rangle \\ &+ (2p-1)\sum_{j=1}^m\Big|g_i(x(s))-g_i(y(s))\Big|^2\Big)ds + 2p\Gamma \\ &+ 2p(2p-1)\sum_{i=1}^m\mathbb{E}\int_0^{t\wedge\theta}|e(s)|^{2p-2}\big|\tilde{R}_2(g_i)\big|^2 ds, \end{aligned} \tag{4.12}$$

in which $\Gamma := \mathbb{E}\int_0^{t\wedge\theta} |e(s)|^{2p-2}\big\langle x(s)-y(s), f(y(s))-f(Y(s))\big\rangle ds$. Now, by applying Assumption 2 and the Young inequality (3.27) we obtain

$$\mathbb{E}(|e(t\wedge\theta|^{2p}) \leq C\Big(\mathbb{E}\int_0^{t\wedge\theta} |e(s)|^{2p}ds + \sum_{i=1}^{m}\mathbb{E}\int_0^{t\wedge\theta}\big(|e(s)|^{2p}+\big|\tilde{R}_2(g_i)\big|^{2p}\big)ds\Big)+2p\Gamma.$$

To approximate Γ, one can write with the help of the Young inequality (3.27) and the Hölder inequality

$$\begin{aligned}\Gamma &\leq C\mathbb{E}\int_0^{t\wedge\theta}\Big(|e(s)|^{2p}+\big|\big\langle x(s)-y(s),\tilde{R}_1(f)\big\rangle\big|^{p}\Big)ds\\ &+\frac{2^{p-1}}{p}\mathbb{E}\int_0^{t\wedge\theta}\Big|\Big\langle x(s)-y(s), f'(Y(s))\big(\sum_{j=1}^{m}\int_{t_{k_s}}^{s} g_j(\overline{Y}(s_1))dB^j(s_1)\big)\Big\rangle\Big|^{p}ds\\ &\leq C\mathbb{E}\int_0^{t\wedge\theta}\big(|e(s)|^{2p}+|\tilde{R}_1(f)|^{2p}\big)ds+\frac{2^{p-1}}{p}J, \end{aligned}\tag{4.13}$$

in which

$$J := \mathbb{E}\int_0^{t\wedge\theta}\Big|\Big\langle x(s)-y(s), f'(Y(s))\Big(\sum_{j=1}^{m}\int_{t_{k_s}}^{s} g_j(\overline{Y}(s_1))dB^j(s_1)\Big)\Big\rangle\Big|^{p}ds. \tag{4.14}$$

Here, k_s is the greatest integer number such that $t_{k_s}\leq s< t_{k_s+1}$. Therefore, it remains to estimate the last term in (4.13). In this regard, by consider $J_{k_s} := f'(Y(s))\big(\sum_{j=1}^{m}\int_{t_{k_s}}^{s} g_j(\overline{Y}(s_1))dB^j(s_1)\big)$ from Lemma 3.6, we have

$$\begin{aligned}\mathbb{E}|J_{k_s}|^{2p} &= \mathbb{E}\Big|\sum_{j=1}^{m}\int_{t_{k_s}}^{s} f'(Y(s))\big(g_j(\overline{Y}(s_1))\big)dB^j(s_1)\Big|^{2p}\\ &\leq m^{2p-1}\sum_{j=1}^{m}\mathbb{E}\Big|\int_{t_{k_s}}^{s} f'(Y(s))\big(g_j(\overline{Y}(s_1))\big)dB^j(s_1)\Big|^{2p}\\ &\leq m^{2p-1}(p(2p-1))^p\Delta^{(p-1)}\sum_{j=1}^{m}\mathbb{E}\int_{t_{k_s}}^{s}\Big|f'(Y(s))\big(g_j(\overline{Y}(s_1))\big)\Big|^{2p}ds\\ &\leq m^{2p-1}(p(2p-1))^p\Delta^{p}\sum_{j=1}^{m}\mathbb{E}\Big|f'(Y_{k_s})g_j(\overline{Y}_{k_s})\Big|^{2p}\leq C\Delta^p. \end{aligned}\tag{4.15}$$

On the other hand, since $t_{k_s}\leq s< t_{k_s+1}$, we have

$$\begin{aligned}x(s)-y(s) &= (x(t_{k_s})-Y_{k_s})+\int_{t_k}^{s}\big[f(x(u)-f(x(t_{k_s}))\big]du\\ &-(s-t_{k_s})f(Y_{k_s})+(s-t_{k_s})f(x(t_{k_s}))\\ &+\sum_{i=1}^{m}\int_{t_k}^{s}\Big[g_i(x(u))-g_i(\overline{Y}(u))-\sum_{j=1}^{m}L^jg_i(\overline{Y}(u))\Delta B^j(u)\Big]dB^i(u).\end{aligned}$$

So with some simplifications, we can deduce

$$\begin{aligned}x(s)-y(s) &= \big(x(t_{k_s})-Y_{k_s}\big)+\int_{t_{k_s}}^{s}\big[f(x(u)-f(x(t_{k_s}))\big]du+\beta_{k_s}\\ &+\sum_{i=1}^{m}\int_{t_{k_s}}^{s}\Big[g_i(x(u))-g_i(y(u))+\tilde{R}_2(g_i)\Big]dB^i(u),\end{aligned}\tag{4.16}$$

in which

$$\beta_{k_s} := (s-t_{k_s})\big(f(x(t_{k_s}))-f(Y_{k_s})\big).\tag{4.17}$$

Inserting the expression (4.16) into (4.14) gives

$$\begin{aligned}
J &\leq 5^{p-1}\mathbb{E}\int_0^{t\wedge\theta}\big|\big\langle\int_{t_{k_s}}^s\big[f(x(u)-f(x(t_{k_s}))\big]du, J_{k_s}\big\rangle\big|^p ds\\
&+5^{p-1}\mathbb{E}\int_0^{t\wedge\theta}\big|\big\langle\sum_{i=1}^m\int_{t_{k_s}}^s\Big[g_i(x(u))-g_i(y(u))\Big]dB^i(u), J_{k_s}\big\rangle\big|^p ds\\
&+5^{p-1}\mathbb{E}\int_0^{t\wedge\theta}\big|\big\langle\sum_{i=1}^m\int_{t_{k_s}}^s\tilde{R}_2(g_i)dB^i(u), J_{k_s}\big\rangle\big|^p ds+\mathbb{E}\int_0^{t\wedge\theta}\big|\big\langle\beta_{k_s}, J_{k_s}\big\rangle\big|^p ds\\
&+5^{p-1}\mathbb{E}\int_0^{t\wedge\theta}\big|\big\langle x(t_{k_s})-Y_{k_s}, J_{k_s}\big\rangle\big|^p ds\\
&:=5^{p-1}(J_1+J_2+J_3+J_4+J_5).
\end{aligned}\tag{4.18}$$

For J_1, we use the Hölder inequality and (4.15) and Lemma 3.5 in [32], to arrive at

$$\begin{aligned}
J_1 &\leq \mathbb{E}\int_0^{t\wedge\theta}\Big(\Big|\int_{t_{k_s}}^s\big[f(x(u))-f(x(t_{k_s}))\big]du\Big|^p\Big|J_{k_s}\Big|^p\Big)ds\\
&\leq \int_0^{t\wedge\theta}\Big(\mathbb{E}\Big|\int_{t_{k_s}}^s\big[f(x(u))-f(x(t_{k_s}))\big]du\Big|^{2p}\Big)^{1/2}\Big(\mathbb{E}\big|J_{k_s}\big|^{2p}\Big)^{1/2}ds\\
&\leq \Delta^{\frac{2p-1}{2}}\int_0^{t\wedge\theta}\Big(\mathbb{E}\int_{t_{k_s}}^s\big|f(x(u))-f(x(t_{k_s}))\big|^{2p}du\Big)^{1/2}\Big(\mathbb{E}\big|J_{k_s}\big|^{2p}\Big)^{1/2}ds\\
&\leq C\Delta^{\frac{2p-1}{2}}\int_0^{t\wedge\theta}\Big(\int_{t_{k_s}}^s\big\|f(x(u))-f(x(t_{k_s}))\big\|^{2p}_{L^{2p}(\Omega;\mathbb{R}^d)}du\Big)^{1/2}\Big(\mathbb{E}\big|J_{k_s}\big|^{2p}\Big)^{1/2}ds\\
&\leq C_{p,T}\Delta^{2p}.
\end{aligned}\tag{4.19}$$

Concerning J_2, we apply the Young inequality with $\varepsilon>0$ that is

$$xy\leq\frac{x^2}{2\varepsilon}+\frac{\varepsilon y^2}{2}.\tag{4.20}$$

So, from (4.20) with $\varepsilon=\Delta^p(h(\Delta))^{2p}$ and the relation (2.5) with $u=\mu^{-1}(h(\Delta))$, we have

$$\begin{aligned}
J_2 &\leq \mathbb{E}\int_0^{t\wedge\theta}\Big(\Big|\sum_{i=1}^m\int_{t_{k_s}}^s\big[g_i(x(u)-g_i(y(u))\big]dB^i(u)\Big|^p\Big|J_{k_s}\Big|^p\Big)ds\\
&\leq \int_0^{t\wedge\theta}\Big(\frac{C_{p,T}}{2\varepsilon}\sum_{i=1}^m\mathbb{E}\Big|\int_{t_{k_s}}^s\big[g_i(x(u)-g_i(y(u))\big]dB^i(u)\Big|^{2p}+\frac{\varepsilon}{2}\mathbb{E}\big|J_{k_s}\big|^{2p}\Big)ds\\
&\leq \int_0^{t\wedge\theta}\Big(C_{p,T}\frac{p(2p-1)}{2\varepsilon}\Delta^{p-1}\sum_{i=1}^m\mathbb{E}\int_{t_{k_s}}^s\big|g_i(x(u)-g_i(y(u))\big|^{2p}du+\frac{\varepsilon}{2}\mathbb{E}\big|J_{k_s}\big|^{2p}\Big)ds\\
&\leq \int_0^{t\wedge\theta}\Big(C_{p,T}m\frac{p(2p-1)}{2}\sup_{0\leq u\leq s}\mathbb{E}\big|e(u)\big|^{2p}+\frac{\Delta^p(h(\Delta))^{2p}}{2}\mathbb{E}\big|J_{k_s}\big|^{2p}\Big)ds\\
&\leq C_{p,T}\int_0^{t\wedge\theta}\sup_{0\leq u\leq s}\mathbb{E}\big|e(u)\big|^{2p}ds+C_{p,T}\Delta^{2p}(h(\Delta))^{2p}.
\end{aligned}\tag{4.21}$$

For J_3, by using Hölder's inequality, Lemma 4.1 and (4.15), we have

$$\begin{aligned}
J_3 &\le \mathbb{E}\int_0^{t\wedge\theta}\Big(\Big|\sum_{i=1}^m\int_{t_{k_s}}^s \tilde{R}_2(g_i)dB^i(u)\Big|^p\big|J_{k_s}\big|^p\Big)ds\\
&\le \int_0^{t\wedge\theta}\Big(\mathbb{E}\Big|\sum_{i=1}^m\int_{t_{k_s}}^s \tilde{R}_2(g_i)dB^i(u)\Big|^{2p}\Big)^{1/2}\big(\mathbb{E}\big|J_{k_s}\big|^{2p}\big)^{1/2}ds\\
&\le \int_0^{t\wedge\theta}\Big(\sum_{i=1}^m\mathbb{E}\Big|\int_{t_{k_s}}^s \tilde{R}_2(g_i)dB^i(u)\Big|^{2p}\Big)^{1/2}\big(\mathbb{E}\big|J_{k_s}\big|^{2p}\big)^{1/2}ds\\
&\le \int_0^{t\wedge\theta}\Big((p(2p-1))^p\Delta^{p-1}\sum_{i=1}^m\mathbb{E}\int_{t_{k_s}}^s\big|\tilde{R}_2(g_i)\big|^{2p}du\Big)^{1/2}\big(\mathbb{E}\big|J_{k_s}\big|^{2p}\big)^{1/2}ds\\
&\le \int_0^{t\wedge\theta}\Big((p(2p-1))^p m\Delta^{3p}(h(\Delta))^{4p}\Big)^{1/2}\big(\mathbb{E}\big|J_{k_s}\big|^{2p}\big)^{1/2}ds \le C\Delta^{2p}(h(\Delta))^{2p}.
\end{aligned}$$

To approximate J_4, we split it into two terms as below:

$$\begin{aligned}
J_4 \le \mathbb{E}\int_0^{t\wedge\theta}\big(\big|\beta_{k_s}\big|^p\big|J_{k_s}\big|^p\big)ds &= \sum_{l=0}^{k_{t\wedge\theta}-1}\mathbb{E}\int_{t_l}^{t_{l+1}}\big(\big|\beta_l\big|^p\big|J_l\big|^p\big)ds + \mathbb{E}\int_{k_{t\wedge\theta}}^{t\wedge\theta}\big(\big|\beta_{k_{t\wedge\theta}}\big|^p\big|J_{k_{t\wedge\theta}}\big|^p\big)ds\\
&:= J_{41} + J_{42}.
\end{aligned} \tag{4.22}$$

Concerning J_{41}, first, we use the Hölder inequality and the relations (4.15) and (2.5) with $u = \mu^{-1}(h(\Delta))$. Then, by the Young inequality (4.20) with $\varepsilon = 8 \times 10^{p-1}$, we have

$$\begin{aligned}
J_{41} &\le \sum_{l=0}^{k_{t\wedge\theta}-1}\int_{t_l}^{t_{l+1}}\big(\mathbb{E}\big|\beta_l\big|^{2p}\big)^{1/2}\big(\mathbb{E}\big|J_l\big|^{2p}\big)^{1/2}ds\\
&\le C\sum_{l=0}^{k_{t\wedge\theta}-1}\int_{t_l}^{t_{l+1}}\Delta^{\frac{p}{2}}\big(\mathbb{E}\big|\beta_l\big|^{2p}\big)^{1/2}ds\\
&\le C\sum_{l=0}^{k_{t\wedge\theta}-1}\int_{t_l}^{t_{l+1}}\Delta^{\frac{3p}{2}}(h(\Delta))^p\big(\mathbb{E}\big|e(t_l)\big|^{2p}\big)^{1/2}ds\\
&\le \Delta^{\frac{p}{4}}\big(\sup_{0\le u\le t\wedge\theta}\mathbb{E}\big|e(u)\big|^{2p}\big)^{1/2}\big(C_{p,T}\Delta^p\big) \le \frac{5}{8}\times 10^{-p}\sup_{0\le u\le t\wedge\theta}\mathbb{E}\big|e(u)\big|^{2p} + C\Delta^{2p}.
\end{aligned}$$

Following the same approach as in the estimation of J_{41}, one can similarly arrive at

$$\begin{aligned}
J_{42} &\le \mathbb{E}\Big(\sum_{l=0}^{k_{t\wedge\theta}-1}\int_{t_l}^{t_{l+1}}\big(\big|\beta_l\big|^p\big|J_l\big|^p\big)ds + \int_{k_{t\wedge\theta}}^{t\wedge\theta}\big(\big|\beta_{k_{t\wedge\theta}}\big|^p\big|J_{k_{t\wedge\theta}}\big|^p\big)ds\Big)\\
&\le \int_0^{t\wedge\theta}\big(\mathbb{E}\big|\beta_{k_s}\big|^{2p}\big)^{1/2}\big(\mathbb{E}\big|J_{k_s}\big|^{2p}\big)^{1/2}ds \le C\int_0^{t\wedge\theta}\Delta^{\frac{p}{2}}\big(\mathbb{E}\big|\beta_{k_s}\big|^{2p}\big)^{1/2}ds\\
&\le C\int_0^{t\wedge\theta}\Delta^{\frac{3p}{2}}(h(\Delta))^p\big(\mathbb{E}\big|e(t_{k_s})\big|^{2p}\big)^{1/2}ds\\
&\le \Delta^{\frac{p}{4}}\big(\sup_{0\le u\le t\wedge\theta}\mathbb{E}\big|e(u)\big|^{2p}\big)^{1/2}\big(C_{p,T}\Delta^p\big) \le \frac{5}{8}\times 10^{-p}\sup_{0\le u\le t\wedge\theta}\mathbb{E}\big|e(u)\big|^{2p} + C\Delta^{2p}.
\end{aligned}$$

Now, it remains to approximate J_5. Similarly, we split J_5 as below:

$$J_5 \leq \mathbb{E}\int_0^{t\wedge\theta} \big|\big\langle x_{t_{k_s}} - Y_{k_s}, J_{k_s}\big\rangle\big|^p ds = \sum_{l=0}^{k_{t\wedge\theta}-1} \mathbb{E}\int_{t_l}^{t_{l+1}} \big|\big\langle x_{t_l} - Y_l, J_l\big\rangle\big|^p ds$$
$$+ \mathbb{E}\int_{k_{t\wedge\theta}}^{t\wedge\theta} \big|\big\langle x_{t_{k_{t\wedge\theta}}} - Y_{k_{t\wedge\theta}}, J_{k_{t\wedge\theta}}\big\rangle\big|^p ds := J_{51} + J_{52}. \tag{4.23}$$

By applying (4.15), the Hölder inequality and the Young inequality (4.20) with $\varepsilon = 8\times 10^{p-1}$, we obtain

$$\begin{aligned}
J_{51} &\leq \sum_{l=0}^{k_{t\wedge\theta}-1} \int_{t_l}^{t_{l+1}} \big(\mathbb{E}\big|x_{t_l} - Y_l\big|^{2p}\big)^{1/2}\big(\mathbb{E}\big|J_l\big|^{2p}\big)^{1/2} ds \\
&\leq \big(\sup_{0\leq u\leq t\wedge\theta} \mathbb{E}\big|e(u)\big|^{2p}\big)^{1/2} \sum_{l=0}^{k_{t\wedge\theta}-1} \int_{t_l}^{t_{l+1}} \big(\mathbb{E}\big|J_l\big|^{2p}\big)^{1/2} ds \\
&\leq \frac{5}{8}\times 10^{-p} \sup_{0\leq u\leq t\wedge\theta} \mathbb{E}\big|e(u)\big|^{2p} + \Big(C\sum_{l=0}^{k_{t\wedge\theta}-1} \int_{t_l}^{t_{l+1}} \big(\mathbb{E}\big|J_l\big|^{2p}\big)^{1/2} ds\Big)^2 \\
&\leq \frac{5}{8}\times 10^{-p} \sup_{0\leq u\leq t\wedge\theta} \mathbb{E}\big|e(u)\big|^{2p} + C\sum_{l=0}^{k_{t\wedge\theta}-1} \Delta \int_{t_l}^{t_{l+1}} \mathbb{E}\big|J_l\big|^{2p} ds \\
&\leq \frac{5}{8}\times 10^{-p} \sup_{0\leq u\leq t\wedge\theta} \mathbb{E}\big|e(u)\big|^{2p} + C\Delta^{p+1}.
\end{aligned}$$

Finally, it results for J_{52}, similarly as above, that

$$\begin{aligned}
J_{52} &\leq \int_{k_{t\wedge\theta}}^{t\wedge\theta} \big(\mathbb{E}\big|x_{t_{k_{t\wedge\theta}}} - Y_{k_{t\wedge\theta}}\big|^{2p}\big)^{1/2}\big(\mathbb{E}\big|J_{k_{t\wedge\theta}}\big|^{2p}\big)^{1/2} ds \\
&\leq \big(\sup_{0\leq u\leq t\wedge\theta} \mathbb{E}\big|e(u)\big|^{2p}\big)^{1/2} \int_{k_{t\wedge\theta}}^{t\wedge\theta} \big(\mathbb{E}\big|J_{k_{t\wedge\theta}}\big|^{2p}\big)^{1/2} ds \\
&\leq \frac{5}{8}\times 10^{-p} \sup_{0\leq u\leq t\wedge\theta} \mathbb{E}\big|e(u)\big|^{2p} + C\Delta^{p+2}.
\end{aligned}$$

By inserting J_i for $i = 1,\ldots,5$ into (4.18) and then (4.13), we have

$$\begin{aligned}
\mathbb{E}(|e(t\wedge\theta|^{2p}) &\leq C\Big(\mathbb{E}\int_0^{t\wedge\theta} \big(|e(s)|^{2p} + |\tilde{R}_1(f)|^{2p}\big) ds + \sum_{i=1}^{m} \mathbb{E}\int_0^{t\wedge\theta} \big|\tilde{R}_2(g_i)\big|^{2p} ds\Big) \\
&\quad + \frac{1}{2}\sup_{0\leq u\leq t\wedge\theta} \mathbb{E}\big|e(u)\big|^{2p} + C\max\{\Delta^{p+1}, \Delta^{2p}(h(\Delta))^{4p}\} \\
&\leq C\mathbb{E}\int_0^{t\wedge\theta} \sup_{0\leq u\leq s} \mathbb{E}\big|e(u)\big|^{2p} ds + \frac{1}{2}\sup_{0\leq u\leq t\wedge\theta} \mathbb{E}\big|e(u)\big|^{2p} \\
&\quad + C\max\big\{\Delta^{p+1}, \Delta^{2p}(h(\Delta))^{4p}\big\}.
\end{aligned}$$

Therefore, we can conclude

$$\sup_{0\leq u\leq t\wedge\theta} \mathbb{E}\big|e(u)\big|^{2p} \leq C\Big(\mathbb{E}\int_0^{t\wedge\theta} \sup_{0\leq u\leq s} \mathbb{E}\big|e(u)\big|^{2p} ds + \max\big\{\Delta^{p+1}, \Delta^{2p}(h(\Delta))^{4p}\big\}\Big),$$

where C is a positive constant independent of the step size Δ. By applying the Gronwall inequality, we see

$$\sup_{0\leq u\leq t\wedge\theta} \mathbb{E}\big|e(u)\big|^{2p} \leq C\max\big\{\Delta^{p+1}, \Delta^{2p}(h(\Delta))^{4p}\big\}.$$

□

To prove the strong convergence rate of the split-step truncated method (3.10), we state the following theorem. It is worth noting that in this context we adopt the idea of the proof from Theorem 2.1 in [9].

Theorem 4.3. *Let the conditions of Assumptions 1 to 3 be satisfied, then there is also a positive real number Δ^* satisfying (3.2). For any $\Delta \in (0, \Delta^*]$ and $q \in (1, +\infty)$, let $R_\Delta^{(q)} := \big(\Delta(h(\Delta))^2\big)^{-1/(q-1)}$. If there exist $0 < \delta \leq \Delta^*$ and $q \in (1, +\infty)$ such that*

$$\mu^{-1}(h(\Delta)) \geq R_\Delta^{(q)}, \quad \forall \Delta \in (0, \delta], \tag{4.24}$$

then, for sufficiently small $\Delta \in (0, \Delta^*]$, there is a positive constant C independent of Δ such that

$$\mathbb{E}|x(T) - y_N|^2 \leq C\Delta^2(h(\Delta))^4, \tag{4.25}$$

for some $N = T/\Delta \in \mathbb{N}$.

Proof. The proof of this theorem is very similar to Theorem 2.1 in [9]. We choose $q \in (1, +\infty)$ and $\delta > 0$ such that the value $R_\Delta^{(q)}$ satisfies the relation (4.24). From now on, let $\Delta \in (0, \delta)$ be a fixed positive number. For this setting, we have

$$\begin{aligned}\mathbb{E}|x(T) - y_N|^2 &= \mathbb{E}\big(|x(T) - y_N|^2 I_{\{\theta > T\}}\big) + \mathbb{E}\big(|x(T) - y_N|^2 I_{\{\theta \leq T\}}\big) \\ &:= \Upsilon_1 + \Upsilon_2.\end{aligned} \tag{4.26}$$

By applying Theorem 4.2 for $p = 1$, we obtain

$$\Upsilon_1 \leq C\Delta^2(h(\Delta))^4. \tag{4.27}$$

Concerning, Υ_2, we use the Young inequality that is

$$x^2 y \leq \frac{\varepsilon}{q} x^{2q} + \frac{q-1}{q\varepsilon^{1/(q-1)}} y^{q/(q-1)}, \quad \forall q \in (1, \infty), \tag{4.28}$$

for any positive ε, see [9] for more details. Therefore, for any $\varepsilon > 0$ and $q > 1$, we have

$$\Upsilon_2 \leq \frac{\varepsilon}{q}\mathbb{E}\big(|x(T) - y_N|^{2q}\big) + \frac{q-1}{q\varepsilon^{1/(q-1)}}\mathbb{P}(\theta_{R_\Delta^{(q)}} \leq T). \tag{4.29}$$

Due to Theorem 2.1 and Lemma 3.4, we can find constant C independent of Δ such that

$$\mathbb{E}\big(|x(T) - y_N|^{2q}\big) \leq C. \tag{4.30}$$

On the other hand, by applying Remark 2.3 and Lemma 3.7, we have

$$\mathbb{P}(\theta_{R_\Delta^{(q)}} \leq T) \leq \mathbb{P}(\tau_{R_\Delta^{(q)}} \leq T) + \mathbb{P}(\gamma_{R_\Delta^{(q)}} \leq T) + \mathbb{P}(\rho_{R_\Delta^{(q)}} \leq T) \leq \frac{3K}{(R_\Delta^{(q)})^{2q}}. \tag{4.31}$$

By setting $\varepsilon = \Delta^2(h(\Delta))^4$ in (4.29), from relations (4.30) and (4.31), we can conclude

$$\Upsilon_2 \leq C\Delta^2(h(\Delta))^4, \tag{4.32}$$

for some constant C independent of Δ. Inserting (4.27) and (4.32) into (4.26) completes the proof. □

5. Numerical results

In this section, we show the efficiency of the proposed method in terms of accuracy and stability. We also numerically compare the split-step truncated Milstein method (3.10) with the method proposed by Guo et al [9]. Accordingly, we consider an example of a strongly nonlinear equation and compute the root mean square error of approximation (RMSE) for a given step size Δ.

Example 5.1. Consider the scalar nonlinear Itô SDE with a one-dimensional Wiener process

$$dx(t) = (x(t) - x^5(t))dt + x^2(t)dB(t), \quad t \geq 0, \quad x(0) = 1. \tag{5.1}$$

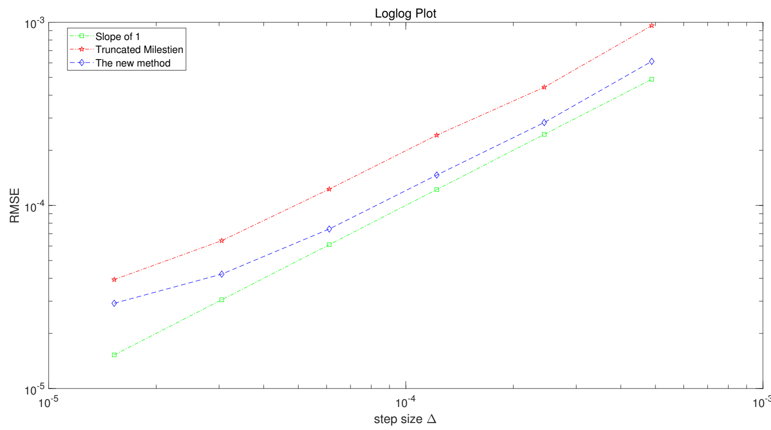

FIGURE 1. The RMSE as a function of Δ to approximate Example 5.1 at time $T = 2$.

Guo and et al. in [9] considered the SDE (5.1) as a test problem where the linear growth condition is violated. They show that Assumptions 1-3 are satisfied with $r = 4$. Concerning (2.5), it is clear

$$\sup_{0<|x|\vee|y|<u} \frac{|f(x)-f(y)|}{|x-y|} \vee \frac{|g(x)-g(y)|}{|x-y|} \leq (5u^4+1). \tag{5.2}$$

Moreover, we have

$$\sup_{|x|<u} \Big(|f(x)| \vee |g(x)| \vee |g'(x)| \Big) \leq u^5 \leq 3u^5, \tag{5.3}$$

for any $u \geq 2$. According to (5.2) and (5.3), we choose $\mu(u) = 3u^5$. On the other hand, for a given $\varepsilon \in (0, 0.2)$, we consider $\Delta^* = \exp(\frac{-\ln(3)}{0.2-\varepsilon})$ and define $h(\Delta) = 3\Delta^{-\varepsilon}$ for $\Delta \in (0, \Delta^*)$. For this setting (3.2) is fulfilled. About (4.24), for any $\varepsilon \in (0, 0.2)$, we choose $q \geq -9 + \frac{5}{\varepsilon}$, which implies

$$\Big(1 + \frac{10}{q-1}\Big)\varepsilon \ln(\Delta) \leq \frac{5}{q-1}\ln(\Delta) + \frac{10}{q-1}\ln(3), \quad \forall \Delta \in (0, \Delta^*). \tag{5.4}$$

Therefore, using elementary calculations we can obtain

$$3\Delta^{-\varepsilon} \geq 3\Big(\big(3^2\Delta^{1-2\varepsilon}\big)^{\frac{-1}{q-1}}\Big)^5. \tag{5.5}$$

So, the property (4.25) in Theorem 4.3 is fulfilled. Therefore, for any $\varepsilon \in (0, 0.2)$, we can deduce

$$\mathbb{E}|x(T) - y_N|^2 \leq 3C_T\Delta^{2-4\varepsilon}, \quad \Delta \in (0, \Delta^*), \tag{5.6}$$

with $N = T/\Delta \in \mathbb{N}$.

To show the efficiency of the split-step method in terms of stability and accuracy, in the following we calculate the RMSE as a function of the step size Δ on the log-log scale in Table 1 and Figure 1, respectively. Since there is no explicit solution for (5.1), we search for a numerical solution with the small step size $\Delta = 2^{-18}$ using the implicit Milstein-Taylor method [29] and use it as a reference solution. We also use the mean of 5000 independent realizations to approximate the expected value at the final time T. In Table 1, we compute the RMSE of the new method (3.10) and the method in [9] for step size $\Delta \in \{2^{-7}, \ldots, 2^{-10}\}$ with $\varepsilon = 0.1$ at the final time $T = 2$. In this table, we indicate the value of the mean square error by "unst" when a method becomes unstable for a certain value of the step size Δ.

From Table 1 we can deduce that the new method is stable at a step size of $\Delta = 2^{-7}$, while the truncated Milstein method [9] is mean square stable for $\Delta \leq 2^{-9}$. From Table 1, it can be seen that the new method has better properties in terms of accuracy and stability compared to the truncated Milstein method [9]. In Figure 1, the RMSE is plotted

Table 1. The RMSE with $\varepsilon = 0.1$ for Example 5.1 at the time $T = 2$.

Step size	The new method	Truncated Milstien method [9]
2^{-7}	7.79×10^{-2}	**unst.**
2^{-8}	2.47×10^{-2}	**unst.**
2^{-9}	4.72×10^{-3}	2.19×10^{-2}
2^{-10}	2.26×10^{-3}	3.53×10^{-3}

as a function of the step size Δ. From this figure, it can be seen that the convergence rate of the new method is very close to one, as expected. Moreover, the figure shows that the new proposed method is more accurate than the method [9].

6. Conclusions

The present study is concerned with the numerical solution of a class of highly nonlinear stochastic differential equations with commutative noise. We have successfully introduced an explicit split-step truncated Milstein method for nonlinear SDEs under the non-global Lipschitz and superlinear growth coefficients. We obtained the moment boundedness and the convergence of the numerical solution under some additional conditions. We proved that the strong convergence rate of the new method can be arbitrarily close to one. Finally, we discussed the efficiency of the present scheme in terms of stability and accuracy by solving an illustrative example.

Acknowledgments

The author would like to thank the reviewers for their valuable comments, which helped me to improve the earlier versions of this paper.

References

[1] A. Abdulle and A. Blumenthal, *Stabilized multilevel Monte Carlo method for stiff stochastic differential equations* , J. Comput. Phys., *251* (2012), 445–460.

[2] E. Buckwar and T. Sickenberger, *A structural analysis of asymptotic mean-square stability for multi-dimensional linear stochastic differential systems*, Applied. Numerical. Mathematics., *62*(7) (2012), 842–859.

[3] K. Burrage and T. Tian, *Predictor-corrector methods of Runge-Kutta type for stochastic differential equations*, SIAM J. Numer. Anal., *40*(4) (2002), 1516–1537.

[4] K. Burrage, P. M. Burrage, and T. Tian, *Numerical methods for strong solutions of stochastic differential equations: an overview* , Proc. R. Soc. Lond. A., *460* (2004), 373–402.

[5] S. Dereich, A. Neuenkirch, and L. Szpruch, *An Euler-type method for the strong approximation of the Cox-Ingersoll-Ross process*, Proc. R. Soc. A., *468* (2012), 1105–1115.

[6] O. Farkhonderooz and D. Ahmadian, *Mean-square stability of a constructed Third-order stochastic Runge–Kutta schemes for general stochastic differential equations*, Comput. methods differ. equ., *10*(3) (2022), 617-638.

[7] A.S. Fatemion Aghda, S. M. Hosseini, and M. Tahmasebi, *Analysis of non-negativity and convergence of solution of the balanced implicit method for the delay Cox-Ingersoll-Ross model* , Appl. Numer. Math., *118* (2017), 249-265.

[8] Q. Guo, W. Liu, X. Mao, and R. Yue, *The partially truncated Euler-Maruyama method and its stability and boundedness*, Appl. Numer. Math., *115* (2017), 235-251.

[9] Q. Guo, W. Liu, X. Mao, and R. Yue, *The truncated Milstein method for stochastic differential equations with commutative noise*, J. Comput. Appl. Math, *338* (2018), 298–310.

[10] A. Haghighi and A. Rößler, *Split-step double balanced approximation methods for stiff stochastic differential equations*, Int. J. Comput. Math., *96*(5) (2019), 1030–1047.

[11] A. Haghighi, S. M. Hosseini, and A. Rößler, *Diagonally drift-implicit Runge-Kutta methods of strong order one for stiff stochastic differential systems*, J. Comput. Appl. Math, *293* (2016), 82-93.

[12] M. Hutzenthaler, A. Jentzen, and P. E. Kloeden, *Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., *467* (2011), 1563–1576.
[13] M. Hutzenthaler, A. Jentzen, and P. E. Kloeden, *Strong convergence of an explicit numerical method for SDEs with non-globally Lipschitz continuous coefficients*, Ann. Appl. Probab. Eng. Sci., *4* (2012), 1611–1641.
[14] D. J. Higham, X. Mao, and A. M. Stuart, *Strong convergence of Euler-type methods for nonlinear stochastic differential equations*, SIAM. J. Numer. Anal., *40*(3) (2002), 1041–1063.
[15] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag, Berlin, 2002.
[16] X. Li and G. Yin, *Explicit Milstein schemes with truncation for nonlinear stochastic differential equations: Convergence and its rate*, J. Comput. Appl. Math, *374* (2020), 112771.
[17] J. Liao, W. Liu, and X. Wang, *Truncated Milstein method for non-autonomous stochastic differential equations and its modification* , J. Comput. Appl. Math., *402* (2022), 113817.
[18] W. Liu and X. Mao, *Strong convergence of the stopped Euler-Maruyama method for nonlinear stochastic differential equations*, Appl. Math. Comput., *223* (2013), 389–400.
[19] F. Mahmoudi and M. Tahmasebi, *The convergence of exponential Euler method for weighted fractional stochastic equations*, Comput. methods differ. equ., *2* (2022), 538–548.
[20] X. Mao *Stochastic Differential Equations and Applications* , second ed., Horwood, Chichester, 2002.
[21] X. Mao and L. Szpruch, *Strong convergence and stability of implicit numerical methods for stochastic differential equations with non-globally Lipschitz continuous coefficients*, J. Comput. Appl. Math., *238* (2013), 14–28.
[22] X. Mao, *The truncated Euler-Maruyama method for stochastic differential equations* , J. Comput. Appl. Math., *290* (2015), 370–384.
[23] G. N. Milstein, *Numerical Integration of Stochastic Differential Equations*, Kluwer Academic, Dordrecht, 2002.
[24] G. N. Milstein and M. V. Tretyakov, *Stochastic Numerics for Mathematical Physics*, Springer-Verlag, Berlin, 2004.
[25] B. Øksendal, *Stochastic Differential Equations, An Introduction With Applications*, sixth ed., in: Universitext, Springer-Verlag, Berlin, 2003.
[26] A. Rößler, *Rooted tree analysis for order conditions of stochastic Runge-Kutta methods for the weak approximation of stochastic differential equations*, Stochastic Anal. Appl., *24*(1) (2006), 97–134.
[27] A. Rößler, *Runge-Kutta methods for the strong approximation of solutions of stochastic differential equations*, SIAM J. Math. Anal., *48*(3) (2010), 922–952.
[28] S. Sabanis, *A note on tamed Euler approximations*, Electron. Comm. Probab., *18* (2013), 1–10.
[29] T. Tian and K. Burrage, *Implicit Taylor methods for stiff stochastic differential equations*, Appl. Numer. Math., *38*(1) (2001), 167–185.
[30] P. Wang and Y. Li, *Split-step forward methods for stochastic differential equations*, J. Comput. Appl. Math., *233*(10) (2010), 2641–2651.
[31] D. J. Wilkinson, *Stochastic Modelling for Systems Biology*, Chapman and Hall/CRC, 2018.
[32] X. Wang and S. Gan, *The tamed Milstein method for commutative stochastic differential equations with non-globally Lipschitz continuous coefficients* , J. Differ. Equations Appl., *19*(3) (2013), 466–490.
[33] W. Zhan, Q. Guo, and Y. Cong, *The truncated Milstein method for super-linear stochastic differential equations with Markovian switching*, Discrete. Continuous. Dyn. Syst. Ser. B., *27*(7) (2022), 3663–3682.