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Hölder estimates of solutions degenerate nonlinear parabolic equations

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Abstract

Hölder estimates of solutions of initial-boundary problem degenerate nonlinear parabolic equations are obtained. Estimates for solutions and parabolic Harnack inequality are proved. Also, one variant of weighted Poincare inequality is shown.

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1. Introduction

Our purpose is to study Hölder estimates of solutions initial-boundary problems of a degenerate nonlinear parabolic equation.

$$\frac{\partial u}{\partial t} = div\left(\omega\left(x\right)\cdot\left|Du\right|^{p-2}Du\right) + f\left(x,t\right), \tag{1.1}$$

where $1 < p < \infty$, $\omega(x)$ is a weight function which satisfied the Muckenhoupt condition and $f(x,t) \in L_2(Q_T)$. We study nonnegative weak solutions of the initial-boundary problem of this type of equations. To more general equations of the type

$$\frac{\partial u}{\partial t} = diva\left(x,t,u,.Du\right), \tag{1.2}$$

where $a(x,t,u,Du)$ is a Caratheodary function and satisfies the standard conditions

$$a\left(x,t,u,Du\right) \geq \gamma^{-1}\omega\left(x\right)\cdot\left|Du\right|^{p}, \tag{1.3}$$

and

$$a\left(x,t,u,Du\right) \leq \gamma\cdot\omega\left(x\right)\cdot\left|Du\right|^{p-1}, \tag{1.4}$$

where $\gamma \in (0,1]$ is positive constant, also can be consider. For simplicity, we consider equation (1.1) in cylindrical domain

$$Q_T = \Omega\times(0,T), T\in(0,\infty),$$

where Ω is an open set in R^n. $\Gamma(Q_T) = (\partial\Omega\times[0,T])\cup(\Omega\times\{(x,t):t=0\})$ is a parabolic boundary of the domain Q_T. For equation (1.1), we consider the initial boundary problem

$$u|_{\Gamma(Q_T)} = 0. \tag{1.5}$$

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Equation of type (1.1) or (1.2) has first been studied by Ladyzhenskaya, Solonnikov, Uraltseva in [13], by Trudinger in [16]. The proof of this type result was based on Moser's work [14]. A subsolution to the p-parabolic type equation studied in [6]. In the elliptic case with Muckenhoupt weight, the problem has been studied by Fabes, Kenig, and Serapioni in [7]. Some problems in the parabolic case have been studied by Chiarenza and Serapioni in [2]. For the heat equation as far as we know that the doubling condition and the Poincare inequality are not only sufficient but necessary conditions for the parabolic Harnack principle on Riemannian manifolds. We show the sufficiency for the general $p \neq 2$ in Euclidian space. An interesting question, in this case, is when this is also a necessity. Moreover, the doubling condition on the Poincare inequality is rather an assumptions on metric spaces (see [9]).

We used a metric space results, which state the doubling property of weights and the weighted Poincare inequality from imply also a Sobolev and a Caccioppoli type inequalities. Let us point out a slightly unexpected phenomenon related to the parabolic equations with BMO coefficients. As so recently, many questions are conserving to this type of equation.

We think that the demonstrated technique will solve this problem as well. Note that the obtained estimate is intrinsic in the sense that the waiting time to take the essential infimum on the right hand side depends on the solution itself. These results extended the results of Di Fazio (see [3–5]).

The paper is organized as follows. We will give in section 1 some information about previous results. In section 2, we give auxiliary results, definitions and the study of parabolic Harnack inequality. In section 3, we give some estimates for solutions, also one variant weighted Poincare inequality. In section 4, we to the study Hölder estimates for solutions and give the proof of Harnack inequality.

2. Preliminaries

Let Ω is an open set in R^n. The Sobolev space $W^1_{p,\omega}(\Omega)$ is defined to be the completion of $C^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W^1_{p,\omega}(\Omega)} = \left(\int_\Omega |u|^p \omega(x)\,dx\right)^{\frac{1}{p}} + \left(\int_\Omega |Du|^p \omega(x)\,dx\right)^{\frac{1}{p}}.$$

A function belongs to the local Sobolev space $W^1_{p,\omega,loc}(\Omega)$ if it belongs to $W^1_{p,\omega}(\Omega')$ for every open subset Ω' of Ω, whose closure is a compact subset of Ω. The Sobolev space $\mathring{W}^1_{p,\omega}(\Omega)$ with zero boundary values is the completion of $C_0^\infty(\Omega)$ with respect to the norm $W^1_{p,\omega}(\Omega)$. For the basic properties of weighted spaces we refer to [6].

We denote by $L_p\left(0,T;\ W^1_{p,\omega}(\Omega)\right)$, the space of functions such that almost every t, $0<t<T$, the function $u(x,t)$ belongs to $W^1_{p,\omega}(\Omega)$ and

$$\int_0^T \int_\Omega (|u(x,t)|^p + |Du(x,t)|^p)\omega(x)\,dxdt < \infty.$$

The definition $L_{p,loc}$ is clear. Function $u(x,t)$ from $L_{p,loc}\left(0,T;\, W^1_{p,\omega,loc}(\Omega)\right)$ is a weak solution to (1.1) in Q_T if

$$\int_0^T \int_\Omega \left(|Du|^{p-2} Du\,\omega(x)\,D\eta - u\frac{\partial \eta}{\partial t} + f(x,t)\,\eta\right) dxdt = 0, \tag{2.1}$$

for all $\eta \in C_0^\infty(Q_T)$.

Also, we say that u is a supersolution to (1.1)), if the integral (2.1) is nonnegative for all $\eta \in C_0^\infty(Q_T)$ with $\eta \geq 0$. If this integral is nonpositive, u is a subsolution. The weight $\omega(x)$ is a doubling, if there exists a constant $C_0 \geq 1$ such that

$$\omega(B(x_0,2R)) \leq C_0 \omega(B(x_0,R)), \tag{2.2}$$

for every $x_0 \in R^n$ and $R>0$, $B(x_0,R)$ the open ball with center x_0 and radius R. Let $0<\sigma\leq 1$, $\tau \in R$. We denote

$$U = B(x_0,R) \times (\tau - R^p, \tau + R^p),$$

$$\sigma U^{+} = B\left(x_0, \sigma R\right) \times \left(\tau + \frac{1}{2}R^p - \frac{1}{2}\left(\sigma R\right)^p, \tau + \frac{1}{2}R^p + \frac{1}{2}\left(\sigma R\right)^p\right),$$

$$\sigma U^{-} = B\left(x_0, \sigma R\right) \times \left(\tau - \frac{1}{2}R^p - \frac{1}{2}\left(\sigma R\right)^p, \tau - \frac{1}{2}R^p + \frac{1}{2}\left(\sigma R\right)^p\right).$$

If we consider the local weak Harnack estimate takes the following form.

Theorem 2.1. *Let $u(x,t)$ be a non-negative weak super solution is $B(x_0, 8R_0) \times (t_0, t_0 + T_0)$. Then there exist constants C_1, C_2 dependent at only known parameters, such that for almost every $t_0 < t_1 < t_0 + T_0$, we have*

$$\fint_{B(x_0,R_0)} \omega(x)\, u(x,t_1)dx \le \left(\frac{C_1 R_0^p}{T_0 + t_0 - t_1}\right)^{\frac{1}{p-2}} + C_2 ess \inf_{Q_T} \left(\omega(x)\, u\right),$$

where

$$Q = B\left(x_0, 4R_0\right) \times \left(t_1 + \frac{T}{2},\, t_1 + T\right),$$

and

$$T = \min\left\{T_0 + t_0 - t_1, C_1 R_0^p \left(\fint_{B(x_0,R_0)} \omega(x)\, u(x,t_1)dx\right)^{2-p}\right\}.$$

Proof. Let $t_0 < t_1 < t_0 + T_0$ be a Lebasgue instant of u. For brevity, we denote

$$N = \fint_{B(x_0,R)} \omega(x)\, u(x,t_1)dx,$$

and assume that $N > 0$. We introduced a scaled function

$$v(x,t) = \frac{M_C}{N} u\left(x_0 + Rx,\, t_1 + \left(\frac{C}{N}\right)^{p-2} R_0^p t\right),$$

where M_C is some constant. The function v is a super solution in $B(0,8) \times \left(0, \left(\frac{N}{M_\varepsilon}\right)^{p-2} \frac{T_0+t_0-t_1}{R_0^p}\right)$ and

$$\fint_{B(0,1)} \omega(x)\, v(x,0)dx = M_C.$$

Let we choose T_h, v_h and T_C, v_C. Suppose that

$$\left(\frac{M_C}{N}\right)^{p-2} \frac{(T_0 + t_0 - t_1)}{R_0^p} \le 2T^*,$$

for some $T^* > \max\{T_h, T_C\}$, or equivalently,

$$N \ge \left(\frac{C_1 R_0^p}{T_0 + t_0 - t_1}\right)^{\frac{1}{p-2}},$$

where $C_1 = 2T^* M_C^{p-2}$. Hence

$$ess \inf_{B(0,2)\times(T_h, 2T_h)} \omega(x)\, v \ge v_h.$$

Then, We scale back and obtain the desired result. □

In the global case, a stronger result holds.

Theorem 2.2. *Let $u(x,t)$ be a non-negative weak super solution to the problem (1.1) in $R^n \times (0, T_0)$. Then there exists a constant C dependent at only known parameters, such that for almost every $0 < t_0 < T_0$, every $x_0 \in R^n$, $R > 0$ and $0 < T < T_0 - t_0$ we have*

$$\fint_{B(x_0,R_0)} \omega(x)\, u(x,t_0)dx \le \left(\frac{CR^p}{T}\right)^{\frac{1}{p-2}} + C\left(\frac{T}{R^p}\right)^{\frac{n}{p}} ess\inf_{Q_T}\left(\omega(x)\, u^{\frac{\lambda}{p}}\right),$$

where $\lambda = n(p-2)+p$ and $Q_T = B(x_0, 2R) \times \left(t_0 + \frac{T}{2}, t_0 + T\right)$.

Theorem 2.3. *Let weight $\omega(x)$ be double and support a weak-weighted Poincare inequality. Let u nonnegative be a weak solution to problem (1.1), (1.5) in U and let $1 < p < \infty$, $0 < \sigma < 1$. Then*

$$ess\sup_{\sigma U^-}(\omega(x)\, u) \le C_1 ess\inf_{\sigma U^+}(\omega(x)\, u), \tag{2.3}$$

where the constant C_1 depends only on p, σ, C_0, C_2.

About Poincare inequality and constant C_2 is said later. Without weight the original proof is due to [11, 13, 16]. Also, it is well-known that the local Hölder continuity of a weak solution is a consequence of the Harnack inequality (also, see [13, 16]). The local Hölder continuity of the solution has been proved in [17] using a different method.

Now we give weak weighted Poincare inequality. The weight is said to support a weak weight Poincare inequality if there exists constants $C_2 > 0$ and $\tau \ge 1$ such that

$$\fint_{B(x_0,R)} \left|u - u_{B(x_0,R)}\right| \omega(x)dx \le C_2 R\left(\fint_{B(x_0,\sigma R)} |Du|^p\, \omega(x)dx\right)^{\frac{1}{p}}, \tag{2.4}$$

for every $u \in W^1_{p,\omega,loc}(R^n)$, $x_0 \in R^n$, $R > 0$. Here we use the notation

$$u_{B(x_0,R)} = \fint_{B(x_0,R)} u\omega(x)dx = \frac{1}{\omega(B(x_0,R))}\int_{B(x_0,R)} u\omega(x)\, dx.$$

If we have possibility $\tau > 1$, is said weak, if $\tau = 1$ is said the weight to support a weighted Poincare inequality. Later by result [9], we use weighted Poincare inequality, as so weight is doubling and space Euclidean. For functions with zero boundary values, following version of Sobolev-type inequality is held. Suppose $u \in \mathring{W}^1_{p,\omega}(B(x_0,R))$. Then

$$\left(\fint_{B(x_0,R)} |u|^k\, \omega(x)\, dx\right)^{\frac{1}{k}} \le C_3 R\left(\fint_{B(x_0,R)} |Du|^p\, \omega(x)\, dx\right)^{\frac{1}{p}}, \tag{2.5}$$

for some $k \ge 2$. k -connected dimension to the weight(see the proof in [2]). The following weighted Poincare inequality is a consequence of the doubling property of weight and in the inequality (2.4) (see [15]).

3. Auxiliary results

Theorem 3.1. *Let $u \in W^1_{p,\omega}(B(x_0,R))$ and $\psi(x) = \left(1 - \frac{|x-x_0|}{R}\right)^\theta$, $\theta > 0$. Then there exists a constant $C_4 = C_4(C_0, C_2, \theta, p)$ such that for all $0 < r < R$*

$$\fint_{B(x_0,r)} |u - u_\psi|^p\, \psi(x)\, \omega(x)\, dx \le C_4 r^p \fint_{B(x_0,r)} |Du|^p\, \psi(x)\, \omega(x)\, dx,$$

where the constant $u_\psi = \frac{\int_{B(x_0,r)} u\psi\omega dx}{\int_{B(x_0,r)} \psi(x)\omega(x)dx}$.

Later, we need the following modification of the lemma due to Bombieri [1].

Lemma 3.2. *Let $\omega(x)$ be a weight of Makenxoupt and θ, C_5, δ be positive constants, $0<\delta<1$, $0<q\leq\infty$. Let U_σ be bounded measurable sets with $U_{\sigma'}\subset U_\sigma$ for $0<\delta\leq\sigma'<\sigma\leq 1$. Moreover, if $q<\infty$, we assume that the doubling condition $\omega(U_1)\leq C_5\omega(U_\delta)$ holds. Let u be a positive measurable function on U_1 which satisfies the reverse Hölder inequality*

$$\left(\fint_{U_{\sigma'}} u^q\omega(x)dx\right)^{\frac{1}{q}}\leq\left(\frac{C_5}{(\sigma-\sigma')^\theta}\fint_{U_\sigma} u^s\omega(x)dx\right)^{\frac{1}{s}},$$

with $0<s<q$. Also, f satisfies

$$\omega\left(\{x\in U_1/\log u>\lambda\}\right)\leq\frac{C_5\omega(U_\delta)}{\lambda^\gamma},$$

for all $\lambda>0$. Then

$$\left(\fint_{U_\delta} u^q\omega(x)dx\right)^{\frac{1}{q}}\leq C_6,$$

where C_6 depends only on θ, γ,δ, q and C_5.

Proof. For proving to denote

$$\eta=\eta(\sigma)=\log\left(\fint_{U_\sigma} u^q\omega(x)dx\right)^{\frac{1}{q}}.$$

From Hölder's inequality we have

$$\begin{aligned}\fint_{U_\sigma} u^s\omega(x)dx&=\frac{1}{\omega(U_\sigma)}\int_{\log u\leq\frac{\eta}{2}} u^s\omega(x)\,dx+\frac{1}{\omega(U_\sigma)}\int_{\log u>\frac{\eta}{2}} u^s\omega(x)\,dx\\&\leq\exp\left(\frac{\eta s}{2}\right)+\left(\fint_{U_\sigma} u^q\omega(x)dx\right)^{\frac{s}{q}}\left(\frac{\omega\left(\{\log u>\frac{\eta}{2}\}\right)}{\omega(U_\sigma)}\right)^{\frac{(q-s)}{q}}\\&\leq\exp\left(\frac{\eta s}{2}\right)+\exp(\eta s)\left(\frac{C_5}{\left(\frac{\eta}{2}\right)^\gamma}\right)^{\frac{(q-s)}{q}}.\end{aligned}$$

□

Let η be so large that $0<\log\left(\frac{\eta^\gamma}{C_5 2^\gamma}\right)\leq q\eta$.

We can obtained lower bound on η depends on C_5, γ, q. If we choose corresponding $0<s<q$, then we have

$$\fint_{U_\sigma} u^s\omega(x)dx\leq 2\exp\left(\frac{\eta s}{2}\right). \tag{3.1}$$

We take a logarithm and use the (3.1)) from which it follows that

$$\eta\geq\frac{C_7}{(\sigma-\sigma')^{\frac{\theta}{\gamma}}},$$

where C_7 depends only on C_5, γ. On the other hand, if $\eta(\sigma)\leq\min\left(C_8,\frac{C_7}{(\sigma-\sigma')^{\frac{\theta}{\gamma}}}\right)$, where C_8 lover boundary, then the doubling implies that

$$\eta(\sigma')\leq\frac{3}{4}\eta(\sigma)+C_9\left(1+\frac{1}{(\sigma-\sigma')^{\frac{\theta}{\gamma}}}\right).$$

The proof follows in a standard way (see [8]). Now we give two estimates for super solution and subsolution of problem (1.1),(1.5).

Lemma 3.3. *Let u - nonnegative is a supersolution of problem (1.1),(1.5) in Q_T and $\varepsilon \neq p-1>0$. Then there exists a constant $C_{10}(p,\varepsilon)$ such that*

$$\int_0^T\int_\Omega |Du|^{p-1} u^{-\varepsilon-1}\varphi^p\omega(x)\,dxdt + \operatorname*{ess\,sup}_{0<t<T}\int_\Omega u^{p-1-\varepsilon}\varphi^p\omega(x)\,dx \le C_{10}\int_0^T\int_\Omega u^{p-1-\varepsilon}|D\varphi|^p\omega(x)\,dxdt + C_{10}\int_0^T\int_\Omega u^{p-1-\varepsilon}\left|\frac{\partial\varphi}{\partial t}\right|\omega(x)\,dxdt,$$

for every $\varphi\in C_0^\infty(Q_T)$ with $\varphi\ge 0$.

Lemma 3.4. *Let u - nonnegative is a subsolution of problem (1.1),(1.5) in Q_T and $\varepsilon>0$. Then there exists a constant $C_{11}(p,\varepsilon)$ such that*

$$\int_0^T\int_\Omega |Du|^{p} u^{\varepsilon-1}\varphi^p\omega(x)\,dxdt + \operatorname*{ess\,sup}_{0<t<T}\int_\Omega u^{p-1+\varepsilon}\varphi^p\omega(x)\,dx \le C_{11}\int_0^T\int_\Omega u^{p-1+\varepsilon}|D\varphi|^p\omega(x)\,dxdt + C_{11}\int_0^T\int_\Omega u^{p-1+\varepsilon}\frac{|\partial\varphi|}{\partial t}\omega(x)\,dxdt,$$

for every $\varphi\in C_0^\infty(Q_T)$ with $\varphi\ge 0$.

This lemma is proving similar. Therefore, we only give the proof of Lemma 3.3.

Proof. of Lemma 3.3

In definition of supersolution of problem (1.1), (1.5) we choose the test function $\eta=u^{-\varepsilon}\varphi^p$ so that

$$D\eta=-\varepsilon u^{\varepsilon-1}\varphi^pDu+u^{-\varepsilon}D(\varphi^p),$$

where $\varphi\in C_0^\infty(Q_T)$. Let $0<\tau_1<\tau_2<T$. We integrate by parts and get

$$-\int_{\tau_1}^{\tau_2}\int_\Omega u\frac{\partial\eta}{\partial t}\omega(x)\,dxdt+\left[\int_\Omega u\eta\omega(x)\,dxdt\right]_{t=\tau_1}^{\tau_2} = \frac{\varepsilon}{p-1-\varepsilon}\int_{\tau_1}^{\tau_2}\int_\Omega\frac{\partial\left(u^{1-\varepsilon}\right)^p}{\partial t}\varphi^p\omega(x)\,dxdt - \int_{\tau_1}^{\tau_2}\int_\Omega u^{1-\varepsilon}\frac{\partial(\varphi^p)}{\partial t}\omega(x)\,dxdt+\left[\int_\Omega u^{1-\varepsilon}\varphi^p\omega(x)\,dx\right]\Bigg|_{t=\tau_1}^{\tau_2}$$
$$\le\frac{p(p-1)}{|p-1-\varepsilon|}\int_{\tau_1}^{\tau_2}\int_\Omega u^{1-\varepsilon}\frac{|\partial\varphi|}{\partial t}\varphi^{p-1}\omega(x)dxdt + \frac{p-1}{p-1-\varepsilon}\left[\int_\Omega u^{1-\varepsilon}\varphi^p\omega(x)dxdt\right]_{\tau_1}^{\tau_2}.$$

□

If we substitute of this η to (2.1) after some calculations we obtain

$$ess \sup_{0<t<T} \int_{\Omega} u^{1-\varepsilon}\varphi^p\omega(x)\,dx \le C_{12}\int_0^T\int_{\Omega} |D\varphi|^p u^{1-\varepsilon}\omega(x)\,dxdt + C_{12}\int_0^T\int_{\Omega} u^{1-\varepsilon}\left|\frac{\partial\varphi}{\partial t}\right|\omega(x)\,dxdt$$
$$\le \int_0^{\tau}\int_{\Omega} |D\varphi|^p u^{1-\varepsilon}\omega(x)\,dxdt + C_{12}\int_0^T\int_{\Omega} u^{1-\varepsilon}\left|\frac{\partial\varphi}{\partial t}\right|\omega(x)\,dxdt.$$

We conclude the same estimate for $\varepsilon > p-1$, if choose $\tau_1 = 0$ and $\tau_2 = \tau$. Now the result follows with the constant C_{11}.

Now we proof a Caccioppoli-type estimate for the logarithm of a super solution. Also we due remark, that if $u > 0$ is a super solution, then $v = u^{-1}$ is a subsolution.

Lemma 3.5. *Let u - nonnegative is a supersolution of problem (1.1),(1.5) in Q_T. Then there exists a constant $C_{13}(p)$ such that*

$$\int_0^T\int_{\Omega} |D(\log u)|^p\varphi^p\omega(x)\,dxdt + ess\sup_{0<t<T}\left|\int_{\Omega}(\log u)\varphi^p\omega(x)\,dx\right| \le C_{13}\int_0^T\int_{\Omega}|D\varphi|^p\omega(x)dxdt$$
$$+ C_{13}\int_0^{\tau}\int_{\Omega}|\log u|\left|\frac{\partial\varphi}{\partial t}\right|\omega(x)dxdt,$$

for every $\varphi \in C_0^\infty(Q_T)$ with $\varphi \ge 0$.

Proof. Let $\eta = u^{1-p}\varphi^p$, where $\varphi \in C_0^\infty(Q_T)$, $\varphi \ge 0$. We integrate by parts, then

$$-\int_{\tau_1}^{\tau_2}\int_{\Omega} u^{p-1}\frac{\partial\eta}{\partial t}\omega(x)dxdt + \left[\int_{\Omega} u^{p-1}\eta\omega(x)dxdt\right]_{t=\tau_1}^{\tau_2} = (p-1)\int_{\tau_1}^{\tau_2}\int_{\Omega}\frac{\partial\log u}{\partial t}\varphi^p\omega(x)dxdt$$
$$-\int_{\tau_1}^{\tau_2}\int_{\Omega}\frac{\partial(\varphi^p)}{\partial t}\omega(x)dxdt + \left[\int_{\Omega}\varphi^p\omega(x)dx\right]\Bigg|_{t=\tau_1}^{\tau_2}$$
$$= -p(p-1)\int_{\tau_1}^{\tau_2}\int_{\Omega}(\log u)\varphi^{p-1}\frac{\partial\varphi}{\partial t}\omega(x)dxdt$$
$$+ (p-1)\left[\int_{\Omega}(\log u)\varphi^p\omega(x)dx\right]_{\tau_1}^{\tau_2},$$

where $0 < \tau_1 < \tau_2 < T$. If we denote $v = \log u$, substitute to (2.1) this η, apply Young's inequality, after some calculation by the standard way as in the proof of Lemma 3.3. the claim follows. □

Also, we show the boundedness of a subsolution.

Lemma 3.6. *Let u - nonnegative is a subsolution of problem (2.1) in Q_T and $0 < \delta < 1$. Then there exist positive constant $C_{14}(p, C_5, T, \delta, C_4)$ and $\theta(p, C_5)$, such that*

$$ess\sup_{\sigma' Q_T}(\omega(x)u) \le \left(\frac{C_{14}}{(\sigma-\sigma')^{\theta}}\right)^{\frac{1}{s}}\left(\fint_{\tau Q_T} u^s\omega(x)dx\right)^{\frac{1}{s}},$$

for all $o < \delta \le \sigma' < \sigma \le 1$ and for all $s > 0$.

Proof. Let the choices of test functions and $\sigma_j = \sigma - (\sigma - \omega')\left(1-\gamma^{-j}\right)$. We obtain from Sobolev's inequality and lemma 3.4 that

$$\underset{\sigma' Q_T}{ess\ \sup}\left(\omega(x)u\right) \le \left(\frac{C_{14}}{(\sigma-\sigma')^{\theta}}\right)^{\frac{1}{s}} \left(\fint_{U_\sigma} u^s\omega(x)dx\right)^{\frac{1}{s}} \fint_{Q_T^{j+1}} u^{\gamma\alpha}\omega(x)dx \le C_{14}\left(\frac{\alpha^p\gamma^{jp}}{(\sigma-\sigma')^p}\int_{Q_T^j} u^{\alpha}\omega(x)\,dxdt\right)^{\gamma}, \tag{3.2}$$

where $\gamma = 2-\frac{p}{k}$, $\alpha = p-1+\varepsilon$, $\varepsilon \ge 1$. Here we fix σ and divide the interval (σ',σ) into k parts by setting $\sigma_0=\sigma$, $\sigma_k=\sigma'$. k - fixed. Also we demote $Q_T^j = \sigma_j Q_T$. Test functions choose with the following properties: $\sup p\,(\varphi_j) < Q_T^j$, $0\le\varphi_j\le 1$ in Q_T^j, $\varphi_j = 1$ in Q_T^{j+1}, $|D\varphi_j| \le C_{15}\frac{\gamma_j}{r(\sigma-\sigma')}$, $|D\varphi_j| \le C_{15}\frac{\gamma_j}{r(\sigma-\sigma')}$ in Q_T^j. In Lemma 3.4, the constant is singular as ε is close to 0. We deliberately avoid this singularity by choosing $\varepsilon\ge 1$. More over, we choose $\alpha_j = p\gamma^j$, $j=0,1\ldots$. We iterate the inequality (3.2) and obtain

$$\left(\fint_{Q_T^0} u^p\omega(x)dx\right)^{\frac{1}{p}} \le \left(\frac{(\sigma-\sigma')}{C_{14}}\right)^{\gamma^{-1}+\gamma^{-2}+\ldots+\gamma^{-k+1}} \prod_{j=0}^{k-1}\gamma^{\frac{-2j}{\gamma^j}}\left(\left(\fint_{Q_T^k} u^{\gamma^k p}\omega(x)dx\right)\right)^{\frac{1}{\gamma^k p}}.$$

Let k tent to infinity and get the result for $s\ge p$ from Hölder inequality. If $s<p$, then we have

$$\begin{aligned}\underset{\sigma' Q_T}{ess\ \sup}\left(\omega(x)u\right) &\le \left(\frac{C_{15}}{(\tau-\tau')^{\theta}}\right)^{\frac{1}{p}}\left(\fint_{\sigma' Q_T} u^p\omega(x)dx\right)^{\frac{1}{p}}\\ &\le \frac{1}{2}\underset{\sigma' Q_T}{ess\ \sup}\left(\omega(x)u\right) + \left(\frac{C_{15}}{(\sigma-\sigma')^{\theta}}\right)^{\frac{1}{s}}\left(\fint_{\sigma Q_T} u^s\omega(x)dx\right)^{\frac{1}{s}},\end{aligned}$$

where we did some calculations and used Young's inequality. By a standard iteration we obtain the result (see, for example [8]). ☐

4. Hölder estimates

Before we give a proof for a weak Harnack inequality, later a proof Theorem 2.3.

Theorem 4.1. *Let u - nonnegative a weak solution of problem (1.1) and (1.5) in U. Then there exist constants $C_{16}\ (p,\,C_5,\,C_4,\,q,\,\delta)$ and $q_0 = (p-1)\left(2-\frac{p}{k}\right)$, $k>p$ such that*

$$\left(\fint_{\delta U^-} u^q\omega(x)dx\right)^{\frac{1}{q}} \le C_{16}\, \underset{\delta U^+}{ess\inf}\, u,$$

for $0<\delta<1$ and $0<q<q_0$.

Here we use the same notation as in Theorem 2.3.

Proof. Let $0<\sigma\le 1$, $\tau\in R$. We consider set

$$\begin{aligned} Q_{T,\tau} &= B\left(x_0,r\right)\left(\tau-Tr^p,\tau+Tr^p\right),\\ \sigma Q_{T,\tau}^{+} &= B\left(x_0,\sigma r\right)\left(\tau,\tau+T\left(\sigma r\right)^p\right),\\ \sigma Q_{T,\tau}^{-} &= B\left(x^0,\sigma r\right)\left(\tau-T\left(\sigma r\right)^p,r\right).\end{aligned}$$

Let u - is nonnegative a supersolution in $Q_{T,\tau}$ and $\varphi(x,t) = \varphi(x) = \left(1-2\frac{|x-x_0|}{(1+\tau)r}\right)$, and $(x,t)\in B(x_0r)(\tau-(\sigma r)^p$, $\tau+(\sigma r)^p)$.

Let $\beta = \int_{B(x_0,r)} (\log u(x,\tau)) \varphi^p(x) \omega(x) dx$. Then we can show that, there exist constants $C_{17}(p, C_5, C_4, \tau, T)$ and $C_{18}(p, C_5, \tau, T)$ such that measure $d\mu = \omega(x) dxdt$ for every $\lambda > 0$ satisfying

$$\mu\left(\left\{(x,t) \in \sigma Q^-_{T,\tau} / \log u(x,t) > \lambda + \beta + C_{18}\right\}\right) \leq \frac{C_{17}}{\lambda^{p-1}} \mu\left(\sigma Q^-_{T,\tau}\right), \tag{4.1}$$

and

$$\mu\left(\left\{(x,t) \in \sigma Q^+_{T,\tau} / \log u(x,t) < -\lambda + \beta - C_{18}\right\}\right) \leq \frac{C_{17}}{\lambda^{p-1}} \mu\left(\sigma Q^+_{T,\tau}\right). \tag{4.2}$$

Now we define $v^+ = u^{-1} \exp(\beta - C_{18})$ and $v^- = u \exp(-\beta - C_{18})$. For the function u, we apply (4.1) and (4.2)

$$\mu\left(\left\{(x,t) \in \frac{1+\delta}{2} U^+ / \log v^+(x,t) > \lambda\right\}\right) \leq \frac{C_{17}}{\lambda^{p-1}} \mu\left(\frac{1+\delta}{2} U^+\right),$$

and

$$\mu\left(\left\{(x,t) \in \frac{1+\delta}{2} U^- / \log v^-(x,t) > \lambda\right\}\right) \leq \frac{C_{17}}{\lambda^{p-1}} \mu\left(\frac{1+\delta}{2} U^-\right).$$

We also used the fact that

$$\mu\left(B(x_0, \sigma R) \times \left(\tau, \tau \pm (\sigma' R)^p\right)\right) \leq C_{17} \mu\left(\delta U^\pm\right),$$

and v^+ is a subsalution in U. Consequently from Lemma 3.6 we get

$$ess \sup_{\sigma' U^+} \left(\omega(x) v^+\right) \leq \left(\frac{C_{17}}{(\sigma - \sigma')^\theta} \fint_{\sigma U^+} \left(v^+\right)^s \omega(x) dxdt\right)^{\frac{1}{s}},$$

when ever $\delta \leq \sigma' < \sigma \leq \frac{(1+\delta)}{2}$ and $s > 0$. If we use Lemma 3.1, it is obtained

$$ess \sup_{\delta U^+} \left(\omega(x) v^+\right) \leq C_{17}. \tag{4.3}$$

Later, we use the following estimation

$$\left(\fint_{\sigma' Q_{T,\tau}} u^q \omega(x) dx\right)^{\frac{1}{q}} \leq \left(\frac{C_{19}}{(\sigma - \sigma')^\theta}\right)^{\frac{1}{s}} \left(\fint_{\sigma Q_{T,\tau}} u^s \omega(x) dx\right)^{\frac{1}{s}}, \tag{4.4}$$

for all $0 < \delta \leq \sigma' < \sigma \leq 1$ and for all $0 < s < q < q_0$, where $q_0 = (p-1)\left(2 - \frac{p}{k}\right)$ and $k < p$. The proof of the fact is based on the use of Sobolev's inequality and Caccioppoli's estimate above.

Now, for v^-, we use estimation (4.4), then

$$\left(\left(\fint_{\sigma' U'} \left(v^-\right)^q \omega(x) dxdt\right)^{\frac{1}{q}}\right)^{\frac{1}{q}} \leq \left(\frac{C_{19}}{(\sigma - \sigma')^\theta} \fint_{\sigma U^-} \left(v^-\right)^s \omega(x) dxdt\right)^{\frac{1}{s}},$$

where $\delta \leq \sigma' < \sigma \leq \frac{(1+\delta)}{2}$ and $0 < s < q < q_0$. Again from Lemma 3.1, we obtain

$$\left(\fint_{\delta U^-} \left(v^-\right)^q \omega(x) dxdt\right)^{\frac{1}{q}} \leq C_{17}.$$

Multiplying this with (4.3) gives

$$\left(\fint_{\delta U^-} u^q \omega(x) dxdt\right)^{\frac{1}{q}} \leq C_{17} ess \inf_{\delta U^+} \left(\omega(x) u\right).$$

This is complete the proof. □

Now, we can prove Theorem 2.3.

Proof. of Theorem 2.3
For proving, we apply Theorem 4.1 with $\delta = \frac{(1+\sigma)}{2}$. Then the result follows from Lemma 3.6. □

Theorem 4.2. *Let weight $\omega(x)$ is doubling and belong to Mucenxoupt weight classes. A_p . Let u - be a weak solution to problem (1.1),(1.5). Then solution belong to Holder classes of functions.*

Proof. It's well known that the local Hölder continuity of a weak solution is a consequence of the Harnack inequality (see [13, 16]). The local Hölder continuity of the solution also has been proved in [17]. The result follows now from Theorem 4.1. □

5. Conclusion

Hölder estimates of solutions of initial-boundary problem degenerate nonlinear parabolic equations is obtained. Estimates for solutions and parabolic Harnack inequality are proved. Also, one variant of weighted Poincare inequality is showen.

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