Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 11, No. 3, 2023, pp. 522-534
DOI:10.22034/cmde.2022.52638.2212

A modified split-step truncated Euler-Maruyama method for SDEs with non-globally Lipschitz continuous coefficients

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Abstract

In this paper, we propose an explicit diffuse the split-step truncated Euler-Maruyama (DSSTEM) method for stochastic differential equations with non-global Lipschitz coefficients. We investigate the strong convergence of the new method under local Lipschitz and Khasiminskii-type conditions. We show that the newly proposed method achieves a strong convergence rate arbitrarily close to half under some additional conditions. Finally, we illustrate the efficiency and performance of the proposed method with numerical results.

Keywords. Local Lipschitz condition, Khasiminskii condition, Truncated method, Split-step method, Strong convergence.
2010 Mathematics Subject Classification. 65C30, 65L07, 65L04.

1. Introduction

Stochastic differential equations (SDEs) are a powerful tool for the modeling real-world problems with uncertainty. These equations have applications in many areas of applied science, including economics, finance, biology, population dynamics, chemistry, epidemiology, physics, and engineering [10, 13, 18, 19]. However, for most nonlinear SDEs, exact solutions are not known. This situation makes numerical methods efficient tools for computing approximate solutions. In recent years, many researchers have developed numerous approximate schemes for solving SDEs for which no analytical solution formula exists [10, 17]. In particular, derivative-free stochastic Runge-Kutta (SRK) methods for strong and weak approximations have been proposed [1, 2, 20]. Recently, new classes of implicit SRK methods with diagonal drift for the strong approximation were introduced by Shahmoradi et al. [22].

However, the numerical methods have been developed mainly for SDEs with the classical global Lipschitz condition. In many applications, the global Lipschitz and linear growth conditions are perturbed, leading to a violation of the convergence properties of most of these methods, such as the Euler-Maruyama (EM) and Milstein methods [8]. It is common to treat the numerical solution of SDEs with non-global Lipschitz coefficients using implicit methods [6, 14]. In these methods, the application of implicit or drift-implicit numerical methods leads to the solution of a large system of nonlinear equations when the dimension of the SDEs is large. Therefore, the explicit numerical methods based on changes in drift and diffusion coefficients have received more attention from researchers. Hutzenthaler et al [9] first proposed tamed EM schemes to approximate SDEs with the global Lipschitz diffusion coefficient and one-sided Lipschitz drift coefficients. Sabanis [21] developed tamed EM schemes for SDEs with nonlinear growth coefficients. Moreover, stopped EM schemes [12], truncated EM schemes [15, 16], truncated Milstein methods [11] and their variants were also developed to solve the strong convergence problem for nonlinear SDEs. However, as far as we know, these modified EM and Milstein methods still cannot treat the stability of a large class of stiff SDEs with nonlinear drift and diffusion coefficients. Although it is common to treat the numerical solution of stiff SDEs with implicit methods [23], there are some classes of split-step methods with extended stability domains that are well suited for solving stiff problems [5]. An original contribution was made by Higham, Mao, and Stuart [6], who developed an implicit split-step variant of the EM method for SDEs with the one-sided Lipschitz drift and global Lipschitz diffusion conditions. As

Received: 21 July 2022 ; Accepted: 31 October 2022.
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implicit methods, Wang and Liu [24] constructed split-step backward balanced Milstein methods for stiff Itô SDEs. These ideas were further developed by Wang and Liu [25] to introduce fully explicit split-step forward methods for solving Itô SDEs with remarkable stability properties. However, the main drawback of most of these methods is that the derivatives of the drift and diffusion coefficients must be evaluated at each step. Among these methods, the drifting split-step Euler (DRSSE) method and the diffused split-step Euler (DISSE) method are the only methods that are simultaneously derivative-free and explicit. Although, for stiff SDEs where the stochastic component plays a significant role in the dynamics, such as large multiplicative noise, it is more recommended to use the DISSE method.

In this paper, we will bring all these ideas together. Based on Mao's truncated EM method [15], as a fully explicit method, we propose a derivative-free diffuse split-step truncated EM method for nonlinear stochastic differential equations with suitable stability properties. We have studied the strong convergence of the new method under local Lipschitz and Khasiminskii-type conditions. We proved that the new method has a strong convergence rate arbitrarily close to half. Finally, we illustrate the efficiency and performance of the proposed method with numerical results.

The remainder of the paper is organized as follows. Section 2 contains some notations and preliminary results on the numerical solution of the truncated EM method. In section 3, we introduce the split-step EM method and its stability properties in the mean square sense for multiplicative SDEs. In section 4, we will drive a diffuse split-step truncated EM method for nonlinear SDEs. We then study convergence rates at a single time point in section 5. We present numerical results in section 6 and our concluding remarks in section 7.

2. Basic concepts and preliminary results

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space a right continuous and increasing filtration $\{\mathcal{F}_t\}_{t\geq 0}$, where $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Here and throughout the paper, if $z \in \mathbb{R}^d$, let $|z| = (z_1^2 + \cdots + z_d^2)^{1/2}$ be the Euclidean norm. If $Q \in \mathbb{R}^{d\times m}$, then $|Q|$ represents the Frobenius norm of the matrix Q, i.e. $|Q| = \sqrt{trace(Q^T Q)}$. Moreover, $s \vee t$ and $s \wedge t$ denote the maximum and minimum of the numbers $s, t \in \mathbb{R}$, respectively, and I_G is the indicator function corresponding for the given set G.

Consider the following d-dimensional Itô SDE

$$dx(t) = f(x(t))dt + g(x(t))dB(t), \quad t \in [0, T], \tag{2.1}$$

with the initial value condition $x(0) = x_0 \in \mathbb{R}^d$. Here, $B(t) = (B_1(t), \ldots, B_m(t))^T$ is a $\mathcal{F}_t$-adapted m-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$. In SDE (2.1), $f : \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are drift and diffusion coefficients, respectively. In the following, for simplicity, we consider numerical methods on a uniform mesh $t_k = k\Delta$ for $k = 1, \ldots, N$ with $\Delta = T/N$ for some $N \in \mathbb{N}$.

One well-known method for approximating the SDE (2.1) is the EM method [10]

$$y(t_{k+1}) = y(t_k) + \Delta f(y(t_k)) + g(y(t_k))\Delta B_k, \tag{2.2}$$

where $\Delta B_k = B(t_{k+1}) - B(t_k)$ and $y(t_k)$ denote the value of the approximation of the exact solution $x(t_k)$ at time t_k for $k = 1, \ldots, N$. As we all know, the EM method (2.2) with order $\frac{1}{2}$ is convergent in the strong sense if the drift and diffusion coefficients satisfy the global Lipschitz condition [10, 18]. However, if this condition is perturbed, it is proved that the EM method is no longer convergent [8]. In Section 4, we propose the DSSTEM method, which is suitable for numerical solutions of a class of SDEs with non-global Lipschitz coefficients. To construct this method, we estimate the growth rate of the coefficients f and g under the following assumptions.

Assumption 1. Suppose there exist real positive constants K_1 and β such that

$$|f(z_1) - f(z_2)|^2 \vee |g(z_1) - g(z_2)|^2 \leq K_1(1 + |z_1|^\beta + |z_2|^\beta)|z_1 - z_2|^2, \tag{2.3}$$

for all $z_1, z_2 \in \mathbb{R}^d$.

From (2.3), we can find the function $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\nu(r) \to +\infty$ as $r \to +\infty$, with strictly increasing continuous properties and

$$\sup_{0<|z_1|\vee|z_2|\leq r} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|} \vee \frac{|g(z_1) - g(z_2)|}{|z_1 - z_2|} \leq \nu(r), \tag{2.4}$$

for any $r \geq 1$. Clearly, $\nu^{-1} : [\nu(0), +\infty) \to (0, +\infty)$ is also a strictly increasing continuous function.

Assumption 2. Suppose that the coefficients satisfy the Khasminskii condition: There is a pair of constants $K_2 > 0$ and $p > 2$ such that

$$z^T f(z) + \frac{p-1}{2}|g(z)|^2 \leq K_2(1 + |z|^2), \tag{2.5}$$

for all $z \in \mathbb{R}^d$.

The following theorem is a known result in the SDEs setting, see, e.g., [13, pp. 59, Theorem 4.1].

Theorem 2.1. *Let Assumptions 1 and 2 hold. Then, the SDE (2.1) with the initial value $x(0) = x_0 \in \mathbb{R}^d$ has a unique global solution $x(t)$. Moreover, there exists a positive constant C, dependent on T, p, and x_0, such that*

$$\mathbb{E}|x(t)|^p \leq C, \qquad \forall t \in [0, T].$$

Remark 2.2. For any real number $R > |x_0|$, consider the stopping time

$$\tau_R := \inf\{t \geq 0,\ |x(t)| \geq R\}. \tag{2.6}$$

Guo et al. in [4] indicated that there exists a positive constant C independent of R such that

$$\mathbb{P}(\tau_R \leq T) \leq \frac{C}{R^p}. \tag{2.7}$$

We will use the fundamental inequality (2.7) to prove the main theorem in section 5.

3. A split-step EM method for SDEs with the global Lipschitz conditions

For SDE (2.1), Wang and Li [25] presented the DISSE method for stiff SDEs, with $Y_\Delta(0) = x_0$ and

$$\begin{cases} \overline{Y}_\Delta(t_k) = Y_\Delta(t_k) + g(Y_\Delta(t_k))\Delta B_k, \\ Y_\Delta(t_{k+1}) = \overline{Y}_\Delta(t_k) + \Delta f(\overline{Y}_\Delta(t_k)), \end{cases} \tag{3.1}$$

for $k = 0, 1, \ldots, N-1$ where $\Delta B_k = B(t_{k+1}) - B(t_k)$. Suppose f and g in (2.1) satisfy the global Lipschitz condition for the constant $K_L > 0$, i.e.

$$|f(z_1) - f(z_2)| \vee |g(z_1) - g(z_2)| \leq K_L|z_1 - z_2|, \quad \forall z_1, z_2 \in \mathbb{R}. \tag{3.2}$$

Then the strong order of convergence of the DISSE method is $\frac{1}{2}$, and it is not suitable for solving stochastic systems with non-global Lipschitz condition, see for example [8].

When we apply (3.1) to the scalar test equation

$$dx(t) = \lambda x(t)dt + \mu x(t)dB(t), \quad t > 0, \quad y(0) = y_0 \neq 0, \tag{3.3}$$

with $2\mathcal{R}(\lambda) + |\mu|^2 < 0$, we have $Y_\Delta(t_{k+1}) = R_k(\lambda\Delta, \mu\sqrt{\Delta}, J)Y_\Delta(t_k)$ where J is the standard Gaussian random variable $J \sim N(0,1)$. Here, $R(p, q, J)$ is called a stability function and is expressed by

$$R_k(p, q, J) = (1 + p)(1 + qJ), \tag{3.4}$$

where $p = \lambda\Delta$ and $q = \mu\sqrt{\Delta}$. If we apply the expectation operation to both sides of (3.4), we also get

$$\mathbb{E}|Y_\Delta(t_{k+1})|^2 = \overline{R}(p, q)\mathbb{E}|Y_\Delta(t_k)|^2, \tag{3.5}$$

where $\overline{R}(p, q) = \mathbb{E}|R_k(p, q, J)|^2$ in (3.5) is called the MS-stability function of the method. Thus, we obviously obtain MS-stability, i.e., $\mathbb{E}|Y_\Delta(t_k)|^2 \to 0$ as $k \to \infty$, if $\overline{R}(p, q) < 1$. The set $R_{MS} = \{(p, q) \in \mathbb{C}^2 : \overline{R}(p, q) < 1\} \subseteq \mathbb{C}^2$ is called the domain of MS-stability of the method. Especially, the domain is called the region of stability in the case of $(p, q) \in \mathbb{R}^2$. With some simple calculations, the MS-stability function of the split-step method (3.1) is given by

$$\overline{R}(p, q) = (1 + p)^2(1 + q^2).$$

The corresponding regions of MS-stability for the two methods EM and DISSE are shown in Figure 1. From Figure 1, we can see that the region of MS-stability for the DISSE method (light gray area) includes the region of MS-stability for the EM method (dark gray area). It is therefore obvious that the DISSE method has a much better result in terms

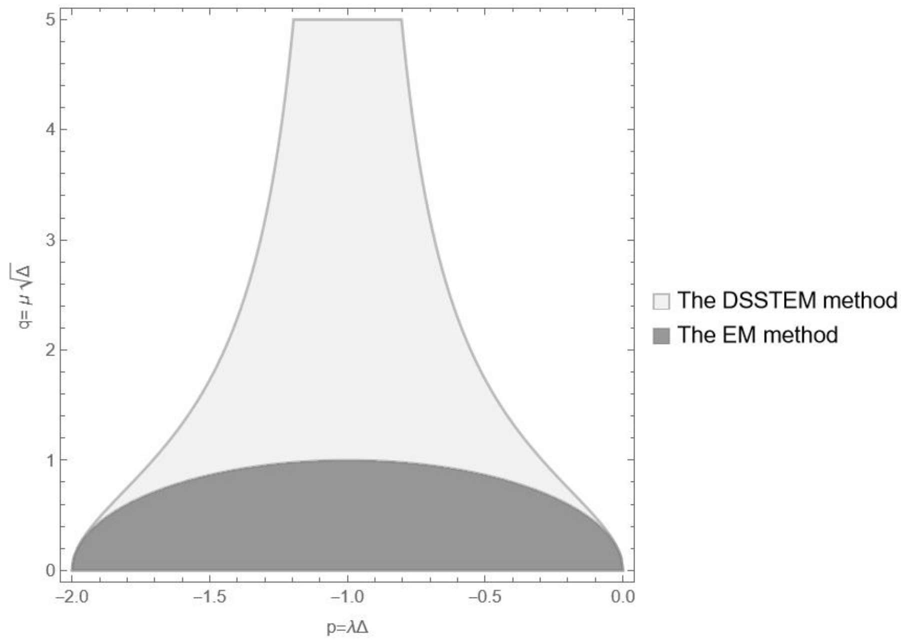

FIGURE 1. Mean-square stability region for the DISSE method (3.1) (light gray surface) and the EM method (dark gray surface).

of MS-stability, at least for the SDE (3.3) that obeys the global Lipschitz condition. This is especially true for SDEs where the stochastic part plays a significant role in their dynamics, such as in the case of large multiplicative noise.

4. The diffused split-step truncated EM method for nonlinear SDEs

In this section, by modifying f and g in (2.1), we propose an explicit diffused split-step truncated Euler-Maruyama method for SDEs with non-global Lipschitz coefficients. In the following, C stands for generic positive real constants that can vary from one place to another and depend on p, T, and x_0 but are independent of the step size Δ.

To construct the new method, we assume that the function ν satisfies the following condition in addition to the properties (2.4)

$$\sup_{|z|\leq r}\Big(|f(z)|\vee|g(z)|\Big)\leq\nu(r), \tag{4.1}$$

for any $r\geq 1$. Due to Assumption 1, the function ν is well defined. Assume that there exists a strictly decreasing function $h:(0,1]\to(0,+\infty)$ such that

$$\Delta^{1/4}h(\Delta)\leq\hat{h},\quad \lim_{\Delta\to 0}h(\Delta)=+\infty,\quad \forall\Delta\in(0,1], \tag{4.2}$$

for a constant $\hat{h}\geq 1$. For example, we can consider $h(\Delta)=\eta_h\Delta^{-\epsilon\omega}$ for suitable positive constants η_h and $\epsilon\in(0,1/4\omega)$ with $\omega>0$. For a given step size $\Delta\in(0,1]$, suppose $\kappa_\Delta:\mathbb{R}^d\to\mathbb{R}^d$ denotes the truncation mapping defined by $\kappa_\Delta(z):=\big(\nu^{-1}(h(\Delta))\wedge|z|\big)\frac{z}{|z|}$. Here, we set $\frac{z}{|z|}=0$ if $z=0$. We can easily show that for all $z_1,z_2\in\mathbb{R}^d$

$$|\kappa_\Delta(z_1)|\leq|z_1|,\qquad |\kappa_\Delta(z_1)-\kappa_\Delta(z_2)|\leq 2|z_1-z_2|. \tag{4.3}$$

In this paper, for a given step size $\Delta\in(0,1]$, we define the truncated functions by

$$f_\Delta(z):=f(\kappa_\Delta(z)),\quad g_\Delta(z):=g(\kappa_\Delta(z)). \tag{4.4}$$

It is obvious from (4.1) and (4.4) that

$$|f_\Delta(z)| \vee |g_\Delta(z)| \le h(\Delta), \quad \forall z \in \mathbb{R}^d. \tag{4.5}$$

Lemma 4.1. *([7]) Suppose (2.5) hold. Then, there exists a constat K_3 such that for all $\Delta \in (0,1]$ and $z \in \mathbb{R}^d$*

$$z^T f_\Delta(z) + \frac{p-1}{2}|g_\Delta(z)| \le K_3(1+|z|^2).$$

Now, according to the definition of the truncated functions (4.4), we introduce the DSSTEM method for the SDE (2.1) with $Y(0) = x_0$ and

$$\begin{cases} Z_\Delta(t_k) = Y_\Delta(t_k) + g_\Delta(Y_\Delta(t_k))\Delta B_k, \\ Y_\Delta(t_{k+1}) = Z_\Delta(t_k) + \Delta f_\Delta(Z_\Delta(t_k)), \end{cases} \tag{4.6}$$

for $k = 0, 1, \ldots, N-1$ where $\Delta B_k = B(t_{k+1}) - B(t_k)$. Here, we form a continuous-time version of the method (4.6). In this regard, first, for any fixed step size $\Delta \in (0,1]$ satisfying in (4.2) set

$$Y_\Delta(t) = \sum_{k=0}^{\infty} Y_\Delta(t_k) I_{[t_k,t_{k+1})}, \quad Z_\Delta(t) = \sum_{k=0}^{\infty} Z_\Delta(t_k) I_{[t_k,t_{k+1})}, \quad \forall t \ge 0. \tag{4.7}$$

We define the continuous-time version of the new method (4.6) as follows

$$y_\Delta(t) = Y_\Delta(t) + \int_{t_k}^{t} f_\Delta(Z_\Delta(\zeta))d\zeta + \int_{t_k}^{t} g_\Delta(Y_\Delta(\zeta))dB(\zeta), \quad \forall t \in [t_k, t_{k+1}). \tag{4.8}$$

From (4.6)-(4.8), we observe that

$$y_\Delta(t) = x_0 + \int_0^t f_\Delta(Z_\Delta(\zeta))d\zeta + \int_0^t g_\Delta(Y_\Delta(\zeta))dB(\zeta), \tag{4.9}$$

for $t \ge 0$. In the following lemma, we show how the values $y_\Delta(t)$ and $Y_\Delta(t)$ can be close to each other.

Lemma 4.2. *For any the step size $\Delta \in (0,1]$ and $\hat{p} > 0$, we obtain*

$$\mathbb{E}(|y_\Delta(t) - Y_\Delta(t)|^{\hat{p}}) \vee \mathbb{E}(|y_\Delta(t) - Z_\Delta(t)|^{\hat{p}}) \le C\Delta^{\hat{p}/2}(h(\Delta))^{\hat{p}}, \quad \forall t \ge 0. \tag{4.10}$$

Proof. First of all, fix $\hat{p} \ge 2$. For a given $t \ge 0$, let $k \ge 0$ be a suitable integer number such that $t_k \le t < t_{k+1}$. From (4.8), by applying Theorem 7.1 in [13], we have

$$\begin{aligned} \mathbb{E}\big|Z_\Delta(t) - Y_\Delta(t)\big|^{\hat{p}} &= \mathbb{E}\Big|\int_{t_k}^{t_{k+1}} g_\Delta(Y_\Delta(\zeta))dB(\zeta)\Big|^{\hat{p}} \\ &\le \Delta^{\frac{\hat{p}-2}{2}}\mathbb{E}\int_{t_k}^{t_{k+1}} \big|g_\Delta(Y(\zeta))\big|^{\hat{p}} d\zeta \le \Delta^{\hat{p}/2}(h(\Delta))^{\hat{p}}. \end{aligned} \tag{4.11}$$

In addition, because of the inequality $|\sum_{i=1}^n \alpha_i|^{\hat{p}} \le n^{\hat{p}-1}\sum_{i=1}^n |\alpha_i|^{\hat{p}}$, from (4.8) we have

$$\mathbb{E}\big|y_\Delta(t) - Y_\Delta(t)\big|^{\hat{p}} \le C\mathbb{E}\Big(\Big|\int_{t_k}^{t} f_\Delta(Z_\Delta(\zeta))d\zeta\Big|^{\hat{p}} + \Big|\int_{t_k}^{t} g_\Delta(Y_\Delta(\zeta))dB(\zeta)\Big|^{\hat{p}}\Big).$$

So, by Theorem 7.1 in [13] (Page 39), we obtain

$$\mathbb{E}\big|y_\Delta(t) - Y_\Delta(t)\big|^{\hat{p}} \le C\Big(\Delta^{\hat{p}-1}\mathbb{E}\int_{t_k}^{t}\Big|f_\Delta(Z_\Delta(\zeta))\Big|^{\hat{p}} d\zeta + \Delta^{\frac{\hat{p}-2}{2}}\mathbb{E}\int_{t_k}^{t}\Big|g_\Delta(Y_\Delta(\zeta))\Big|^{\hat{p}} d\zeta\Big).$$

Now, by applying the relation (4.5), we can conclude

$$\mathbb{E}\big|y_\Delta(t) - Y_\Delta(t)\big|^{\hat{p}} \le C\big(\Delta^{\hat{p}}(h(\Delta))^{\hat{p}} + \Delta^{\hat{p}/2}(h(\Delta))^{\hat{p}}\big). \tag{4.12}$$

On the other hand, for any $\hat{p} \in (0,2)$ by the Hölder inequality and (4.12), we have

$$\mathbb{E}\big|y_\Delta(t) - Y_\Delta(t)\big|^{\hat{p}} \le \Big(\mathbb{E}\big|y_\Delta(t) - Y_\Delta(t)\big|^2\Big)^{\hat{p}/2} \le C\Big(\Delta(h(\Delta))^2\Big)^{\hat{p}/2} = C\Delta^{\hat{p}/2}(h(\Delta))^{\hat{p}}.$$

Due to the same reason from (4.11), we can prove

$$\mathbb{E}(|y_\Delta(t)-Z_\Delta(t)|^{\hat{p}})\leq C\Delta^{\hat{p}/2}(h(\Delta))^{\hat{p}}.$$

So, the proof is completed. □

Lemma 4.3. *Suppose Assumption 2 holds. Then for any $T>0$ and $\Delta\in(0,1]$ satisfying in (4.2), we have*

$$\sup_{0<\Delta\leq 1}\Big(\sup_{0\leq t\leq T}\mathbb{E}|y_\Delta(t)|^p\Big)\leq C. \tag{4.13}$$

Proof. For any fix $\Delta\in(0,1]$ and $T>0$, from (4.9) and the Itô formula, we have

$$\begin{aligned}\mathbb{E}|y_\Delta(t)|^p &\leq \mathbb{E}|y_\Delta(0)|^p+p\mathbb{E}\int_0^t|y_\Delta(\zeta)|^{p-2}\Big(y_\Delta^T(\zeta)f_\Delta(Z_\Delta(\zeta))+\frac{p-1}{2}|g_\Delta(Y_\Delta(\zeta))|^2\Big)d\zeta\\ &=|x_0|^p+p\mathbb{E}\int_0^t|y_\Delta(\zeta)|^{p-2}\Big(Y_\Delta^T(\zeta)f_\Delta(Y_\Delta(\zeta))+\frac{p-1}{2}|g_\Delta(Y_\Delta(\zeta))|^2\Big)d\zeta\\ &+p\mathbb{E}\int_0^t|y_\Delta(\zeta)|^{p-2}\Big(\big(y_\Delta(\zeta)-Y_\Delta(\zeta)\big)^Tf(Y_\Delta(\zeta))\Big)d\zeta\\ &+p\mathbb{E}\int_0^t|y_\Delta(\zeta)|^{p-2}\Big(y_\Delta^T(\zeta)\big(f_\Delta(Z_\Delta(\zeta))-f_\Delta(Y_\Delta(\zeta))\big)\Big)d\zeta.\end{aligned}$$

By Lemmas 4.1 and 4.2 and the Young inequality that is

$$z_1^{p-2}z_2\leq\frac{p-2}{p}z_1^p+\frac{2}{p}z_2^{p/2},\quad\forall z_1,z_2\geq 0, \tag{4.14}$$

we can write

$$\begin{aligned}\mathbb{E}|y_\Delta(t)|^p&\leq\mathbb{E}|x_0|^p+(p-2)(K_3+2)\mathbb{E}\int_0^t|y_\Delta(\zeta)|^pd\zeta+2K_3\mathbb{E}\int_0^t\big(1+|Y_\Delta(\zeta)|^2\big)^{p/2}d\zeta\\ &+2\mathbb{E}\int_0^t|y_\Delta(\zeta)-Y_\Delta(\zeta)|^{p/2}|f(Y_\Delta(\zeta))|^{p/2}d\zeta+2\mathbb{E}\int_0^t|y_\Delta(\zeta)|^{p/2}|f_\Delta(Z_\Delta(\zeta))-f(Y_\Delta(\zeta))|^{p/2}d\zeta.\end{aligned}$$

Therefore, we can write

$$\mathbb{E}|y_\Delta(t)|^p\leq C_1+C_2\int_0^t\Big(\mathbb{E}|y_\Delta(\zeta)|^p+\mathbb{E}|Y_\Delta(\zeta)|^p\Big)d\zeta+\Pi_1+\Pi_2, \tag{4.15}$$

for some positive constants C_1 and C_2 in which

$$\Pi_1=2\mathbb{E}\int_0^t|y_\Delta(\zeta)-Y_\Delta(\zeta)|^{p/2}|f(Y_\Delta(\zeta))|^{p/2}d\zeta,$$

and

$$\Pi_2=2\mathbb{E}\int_0^t|y_\Delta(\zeta)|^{p/2}|f_\Delta(Z_\Delta(\zeta))-f(Y_\Delta(\zeta))|^{p/2}d\zeta.$$

Next, we try to estimate the values Π_1 and Π_2. Concerning Π_1, we use (4.10) and Lemma 4.2 to arrive at

$$\Pi_1\leq 2(h(\Delta))^{p/2}\int_0^T\mathbb{E}|y_\Delta(\zeta)-Y_\Delta(\zeta)|^{p/2}d\zeta\leq C_3T\Delta^{p/4}(h(\Delta))^p\leq C_3T\hat{h}^p, \tag{4.16}$$

for a positive constant C_3. By the Young inequality (4.14), Lemma 4.2 with the relations (2.4) and (4.3), we can approximate Π_2 as below

$$\begin{aligned}\Pi_2 &\leq \int_0^t \mathbb{E}|y_\Delta(\zeta)|^p d\zeta + \mathbb{E}\int_0^T |f_\Delta(Z_\Delta(\zeta)) - f_\Delta(Y_\Delta(\zeta))|^p d\zeta \\ &\leq \int_0^t \mathbb{E}|y_\Delta(\zeta)|^p d\zeta + (h(\Delta))^p \int_0^T \mathbb{E}|\kappa_\Delta(Z_\Delta(\zeta)) - \kappa_\Delta(Y_\Delta(\zeta))|^p d\zeta \\ &\leq \int_0^t \mathbb{E}|y_\Delta(\zeta)|^p d\zeta + 2T\Delta^p (h(\Delta))^{2p} \leq \int_0^t \mathbb{E}|y_\Delta(\zeta)|^p d\zeta + C_4,\end{aligned} \tag{4.17}$$

with a constant $C_4 \in \mathbb{R}^+$. Inserting (4.16) and (4.17) into (4.15) leads to

$$\begin{aligned}\mathbb{E}|y_\Delta(t)|^p &\leq A_1 + A_2 \int_0^t (\mathbb{E}|y_\Delta(\zeta)|^p + \mathbb{E}|Y_\Delta(\zeta)|^p) d\zeta \\ &\leq A_1 + A_2 \int_0^t \sup_{0\leq s\leq \zeta} (\mathbb{E}|y_\Delta(s)|^p) d\zeta,\end{aligned}$$

for some positive real constants A_1 and A_2. In the above inequality as the sum of the right-hand-side terms are increasing functions of t, we can assume

$$\sup_{0\leq s\leq t} \mathbb{E}|y_\Delta(s)|^p \leq A_1 + A_2 \int_0^t \Big(\sup_{0\leq s\leq \zeta} \mathbb{E}|y_\Delta(s)|^p d\zeta \Big).$$

By the Gronwall inequality, we have

$$\sup_{0\leq \zeta\leq t} \mathbb{E}|y_\Delta(\zeta)|^p \leq C,$$

which gives us the required assertion. □

Remark 4.4. Let Assumption 2 hold and Δ in $(0, 1]$ be fixed. For any $t \in [0, T]$, there exists integer number k such that $t \in [t_k, t_{k+1})$ and $Z_\Delta(t) = Z_\Delta(t_k)$. From (4.2) and (4.5), we can conclude

$$|Z_\Delta(t)|^p \leq 2^{p-1}\Big(|Y_\Delta(t)|^p + (h(\Delta))^p |\Delta B_k|^p \Big).$$

Since

$$\mathbb{E}|\Delta B(t_k)|^p = \frac{2^{p/2}\Gamma(\frac{p+1}{2})}{\sqrt{\pi}} \Delta^{p/2},$$

from (4.13), we have

$$\mathbb{E}|Z_\Delta(t)|^p \leq 2^{p-1}\Big(\sup_{0\leq\Delta\leq 1}\big(\sup_{0\leq t\leq T} \mathbb{E}|Y_\Delta(t)|^p \big) + \frac{2^{p/2}\Gamma(\frac{p+1}{2})}{\sqrt{\pi}} \hat{h}^p \Big),$$

which implies

$$\sup_{0\leq\Delta\leq 1}\big(\sup_{0\leq t\leq T} \mathbb{E}|Z_\Delta(t)|^p \big) < \infty. \tag{4.18}$$

5. Convergence properties

In this section, we are going to study the convergence rates of the DSSTEM method (4.6). Accordingly, we need an additional condition.

Assumption 3. Assume that there exist real positive constants K_4 and $q > 2$ such that

$$(z_1 - z_2)^T \big(f(z_1) - f(z_2) \big) + \frac{q-1}{2} |g(z_1) - g(z_2)|^2 \leq K_4 |z_1 - z_2|^2, \ \ \forall z_1, z_2 \in \mathbb{R}^d.$$

In addition to (2.6), for any real number $R > |x_0|$, we define two stopping times

$$\gamma_R := \inf\{t \geq 0, |y_\Delta(t)| \geq R\}, \quad \text{and} \quad \overline{\gamma}_R := \inf\{t \geq 0, |Z_\Delta(t)| \geq R\}.$$

Lemma 5.1. *Let Assumption 2 hold and $R > |x_0|$ be fixed. Then, for any sufficiently small step size $\Delta \in (0, 1]$, there exist positive constant K' independent of R and Δ such that*

$$\mathbb{P}(\gamma_R \leq T) \vee \mathbb{P}(\overline{\gamma}_R \leq T) \leq \frac{K'}{R^p}.$$

Proof. Replacing t by $\gamma_R \wedge T$ in Lemma 4.3, we can derive that

$$\mathbb{E}(y_\Delta(\gamma_R \wedge T)) \leq K_1',$$

for a positive real constant K_1'. Therefore,

$$R^p \mathbb{P}(\gamma_R \leq T) = \mathbb{E}\Big(|y_\Delta(\gamma_R)|^p I_{\{\gamma_R \leq T\}}\Big) \leq \mathbb{E}|y_\Delta(\gamma_R \wedge T)|^p \leq K_1'.$$

On the other hand, by (4.7) and (4.18), one can similarly arrive at

$$\mathbb{P}(\overline{\gamma}_R \leq T) \leq \frac{K_2'}{R^p},$$

for a positive constant K_2' which completes the proof. □

Theorem 5.2. *Consider an arbitrary given real number $R > |x_0|$. If the coefficients of the SDE (2.1) satisfy in Assumptions 1-3 such that $2p > q\beta$ and $p > q > 2$. Then for any desired step size $\Delta \in (0, 1]$ with property $\nu^{-1}(h(\Delta)) \geq R$ and for any $\tilde{q} \in [2, q)$, we have*

$$\mathbb{E}(|e_\Delta(t \wedge \theta_{\Delta,R})|^{\tilde{q}}) \leq C\Delta^{\tilde{q}/2}(h(\Delta))^{\tilde{q}}, \tag{5.1}$$

for all positive T. Here, $e_\Delta(t) := x(t) - y_\Delta(t)$ and $\theta_{\Delta,R} = \theta := \tau_R \wedge \gamma_R \wedge \overline{\gamma}_R$.

Proof. We try to estimate $e_\Delta(t \wedge \theta)$ for the approximation solution $y_\Delta(t)$. In this regard, from relations (2.1) and (4.9), we can write

$$e_\Delta(t \wedge \theta) = \int_0^{t\wedge\theta} \Big(f(x(\zeta)) - f_\Delta(Z_\Delta(\zeta))\Big) d\zeta + \int_0^{t\wedge\theta} \Big(g(x(\zeta)) - g_\Delta(Y_\Delta(\zeta))\Big) dB(\zeta).$$

By Itô formula for any $0 \leq t \leq T$, we have

$$\mathbb{E}|e_\Delta(t \wedge \theta)|^{\tilde{q}} \leq \mathbb{E}\int_0^{t\wedge\theta} \tilde{q}|e_\Delta(\zeta)|^{\tilde{q}-2}\Big(e_\Delta^T(\zeta)\big[f(x(\zeta)) - f_\Delta(Z_\Delta(\zeta))\big] + \frac{\tilde{q}-1}{2}|g(x(\zeta)) - g_\Delta(Y_\Delta(\zeta))|^2\Big) d\zeta. \tag{5.2}$$

For $\zeta \in [0, t \wedge \theta]$, we see that $Y_\Delta(\zeta) \vee Z_\Delta(\zeta) \leq R$. But we have the condition $\nu^{-1}(h(\Delta)) \geq R$, so $Y_\Delta(\zeta) \vee Z_\Delta(\zeta) \leq \nu^{-1}(h(\Delta))$. Recalling the truncated functions f_Δ and g_Δ, we can conclude

$$f_\Delta(Z_\Delta(\zeta)) = f(Z_\Delta(\zeta)),\ g_\Delta(Y_\Delta(\zeta)) = g(Y_\Delta(\zeta)), \tag{5.3}$$

for all $\zeta \in [0, t \wedge \theta]$. By the Young inequality with $\varepsilon = \frac{q-\tilde{q}}{\tilde{q}-1}$ that is

$$2z_1z_2 \leq \varepsilon z_1^2 + \frac{z_2^2}{\varepsilon}, \quad \forall z_1, z_2 \geq 0,$$

and (5.3), we can conclude that

$$\begin{aligned} \big|g(x(\zeta)) - g_\Delta(Y_\Delta(\zeta))\big|^2 &\leq \big|g(x(\zeta)) - g(y_\Delta(\zeta))\big|^2 + \big|g(y_\Delta(\zeta)) - g(Y_\Delta(\zeta))\big|^2 \\ &\quad + 2\big|g(x(\zeta)) - g(y_\Delta(\zeta))\big|\big|g(y_\Delta(\zeta)) - g(Y_\Delta(\zeta))\big| \\ &\leq \Big(1 + \frac{q-\tilde{q}}{\tilde{q}-1}\Big)\big|g(x(\zeta)) - g(y_\Delta(\zeta))\big|^2 + \Big(1 + \frac{\tilde{q}-1}{q-\tilde{q}}\Big)\big|g(y_\Delta(\zeta)) - g(Y_\Delta(\zeta))\big|^2. \end{aligned} \tag{5.4}$$

Therefore, by inserting (5.4) into (5.2), we have

$$\mathbb{E}|e(t \wedge \theta)|^{\tilde{q}} \leq \Pi_3 + \Pi_4, \tag{5.5}$$

where

$$\Pi_3 = \mathbb{E}\int_0^{t\wedge\theta} \tilde{q}|e_\Delta(\zeta)|^{\tilde{q}-2}\Big(e_\Delta^T(\zeta)\big[f(x(\zeta)) - f(y_\Delta(\zeta))\big] + \frac{q-1}{2}\big|g(x(\zeta)) - g(y_\Delta(\zeta))\big|^2\Big)d\zeta,$$

and

$$\Pi_4 = \mathbb{E}\int_0^{t\wedge\theta} \tilde{q}|e_\Delta(\zeta)|^{\tilde{q}-2}\Big(e_\Delta^T(\zeta)\big[f(y_\Delta(\zeta)) - f(Z_\Delta(\zeta))\big] + \frac{(\tilde{q}-1)(q-1)}{2(q-\tilde{q})}\big|g(y_\Delta(\zeta)) - g(Y_\Delta(\zeta))\big|^2\Big)d\zeta.$$

Clearly, from Assumption 3, we obtain

$$\Pi_3 \le \tilde{q}K_4\mathbb{E}\int_0^t |e_\Delta(\zeta\wedge\theta)|^{\tilde{q}}d\zeta. \tag{5.6}$$

To approximate Π_4, by the Young inequality and Assumption 1, we can write

$$\Pi_4 \le \frac{\tilde{q}}{2}\mathbb{E}\int_0^t |e_\Delta(\zeta\wedge\theta)|^{\tilde{q}}d\zeta + \Pi_{41} + \Pi_{42}, \tag{5.7}$$

where

$$\Pi_{41} = \frac{\tilde{q}}{2}\mathbb{E}\int_0^{t\wedge\theta} |e_\Delta(\zeta)|^{\tilde{q}-2}\big(1 + |y_\Delta(\zeta)|^\beta + |Z_\Delta(\zeta)|^\beta\big)|y_\Delta(\zeta) - Z_\Delta(\zeta)|^2 d\zeta,$$

and

$$\Pi_{42} = \frac{(\tilde{q}-1)(q-1)}{2(\tilde{q}-q)}\mathbb{E}\int_0^{t\wedge\theta} |e_\Delta(\zeta)|^{\tilde{q}-2}\big(1 + |y_\Delta(\zeta)|^\beta + |Y_\Delta(\zeta)|^\beta\big)|y_\Delta(\zeta) - Y_\Delta(\zeta)|^2 d\zeta.$$

About Π_{41}, we first use Young's inequality. Since $p > q > 2$ and $2p > q\beta$, by the Hölder inequality and Lemmas 4.2 and 4.3, we can write

$$\begin{aligned}
\Pi_{41} &\le C_1\mathbb{E}\int_0^t |e_\Delta(\zeta\wedge\theta)|^{\tilde{q}}d\zeta + C_2\mathbb{E}\int_0^{t\wedge\theta}\big(1 + |y_\Delta(\zeta)|^{\frac{\beta\tilde{q}}{2}} + |Z_\Delta(\zeta)|^{\frac{\beta\tilde{q}}{2}}\big)|y_\Delta(\zeta) - Z_\Delta(\zeta)|^{\tilde{q}}d\zeta,\\
&\le C_1\mathbb{E}\int_0^t |e_\Delta(\zeta\wedge\theta)|^{\tilde{q}}d\zeta + C_2\int_0^T\Big(\mathbb{E}|y_\Delta(\zeta) - Z_\Delta(\zeta)|^{\frac{2p\tilde{q}}{2p-\beta\tilde{q}}}\Big)^{\frac{2p-\beta\tilde{q}}{2p}}\Big(1 + \mathbb{E}|y_\Delta(\zeta)|^p + \mathbb{E}|Z_\Delta(\zeta)|^p\Big)^{\frac{\beta\tilde{q}}{2p}}d\zeta\\
&\le C_1\mathbb{E}\int_0^t |e_\Delta(\zeta\wedge\theta)|^{\tilde{q}}d\zeta + C_2\Delta^{\tilde{q}/2}(h(\Delta))^{\tilde{q}}.
\end{aligned} \tag{5.8}$$

In the same way as above by the Hölder inequality, we have

$$\Pi_{42} \le C_3\mathbb{E}\int_0^t |e_\Delta(\zeta\wedge\theta)|^{\tilde{q}}d\zeta + C_4\Delta^{\tilde{q}/2}(h(\Delta))^{\tilde{q}}, \tag{5.9}$$

where C_3 and C_4 are positive constants independent of the step size Δ. By substituting (5.6)-(5.9) into (5.5) and applying Gronwall's inequality, the proof is thus completed. □

To prove the strong convergence rate of the DSSTEM method (4.6), we present the following theorem.

Theorem 5.3. *Let Assumptions 1, 2, and 3 be fulfilled with $2p > q\beta$ and $p > q > 2$. For any $\tilde{q} \in [2, q)$, let $R_\Delta^{(\tilde{q})} := \big(\Delta^{\tilde{q}/2}(h(\Delta))^{\tilde{q}}\big)^{-1/(p-\tilde{q})}$. If there exist $0 < \delta \le 1$ and $\tilde{q} \in [2, q)$ such that*

$$\nu^{-1}(h(\Delta)) \ge R_\Delta^{(\tilde{q})}, \quad \forall\Delta \in (0, \delta]. \tag{5.10}$$

Then, for sufficiently small $\Delta \in (0, \delta]$, we have

$$\mathbb{E}|x(T) - y_\Delta(t_N)|^{\tilde{q}} \le C\Delta^{\tilde{q}/2}(h(\Delta))^{\tilde{q}}, \tag{5.11}$$

where C is a positive real constant independent of Δ and $N = T/\Delta \in \mathbb{N}$.

Proof. We choose $\tilde{q} \in [2, q)$ and $\delta > 0$ such that the value $R_\Delta^{(\tilde{q})}$ satisfy the relation (5.10). From now on, let $\Delta \in (0, \delta)$ be a fixed positive number. For this setting we have

$$\begin{aligned}\mathbb{E}|x(T) - y_\Delta(t_N)|^{\tilde{q}} &= \mathbb{E}\big(|x(T) - y_\Delta(t_N)|^{\tilde{q}} I_{\{\theta_{\Delta, R_\Delta^{(\tilde{q})}} > T\}}\big) + \mathbb{E}\big(|x(T) - y_\Delta(t_N)|^{\tilde{q}} I_{\{\theta_{\Delta, R_\Delta^{(\tilde{q})}} \le T\}}\big) \\ &:= \Upsilon_1 + \Upsilon_2. \end{aligned} \tag{5.12}$$

By applying (5.1) in Theorem 5.2, we obtain

$$\Upsilon_1 \le C\Delta^{\tilde{q}/2}(h(\Delta))^{\tilde{q}}. \tag{5.13}$$

Concerning, Υ_2, we use Young's inequality, which reads

$$z_1^{\tilde{q}} z_2 \le \frac{\varepsilon \tilde{q}}{p} z_1^p + \frac{p - \tilde{q}}{p\varepsilon^{\tilde{q}/(p-\tilde{q})}} z_2^{p/(p-\tilde{q})}, \quad \forall \tilde{q} \in [2, q),$$

for any positive ε and $z_1, z_2 \in [0, +\infty)$, see [4] for more details. Therefore, we have

$$\Upsilon_2 \le \frac{\varepsilon \tilde{q}}{p} \mathbb{E}\big(|x(T) - y_\Delta(t_N)|^p\big) + \frac{p - \tilde{q}}{p\varepsilon^{\tilde{q}/(p-\tilde{q})}} \mathbb{P}(\theta_{\Delta, R_\Delta^{(\tilde{q})}} \le T), \tag{5.14}$$

for any $\varepsilon > 0$ and $\tilde{q} \in [2, q)$. Due to Theorem 2.1 and Lemma 4.3, there is a constant C independent of Δ such that

$$\mathbb{E}\big(|x(T) - y_\Delta(t_N)|^p\big) \le C. \tag{5.15}$$

On the other hand, by applying Remark 2.2 and Lemma 5.1 we have

$$\mathbb{P}(\theta_{\Delta, R_\Delta^{(\tilde{q})}} \le T) \le \mathbb{P}(\tau_{R_\Delta^{(\tilde{q})}} \le T) + \mathbb{P}(\gamma_{R_\Delta^{(\tilde{q})}} \le T) + \mathbb{P}(\overline{\gamma}_{R_\Delta^{(\tilde{q})}} \le T) \le \frac{C}{(R_\Delta^{(\tilde{q})})^p}. \tag{5.16}$$

By setting $\varepsilon = \Delta^{\tilde{q}/2}(h(\Delta))^{\tilde{q}}$ in (5.14), from relations (5.15) and (5.16), we can conclude

$$\Upsilon_2 \le C\Big(\frac{\tilde{q}}{p}\Delta^{\tilde{q}/2}(h(\Delta))^{\tilde{q}} + \frac{p - \tilde{q}}{p\Big(\Delta^{\tilde{q}/2}(h(\Delta))^{\tilde{q}}\Big)^{\tilde{q}/(p-\tilde{q})}} \Big(\Delta^{\tilde{q}/2}(h(\Delta))^{\tilde{q}}\Big)^{p/(p-\tilde{q})}\Big) \le C\Delta^{\tilde{q}/2}(h(\Delta))^{\tilde{q}}, \tag{5.17}$$

for some constant C independent of Δ. Inserting (5.13) and (5.17) into (5.12) completes the proof. □

6. Numerical results

In this section, we illustrate the efficiency of the proposed method in terms of accuracy and stability. We also compare the DSSTEM method (4.6) with the truncated EM method in [15] and the partially truncated EM method in [3]. Accordingly, we consider an example of strongly nonlinear equations and compute the root mean square error of approximation (RMSE) for a given step size Δ, defined by

$$\Big(\mathbb{E}|x(T) - y_\Delta(T)|^2\Big)^{1/2} \approx \Big(\frac{1}{2000}\sum_{i=1}^{2000}\big|x^{(i)}(T) - y_\Delta^{(i)}(T)\big|^2\Big)^{1/2}. \tag{6.1}$$

Example 6.1. Consider the scalar nonlinear Itô SDE with a one-dimensional Wiener process

$$dx(t) = (x(t) - x^5(t))dt + x^2(t)dB(t), \quad t \ge 0, \quad x(0) = 1. \tag{6.2}$$

We consider the SDE (6.2) as a test problem where the linear growth condition is violated. It is obvious that Assumption 1 with $\beta = 8$ and Assumption 3 are satisfied for any $q \ge 2$. Regarding Assumption 2: For any $p > 3$, we can write

$$zf(z) + \frac{p-1}{2}|g(z)|^2 = |z|^2 - |z|^6 + \frac{p-1}{2}|z|^4 \le C(1 + z^2), \quad \forall z \in \mathbb{R}.$$

Here, C is a suitable positive constant in $\mathbb{R}$. Concerning (2.4), it is clear

$$\sup_{0<|z_1|\vee|z_2|<r} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|} \vee \frac{|g(z_1) - g(z_2)|}{|z_1 - z_2|} \le (5r^4 + 1). \tag{6.3}$$

Moreover, about (4.1) we have

$$\sup_{|z|\le r}\big(|f(z)|\vee|g(z)|\big)\le r^5\le 3r^5, \tag{6.4}$$

for any $r\geq 1$. According to (6.3) and (6.4), we choose $\nu(r)=3r^5$. On the other hand, for a given $\varepsilon\in(0,0.25)$, we consider $h(\Delta)=\eta_h\Delta^{-\varepsilon}$ for $\eta_h\geq 3$. For this setting (4.2) is fulfilled. About (5.10), for any $\varepsilon\in(0,0.25)$ and $\tilde q\geq 2$, we choose $p\geq(\frac{5\tilde q}{2\varepsilon}-4\tilde q)\vee\tilde q\vee 4$, which implies

$$\Big(1+\frac{5\tilde q}{p-\tilde q}\Big)\varepsilon\ln(\Delta)\le\frac{2.5\tilde q}{p-\tilde q}\ln(\Delta)+\frac{5\tilde q}{p-\tilde q}\ln(\eta_h)+\ln(\frac{\eta_h}{3}),\ \forall\Delta\in(0,1).$$

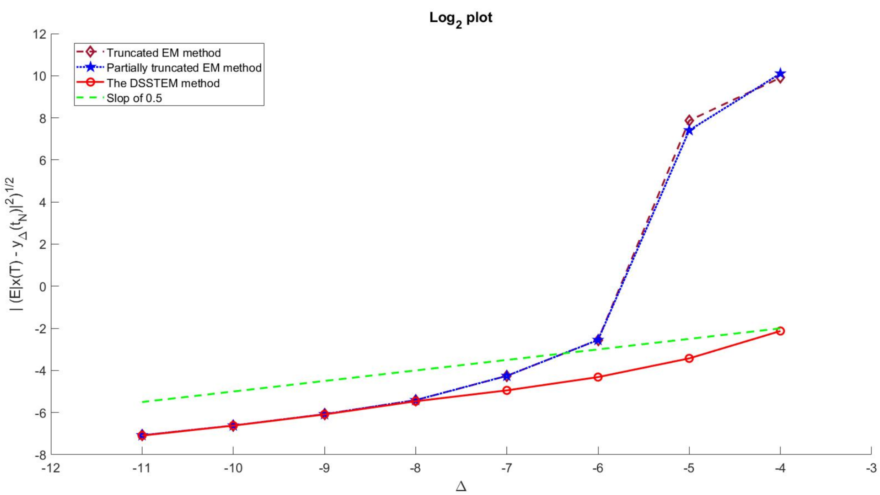

FIGURE 2. The RMSE as a function of step size Δ for the approximation of Example 6.1 at time $T=4$ with parameters $\eta_h=3\times10^5$, $\varepsilon=0.1$ and $\tilde q=2$.

Therefore, using elementary calculations, we can obtain

$$\frac{\eta_h}{3}\Delta^{-\varepsilon}\geq\Big(\big(\eta_h^{\tilde q}\Delta^{\tilde q/2-\tilde q\varepsilon}\big)^{\frac{-1}{p-\tilde q}}\Big)^5.$$

So, the property (5.11) in Theorem 5.3 is fulfilled and for any $\varepsilon\in(0,0.25)$, we can deduce

$$\mathbb{E}|x(T)-y(t_N)|^{\tilde q}\le C_T\Delta^{\tilde q/2-\tilde q\varepsilon},\ \ \Delta\in(0,1), \tag{6.5}$$

with $N=T/\Delta\in\mathbb{N}$. To show the efficiency of the DSSTEM method in terms of stability and accuracy, we calculate the RMSE (6.1) as a function of the step size Δ for different values of T and η_h in Figures 2-4. Since there is no explicit solution for (6.2), we search for a numerical solution with a small step size $\Delta=2^{-23}$ using the implicit Milstein-Taylor method [23] and use it as a reference solution. We also use the mean of 2000 independent realizations to approximate the expected value at the final time T. In Figures 2-4, we can see that the convergence rate of the new method is very close to half of the expected convergence rate in (6.5), and that the new method has better properties in terms of accuracy and stability than the truncated EM method in [15] and the partially truncated EM method in [3].

The simulation results presented in Figures 2-4 clearly show that the proposed method is an efficient one in terms of MS stability and accuracy.

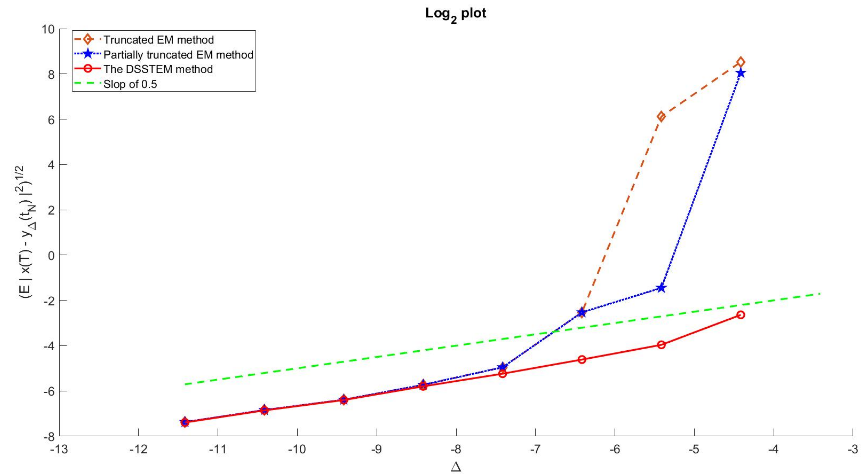

FIGURE 3. The RMSE as a function of step size Δ for the approximation of Example 6.1 at time $T = 6$ with parameters $\eta_h = 3 \times 9^5$, $\varepsilon = 0.1$ and $\tilde{q} = 2$.

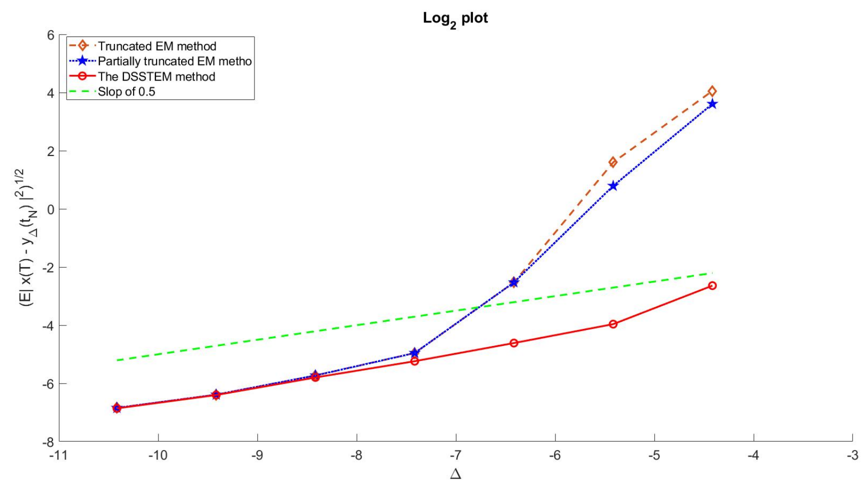

FIGURE 4. The RMSE as a function of step size Δ for the approximation of Example 6.1 at time $T = 6$ with parameters $\eta_h = 3 \times 5^5$, $\varepsilon = 0.1$ and $\tilde{q} = 2$.

7. Conclusion

In this paper, we present the diffuse split-step truncated Euler-Maruyama method for nonlinear stochastic differential equations. We prove the boundedness of the moment and the convergence of the numerical solution under some additional conditions. We proved that the strong convergence rate of the proposed method can be arbitrarily close to half. Finally, we have confirmed the advantages of the new method in numerical experiments.

Acknowledgment

The author would like to thank the reviewers for their valuable comments, which helped me to improve the earlier versions of this paper.

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