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Steady state bifurcation in a cross-diffusion prey-predator model

Marzieh Farshid and Yaghoub Jalilian*
[1]Department of Mathematics, Razi University, Kermanshah, Iran.

Abstract

In this paper, we study the bifurcation of nontrivial steady state solutions for a cross-diffusion prey-predator model with homogeneous Neumann boundary conditions. The existence of positive steady state solutions near a bifurcation point is proved using a crossing curve bifurcation theorem. We consider a situation where the transversality condition is not satisfied. Unlike the case in saddle-node bifurcation, the solution set is a pair of transversally intersecting curves.

Keywords. Steady state bifurcation, Cross diffusion, Prey-predator model.
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1. Introduction

In this paper, we consider the following cross-diffusion predator-prey system with group defense behavior of the prey species

$$\begin{cases} u_t - \mu\Delta u = ru(1-\frac{u}{k}) - \dfrac{muv}{u^p+c}, & (x,t) \in \Omega \times (0,\infty), \\ v_t - \Delta[(\alpha+\beta u)v] = (d - \dfrac{e}{u+a})v^2, & (x,t) \in \Omega \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, & x \in \Omega, \\ \partial_\nu u = \partial_\nu v = 0, & (x,t) \in \partial\Omega \times (0,\infty), \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, u and v are the densities of the prey and the predator and ν is the outward unit normal vector on $\partial\Omega$. The parameter r is the intrinsic growth rate of the prey species, k is the environmental carrying capacity of the prey, m is the maximum predation rate and, c is the protection provided to the prey population by the environment, d is the reproduction rate of the generalist predator, e is the maximum rate of death in predator population and a is the residual loss of predator species due to severe scarcity of the prey species. The functional response

$$g(u) = \frac{mu}{u^p+c},\ p > 1,$$

represents group defense in the prey [13]. Also, the coefficients μ and α are the diffusion of the prey and predator respectively. μ and α are assumed to be only positive constants, while β can be non-negative constant. The nonlinear cross-diffusion term $\beta\Delta(uv)$ describes the tendency of predators to move away from high-density areas of prey species.

Some researchers studied the steady state bifurcation from a simple eigenvalue of the predator-prey system. For example, Li in [6] investigated the global steady-state bifurcation in a cross-diffusion predator-prey system with prey-taxis and Holling type II functional response. Kong and Lu in [4] discussed the global steady-state bifurcation and the stability of bifurcating solutions for a general reaction-diffusive predator-prey system with prey-taxis. To see more research papers about the steady state bifurcation from a simple eigenvalue we refer the reader to [3, 5, 7, 8, 11, 15, 17].

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* Corresponding author. Email: y.jalilian@razi.ac.ir.

Further, in [1, 16] one can find some results on the steady state bifurcation for two dimensional kernels. The existence and stability of the steady state bifurcation were examined in a general reaction-diffusion equation with a nonlocal delay effect in [18]. To see some numerical simulations of bifurcations, we refer the reader to [10, 12, 14].

The main purpose of this paper is to study the local steady state bifurcation to the cross-diffusion prey-predator system (1.1). We take c as the bifurcation parameter and we study a situation where the transversality condition is violated. In this case, the Crandall and Rabinowitz theorem is not applicable and we use a crossing curve bifurcation theorem to prove the existence of two intersecting curves of positive steady state solutions to the system (1.1).

The rest of the paper is organized as follows: In section 2, we give some preliminary results which are needed in the next section. In section 3, we study the local steady state bifurcation from a simple eigenvalue for system (1.1).

2. Preliminary

System (1.1) has the following equilibrium points

$$(0,0),\ (k,0),\ U_*=(u_*,v_*)=\left(\frac{e}{d}-a,\frac{r}{mk}(k+a-\frac{e}{d})((\frac{e}{d}-a)^p+c)\right), \tag{2.1}$$

u_* and v_* are positive constant solutions of system (1.1) if and only if

$$0<\frac{e}{d}-a<k. \tag{2.2}$$

From now on we assume that the inequalities (2.2) hold. The linearized system of (1.1) at U_* is

$$\begin{pmatrix}u_t\\v_t\end{pmatrix}:=L\begin{pmatrix}u\\v\end{pmatrix}=(D\Delta+J(U_*))\begin{pmatrix}u\\v\end{pmatrix},$$

with domain

$$X=\{(u,v)\in\mathbb{H}^2(\Omega)\times\mathbb{H}^2(\Omega):\partial_\nu u=\partial_\nu v=0\},$$

where

$$D:=\begin{pmatrix}\mu & 0\\ \beta v_* & \alpha+\beta u_*\end{pmatrix},\ J:=\begin{pmatrix}\dfrac{-ru_*}{k}+\dfrac{pu_*^pr(1-\frac{u_*}{k})}{u_*^p+c} & -\dfrac{mu_*}{u_*^p+c}\\ \dfrac{d^2v_*^2}{e} & 0\end{pmatrix}.$$

Denote the eigenvalues of the operator $-\Delta$ with homogeneous Neumann boundary conditions on $\partial\Omega$ by $0=\lambda_0<\lambda_1\le\lambda_2\le...\le\lambda_n\le...$, and let φ_n be the normalized eigenfunction corresponding to λ_n. Then λ is an eigenvalue of the operator L if and only if λ is an eigenvalue of the matrix $L_n=-\lambda_nD+J(U_*)$ for $n\ge0$. The characteristic equation of L_n can be expressed by

$$\lambda^2-T_n(c)\lambda+D_n(c)=0, \tag{2.3}$$

where

$$T_n=-(\mu+\alpha+\beta u_*)\lambda_n-\frac{ru_*}{k}+\frac{pu_*^pr(1-\frac{u_*}{k})}{u_*^p+c},$$

$$D_n=\mu(\alpha+\beta u_*)\lambda_n^2-\left((\alpha+\beta u_*)(\frac{-ru_*}{k}+\frac{pu_*^pr(1-\frac{u_*}{k})}{u_*^p+c})+\frac{\beta mu_*v_*}{u_*^p+c}\right)\lambda_n+\frac{d^2mu_*v_*^2}{e(u_*^p+c)}.$$

We can rewrite D_n as

$$D_n:=\frac{1}{me(u_*^p+c)}(B_1c^2+B_{2n}c+B_{3n}),$$

where

$$\begin{aligned} B_1 &= d^2 u_* r^2 (1-\frac{u_*}{k})^2, \\ B_{n2} &= 2d^2 u_*^{p+1} r^2 (1-\frac{u_*}{k})^2 + em\mu(\alpha+\beta u_*)\lambda_n^2 + meru_*\left(\frac{\alpha+2\beta u_*}{k}-\beta\right)\lambda_n, \\ B_{n3} &= d^2 u_*^{2p+1} r^2 (1-\frac{u_*}{k})^2 - meu_*^p(\alpha+\beta u_*)\lambda_n\left(pr(1-\frac{u_*}{k})-\mu\lambda_n\right) + meru_*^{p+1}\left(\frac{\alpha+2\beta u_*}{k}-\beta\right)\lambda_n. \end{aligned} \tag{2.4}$$

We have $B_1 > 0$. If $B_{n3} < 0$, then the equation $D_n = 0$ has the positive solution $c_*^n = \dfrac{-B_{n2}+\sqrt{B_{n2}^2-4B_1B_{n3}}}{2B_1}$. Also the next solution is negative. It is easy to see that $B_{n3} < 0$ when

$$\frac{u_*^{p+1} d^2 r (1-\frac{u_*}{k})}{mep(\alpha+\beta u_*)} < \lambda_n < \frac{\beta u_* r(k-2u_*) - ru_*\alpha}{k\mu(\alpha+\beta u_*)}. \tag{2.5}$$

So, 0 is an eigenvalue of the linearized operator L at (U_*, c_*^n).

3. The steady state bifurcation

In this section, we investigate the existence of a steady state bifurcation at (U_*, c_*^n) for system (1.1) by a crossing curve bifurcation theorem. Let

$$X = \{(u,v) \in \mathbb{H}^2(\Omega)\times\mathbb{H}^2(\Omega) : \partial_\nu u = \partial_\nu v = 0\},$$
$$Y = \mathbb{L}^2(\Omega)\times\mathbb{L}^2(\Omega).$$

Also, $(f,g)_Y = \int_\Omega f^T g dx$ denotes the inner product in $\mathbb{L}^2(\Omega)\times\mathbb{L}^2(\Omega)$. Set

$$F(U,c) := \begin{pmatrix} \mu\Delta u + ru(1-\frac{u}{k}) - \dfrac{muv}{u^p+c} \\ \Delta[(\alpha+\beta u)v] + (d-\dfrac{e}{u+a})v^2 \end{pmatrix}.$$

Then $F(U_*, c_*^n) = 0$ and

$$F_U(U,c) := \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{pmatrix},$$

where

$$\begin{aligned} \gamma_1 &= \mu\Delta + r(1-\frac{2u}{k}) - \frac{mv(u^p+c)-pmu^p v}{(u^p+c)^2}, \\ \gamma_2 &= -\frac{mu}{(u^p+c)}, \\ \gamma_3 &= \beta(\Delta v + v\Delta + 2\nabla v.\nabla) + \frac{ev^2}{(u+a)^2}, \\ \gamma_4 &= (\alpha+\beta u)\Delta + \beta\Delta u + 2\beta\nabla u.\nabla + 2v(d-\frac{e}{u+a}), \end{aligned}$$

and

$$F_c(U,c) := \begin{pmatrix} \dfrac{muv}{(u^p+c)^2} \\ 0 \end{pmatrix}, \tag{3.1}$$

$$F_{cU}(U,c) := \begin{pmatrix} \dfrac{mv(u^p+c)-2pmu^p v}{(u^p+c)^3} & \dfrac{mu}{(u^p+c)^2} \\ 0 & 0 \end{pmatrix}.$$

As we can see in (2.1), U_* depends on c_*^n. The transversality condition

$$F_c(U_*, c_*^n) \notin R(F_U(U_*, c_*^n)), \tag{3.2}$$

is used in the saddle-node bifurcation theorem [2, Theorem 3.2]. In the sequel, we show that F does not satisfy (3.2). Hence we use the following crossing curve bifurcation theorem [9].

Theorem 3.1. *Let $F : O \longrightarrow Y$ be a $\mathbb{C}^2$ mapping where $O \subset X \times \mathbb{R}$ is a neighborhood of $(U_0, \lambda_0) \in X \times \mathbb{R}$. Suppose that $F(U_0, \lambda_0) = 0$ and F satisfies*

- **(F1)** *$dimN(F_U(U_0, \lambda_0)) = codimR(F_U(U_0, \lambda_0)) = 1$, and $N(F_U(U_0, \lambda_0)) = span\{w_0\}$,*
- **(F2)** *$F_\lambda(U_0, \lambda_0) \in R(F_U(U_0, \lambda_0))$.*

Let $X = N(F_U(U_0, \lambda_0)) \bigoplus Z$ be a fixed splitting of X, let $v_1 \in Z$ be the unique solution of

$$F_\lambda(U_0, \lambda_0) + F_U(U_0, \lambda_0)[v] = 0,$$

and let $l \in Y^*$ such that $R(F_U(U_0, \lambda_0)) = \{v \in Y : \langle l, v\rangle = 0\}$. We assume that the matrix (all derivatives are evaluated at (U_0, λ_0))

$$H_0 = H_0(U_0, \lambda_0) \equiv \begin{pmatrix} \langle l, F_{\lambda\lambda} + 2F_{\lambda U}[v_1] + F_{UU}[v_1]^2\rangle & \langle l, F_{\lambda U}[w_0] + F_{UU}[w_0, v_1]\rangle \\ \langle l, F_{\lambda U}[w_0] + F_{UU}[w_0, v_1]\rangle & \langle l, F_{UU}[w_0]^2\rangle \end{pmatrix},$$

is non-degenerate, i.e., $det(H_0) \neq 0$.

1. *If H_0 is definite, i.e. $det(H_0) > 0$, then the solution set of $F(U, \lambda) = 0$ near $(U, \lambda) = (U_0, \lambda_0)$ is $\{(U_0, \lambda_0)\}$.*
2. *If H_0 is indefinite, i.e. $det(H_0) < 0$, then the solution set of $F(U, \lambda) = 0$ near $(U, \lambda) = (U_0, \lambda_0)$ is the union of two intersecting $\mathbb{C}^1$ curves, and the two curves are in form of $(U_i(s), \lambda_i(s)) = (U_0 + \eta_i s w_0 + s y_i(s), \lambda_0 + \mu_i s + s\theta_i(s))$, $i = 1, 2$, where $s \in (-\delta, \delta)$ for some $\delta > 0$, (μ_1, η_1) and (μ_2, η_2) are non-zero linear independent solutions of the equation*

$$\langle l, F_{\lambda\lambda} + 2F_{\lambda U}[v_1] + F_{UU}[v_1]^2\rangle\mu^2 + 2\langle l, F_{\lambda U}[w_0] + F_{UU}[w_0, v_1]\rangle\eta\mu + \langle l, F_{UU}[w_0]^2\rangle\eta^2 = 0, \tag{3.3}$$

   where $\theta_i(s)$, $y_i(s)$ are some functions defined on $s \in (-\delta, \delta)$ which satisfy $\theta_i(0) = \theta_i'(0) = 0$, $y_i(s) \in Z$, and $y_i(0) = y_i'(0) = 0$, $i = 1, 2$.

Our main result is as follows:

Theorem 3.2. *Let (2.2) and (2.5) for a fixed $n \in \mathbb{N}$ be satisfied. Assume λ_n is a simple eigenvalue of Δ with zero Neumann boundary conditions, $c_*^n \neq c_*^m$, $D_m(c_*^n) \neq 0$ for $n \neq m$ and $\int_\Omega \varphi_n dx = \int_\Omega \varphi_n^3 dx = 0$. Then the steady state solution set of system (1.1) near (U_*, c_*^n) consists two curves in form of $\Gamma_i = \{(U_i(s), c_i(s)) : s \in (-\delta, \delta)\}$, $i = 1, 2$ for some $\delta > 0$ where*

$$(U_1(s), c_1(s)) = \Big(U_* + s y_1(s), c_*^n + s + s\theta_1(s)\Big),$$

$$(U_2(s), c_2(s)) = \Big(U_* + s\phi_n + s y_2(s), c_*^n + s\theta_2(s)\Big),$$

and also $\theta_i(s)$ and $y_i(s)$ are some functions on $s \in (-\delta, \delta)$ which satisfy $\theta_i(0) = \theta_i'(0) = 0$ and $y_i(0) = y_i'(0) = 0$, $i = 1, 2$.

Proof. It is easy to see that F is a C^2 mapping in a neighborhood of each coexistence equilibrium point. Since for a fixed positive integer n, λ_n is a simple eigenvalue, $c_*^n \neq c_*^m$ and $D_m(c_*^n) \neq 0$ for $n \neq m$, we get

$$ker F_U(U_*, c_*^n) = span\{\phi_n\},$$

where

$$\phi_n := \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \dfrac{-\beta e v_* \lambda_n + d^2 v_*^2}{(\alpha + \beta u_*) e \lambda_n}, \end{pmatrix} \varphi_n.$$

Here, φ_n is the eigenfunction corresponding to the eigenvalue λ_n. On the other hand, the adjoint operator of $F_U(U_*, c_*^n)$ is given by

$$F_U^*(U_*, c_*^n) = \begin{pmatrix} \mu\Delta + r(1 - \frac{2u_*}{k}) - \dfrac{mv_*(u_*^p + c_*^n) - pmu_*^p v_*}{(u_*^p + c_*^n)^2} & \beta v_*\Delta + \dfrac{ev_*^2}{(u_* + a)^2} \\ -\dfrac{mu_*}{(u_*^p + c_*^n)} & (\alpha + \beta u_*)\Delta + 2v_*(d - \dfrac{e}{u_* + a}) \end{pmatrix}.$$

Then, we have

$$ker(F_U^*(U_*, c_*^n)) = span\{\phi_n^*\},$$

where

$$\phi_n^* := \begin{pmatrix} \dfrac{-(\alpha + \beta u_*)(u_*^p + c)\lambda_n}{mu_*} \\ 1 \end{pmatrix} \varphi_n.$$

By $R(F_U(U_*, c_*^n)) = (ker(F_U^*(U_*, c_*^n)))^\perp$, we find

$$dimker F_U(U_*, c_*^n) = codim R(F_U(U_*, c_*^n)) = 1,$$

and

$$R(F_U(U_*, c_*^n)) = \Big\{ g = (g_1, g_2) \in Y : \langle \phi_n^*, g \rangle = 0 \Big\}.$$

Also, by the condition $\int_\Omega \varphi_n dx = 0$, we have

$$\begin{aligned} \langle \phi_n^*, F_c(U_*, c_*^n) \rangle &= \int_\Omega [(\frac{mu_*v_*}{(u_*^p + c_*^n)^2})(\frac{-(\alpha + \beta u_*)(u_*^p + c_*^n)\lambda_n}{mu_*})]\varphi_n dx \\ &= -\frac{r\lambda_n}{m}(1 - \frac{u_*}{k})(\alpha + \beta u_*) \int_\Omega \varphi_n dx = 0. \end{aligned}$$

Then

$$F_c(U_*, c_*^n) \in R(F_U(U_*, c_*^n)).$$

Therefore, (**F1**) and (**F2**) are satisfied. Set

$$\zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{r}{m}(1 - \dfrac{u_*}{k}) \end{pmatrix}.$$

Then

$$F_c(U_*, c_*^n) + F_U(U_*, c_*^n)[\zeta] = \begin{pmatrix} \Big(-\dfrac{ru_*}{k} + \dfrac{pu_*^p r(1 - \frac{u_*}{k})}{u_*^p + c_*^n} \Big)\zeta_1 - \dfrac{mu_*}{u_*^p + c_*^n}\zeta_2 + \dfrac{mu_*v_*}{(u_*^p + c_*^n)^2} \\ \dfrac{ev_*^2}{(u_*^p + c_*^n)^2}\zeta_1 \end{pmatrix} = 0.$$

By calculation, we have

$$F_{UU}(U_*, c_*^n)[\phi_n]^2 = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \varphi_n^2 + \begin{pmatrix} 0 \\ e_3 \end{pmatrix} |\nabla\varphi_n|^2,$$

where

$$\begin{aligned}
e_1 =&\frac{-2r}{k}+\frac{mv_*pu_*^{p-1}((u_*^p+c_*^n)(p+1)-2pu_*^p)}{(u_*^p+c_*^n)^3}-\frac{2m(u_*^p+c_*^n-pu_*^p)}{(u_*^p+c_*^n)^2}(\frac{-\beta ev_*\lambda_n+d^2v_*^2}{(\alpha+\beta u_*)e\lambda_n}),\\
e_2 =&(-4\beta\lambda_n+\frac{4d^2v_*}{e})(\frac{-\beta ev_*\lambda_n+d^2v_*^2}{(\alpha+\beta u_*)e\lambda_n})-\frac{2d^3v_*^2}{e^2},\\
e_3 =&4\beta(\frac{-\beta ev_*\lambda_n+d^2v_*^2}{(\alpha+\beta u_*)e\lambda_n}).
\end{aligned}$$

By the assumption $\int_\Omega \varphi_n^3dx=0$, and the integration by part, we have

$$\int_\Omega |\nabla\varphi_n|^2\varphi_n dx=\int_\Omega \frac{-1}{2}\varphi_n^2\Delta\varphi_n dx=\frac{\lambda_n}{2}\int_\Omega \varphi_n^3dx=0.$$

Therefore

$$\begin{aligned}
\langle l, F_{UU}[\phi_n]^2\rangle=\langle \phi_n^*, F_{UU}[\phi_n]^2\rangle&=\int_\Omega [e_1(\frac{-(\alpha+\beta u_*)(u_*^p+c_*^n)\lambda_n}{mu_*})+\frac{e_3\lambda_n}{2}+e_2]\varphi_n^3dx\\
&=\left(e_1(\frac{-(\alpha+\beta u_*)(u_*^p+c_*^n)\lambda_n}{mu_*})+\frac{e_3\lambda_n}{2}+e_2\right)\int_\Omega \varphi_n^3dx=0.
\end{aligned}$$

Also, we obtain that

$$\begin{aligned}
&F_{cc}+2F_{cU}[\zeta]+F_{UU}[\zeta]^2=\begin{pmatrix}-\frac{2mu_*v_*}{(u_*^p+c_*^n)^3}\\ 0\end{pmatrix}+2\begin{pmatrix}(\frac{mv_*(u_*^p+c_*^n)-2pmu_*^pv_*}{(u_*^p+c_*^n)^3})\zeta_1+(\frac{mu_*}{(u_*^p+c_*^n)^2})\zeta_2\\ 0\end{pmatrix}\\
&+\begin{pmatrix}(\frac{-2r}{k}+\frac{mv_*pu_*^{p-1}((u_*^p+c_*^n)(p+1)-2pu_*^p)}{(u_*^p+c_*^n)^3})\zeta_1^2-(\frac{2m(u_*^p+c_*^n-pu_*^p)}{(u_*^p+c_*^n)^2})\zeta_1\zeta_2\\ -(\frac{2d^3v_*^2}{e^2})\zeta_1^2+(\frac{4d^2v_*}{e})\zeta_1\zeta_2\end{pmatrix}\\
&=0,
\end{aligned}$$

and

$$\begin{aligned}
&F_{cU}[\phi_n]+F_{UU}[\phi_n,\zeta]=\begin{pmatrix}(\frac{mv_*(u_*^p+c_*^n)-2pmu_*^pv_*}{(u_*^p+c_*^n)^3})\phi_1+(\frac{mu_*}{(u_*^p+c_*^n)^2})\phi_2\\ 0\end{pmatrix}\\
&+\begin{pmatrix}(\frac{-2r}{k}+\frac{mv_*pu_*^{p-1}((u_*^p+c_*^n)(p+1)-2pu_*^p)}{(u_*^p+c_*^n)^3})\zeta_1\phi_1-(\frac{m(u_*^p+c_*^n-pu_*^p)}{(u_*^p+c_*^n)^2})(\zeta_1\phi_2+\zeta_2\phi_1)\\ -(\frac{2d^3v_*^2}{e^2})\zeta_1\phi_1+(\frac{2d^2v_*}{e})(\zeta_1\phi_2+\zeta_2\phi_1)+\beta(\zeta_2\Delta\phi_1+\zeta_1\Delta\phi_2)\end{pmatrix}\\
&=\begin{pmatrix}-\frac{pmu_*^pv_*}{(u_*^p+c_*^n)^3}+\frac{mu_*}{(u_*^p+c_*^n)^2}(\frac{-\beta ev_*\lambda_n+d^2v_*^2}{(\alpha+\beta u_*)e\lambda_n})\\ (-\beta\lambda_n+\frac{2d^2v_*}{e})\zeta_2\end{pmatrix}\varphi_n.
\end{aligned}$$

Then

$$\langle l, F_{cc}+2F_{cU}[\zeta]+F_{UU}[\zeta]^2\rangle=\langle \phi_n^*, F_{cc}+2F_{cU}[\zeta]+F_{UU}[\zeta]^2\rangle=0.$$

Using this fact that φ_n is a normalized eigenfunction, that is $\int_\Omega \varphi_n^2 dx = 1$, we have

$$\begin{aligned}\langle l, F_{cU}[\phi_n] + F_{UU}[\phi_n,\zeta]\rangle &= \langle \phi_n^*, F_{cU}[\phi_n] + F_{UU}[\phi_n,\zeta]\rangle \\ &= \Bigg[\Bigg(- \frac{pmu_*^p v_*}{(u_*^p + c_*^n)^3} + \frac{mu_*}{(u_*^p + c_*^n)^2}(\frac{-\beta e v_* \lambda_n + d^2 v_*^2}{(\alpha+\beta u_*)e\lambda_n})\Bigg) \\ &\times (\frac{-(\alpha+\beta u_*)(u_*^p + c_*^n)\lambda_n}{mu_*}) + (-\beta\lambda_n + \frac{2d^2 v_*}{e})\zeta_2\Bigg] \int_\Omega \varphi_n^2 dx \\ &= \frac{pu_*^{p-1} v_* \lambda_n (\alpha+\beta u_*)}{(u_*^p + c_*^n)^2} + \frac{d^2 v_*^2}{e(u_*^p + c_*^n)} =: M > 0.\end{aligned}$$

Therefore,

$$H_0(U_*, c_*^n) = \begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix},$$

and $det(H_0) = -M^2 < 0$. It is easy to see that $(\mu_1, \eta_1) = (1,0)$ and $(\mu_2,\eta_2) = (0,1)$ are non-zero linear independent solutions of (3.3). So by Theorem 3.1, the proof is complete. □

Example 3.3. Consider system (1.1) with $\Omega = (0,\pi)$ and

$$\begin{aligned}&r = 3.19,\ k = 46,\ m = 11.04,\ p = 2.6,\ e = 5.1, \\ &d = 1.2,\ a = 1.8,\ ,\alpha = 1.03,\ \beta = 13.8,\ \mu = 2.75.\end{aligned} \tag{3.4}$$

According to (3.4), we have

$$\begin{aligned}D_n(c) &= \frac{0.5715c^2 + c(11.746 + 95.81\lambda_n^2 - 96.1901\lambda_n) + 60.3524 + 984.561\lambda_n^2 - 3799.7486\lambda_n}{(10.2761 + c)}, \\ T_n(c) &= \frac{-c(0.17 + 37.59\lambda_n) - 386.2817\lambda_n + 78.9452}{(10.2761 + c)}.\end{aligned}$$

The only eigenvalue satisfying (2.5), is $\lambda_1 = 1$. Then the positive solution of the equation $D_1(c) = 0$ is $c_*^1 = 60.1924$. Furthermore we obtain

$$\begin{aligned}D_n(c_*^1) &= 95.81\lambda_n^2 - 136.084\lambda_n + 40.274, \\ T_n(c_*^1) &= -37.59\lambda_n + 0.975.\end{aligned}$$

So, $T_1(c_*^1) = -36.615$ and $D_0(c_*^1) = 40.274$. Also $D_{n+1}(c_*^1) > D_n(c_*^1) > 0$, for $n > 1$. Then we get the following positive bifurcation point

$$(U_*, c_*^1) = \big(2.45, 0.0575, 60.1924\big).$$

Using Theorem 3.2, the solution set of the system near the bifurcation point $(2.45, 0.0575, 60.1924)$ is two curves $\Gamma_1 = \{(U_1(s), c_1(s)) : s \in (-\delta,\delta)\}$ and $\Gamma_2 = \{(U_2(s), c_2(s)) : s \in (-\delta,\delta)\}$ where

$$\begin{aligned}U_1(s) &= (2.45, 0.0575) + s y_1(s), \\ c_1(s) &= 60.1924 + s + s\theta_1(s), \\ U_2(s) &= (2.45, 0.0575) + s(1, -0.0227)\sqrt{\frac{2}{\pi}}\cos x + s y_2(s), \\ c_2(s) &= 60.1924 + s\theta_2(s).\end{aligned}$$

Here θ_i and y_i for $i = 1,2$ are functions which satisfy $\theta_i(0) = \theta_i'(0) = 0$ and $y_i(0) = y_i'(0) = 0$.

4. Conclusion

In this paper we investigated the existence of positive steady state solutions for cross-diffusion prey-predator model (1.1) under homogeneous Neumann boundary conditions. We considered the prey environmental protection as a bifurcation parameter. It is proved that the transversality condition

$$F_c(U_*, c_*^n) \notin R(F_U(U_*, c_*^n)),$$

is not satisfy when $\int_\Omega \varphi_n dx = 0$ where φ_n is an eigenfunction corresponding to the eigenvalue λ_n. Therefore the saddle-nod bifurcation theorem is not applicable for system (1.1). Using a crossing curve bifurcation theorem we established that the steady state solution set of system (1.1) near the constant coexistence equilibrium (U_*, c_*^n) is the union of two intersecting $\mathbb{C}^1$ curves of the following forms

$$(U_1(s), c_1(s)) = \Big(U_* + sy_1(s), c_*^n + s + s\theta_1(s)\Big), \ s \in (-\delta, \delta),$$

$$(U_2(s), c_2(s)) = \Big(U_* + s\phi_n + sy_2(s), c_*^n + s\theta_2(s)\Big), \ s \in (-\delta, \delta),$$

where $\theta_i(s)$ and $y_i(s)$ are some functions on $(-\delta, \delta)$ which satisfy $\theta_i(0) = \theta_i^{'}(0) = 0$ and $y_i(0) = y_i^{'}(0) = 0$, $i = 1, 2$. For the future work one can investigate the stability of the above steady state solutions. To do this we suggest to use a perturbation of the zero eigenvalue of the operator $F_U(U_*, c_*^n)$.

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