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Haar wavelet-based valuation method for pricing European options

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Abstract

A numerical method based on the Haar wavelet is introduced in this study for solving the partial differential equation which arises in the pricing of European options. In the first place, and due to the change of variables, the related partial differential equation (PDE) converts into a forward time problem with a spatial domain ranging from 0 to 1. In the following, the Haar wavelet basis is used to approximate the highest derivative order in the equation concerning the spatial variable. Then the lower derivative orders are approximated using the Haar wavelet basis. Finally, by substituting the obtained approximations in the main PDE and doing some computations using the finite differences approach, the problem reduces to a system of linear equations that can be solved to get an approximate solution. The provided examples demonstrate the effectiveness and precision of the method.

Keywords. European option, Haar wavelet, Option pricing.
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1. Introduction

Options are some of the relatively new tools in the financial markets, which due to their nature and type of use have attracted the main attention of market traders (Hedgers, Speculators, and Arbitrageurs)[11]. In general, Options can be divided into two categories: "Call Options" and "Put Options". The call option actually gives the holder the right, not the obligation, to buy the asset subject to the contract at a specified price and on or before a specified date. Similarly, a put option gives the holder the right to sell the asset subject to the contract at a specified price and on or before a specific date. The price mentioned in the contract is called the strike price or the exercise price and the date mentioned in the contract, the so-called expiration date or maturity date. The options are divided into European and American types. The European type is only applicable on the expiration date whereas, the American option can be applied at any time before the expiration date or on the expiration date. The unofficial history of the first trading options dates back to the early 18th century in Europe and the United States, but were officially exchanged in 1973 by the Chicago Stock Exchange and then in 1975 by the US Stock Exchange and the Philadelphia Stock Exchange Launched. Later, following the explosion of this financial instrument, the exchange of options on foreign exchange, futures contracts, bonds, and various indicators such as S&P 100, S&P 500, and NAZDAQ were also created.

Dependence and influence of options on factors such as underlying asset, maturity, strike price, interest rate, and the volatility of the underlying asset, have made the fair pricing of options always one of the main challenges facing researchers interested in this issue. To learn more about the scope of options and how to price them, see [10, 17, 21, 24].

Wavelets, and specifically Haar Wavelets, have been widely used to solve a variety of problems, such as differential equations, integral equations, integral-differential equations, Calculus of variations, and fractional calculus.

The Regularized Long Wave (RLW) equation is numerically solved using the Higher Order Haar wavelet method and the traditional Haar wavelet method [6]. In [22], the Haar wavelet method is used to solve a classic one-dimensional nonlinear eigenvalue problem of the well-known Gelfand elliptic BVP. The generalized regularized long wave (GRLW)

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model is simulated using Scale-3 Haar wavelets (S3HWs)[15]. To explore and analyze the physical and numerical features of two-parameter singularly perturbed problems with Robin boundary conditions, a highly accurate wavelet-based approximation is proposed in [13]. The book "A basic course in wavelets with Fourier analysis" is an excellent resource for mathematicians, signal processing engineers, and scientists interested in wavelet theory and Fourier analysis [5]. Haar wavelets also were used for finding the numerical solution of stochastic Volterra integral equations [23], oscillating magnetic fields integro-differential equations [12], nonlinear Fredholm and Volterra integral equations [4], lumped and distributed-parameter systems [7], the system of integral equations [26], fractional integro-differential equations [2], nonlinear integral and integro-differential equations of first and higher orders [20], systems of PDEs [3], fractional oscillation equations [18], time fractional Riesz space telegraph equation with separable solution [1], a class of generalized Burgers equation [25], Emden–Fowler type equations [19], nonlinear Drinfel'd–Sokolov (DS) system of partial differential equations [9] and cubic nonlinear Schrodinger equations [16]. For some other applications of Haar wavelets ones can refer to [14].

2. Haar Wavelets

The two functions that always play a fundamental role in wavelet analysis are the scaling function φ and the wavelet function ψ. These two functions produce a family of functions which are used for the decomposition of a signal (function). In some cases, the scaling function is called the father wavelet, and the wavelet function is called the mother wavelet [5]. This is how the Haar scaling function is defined:

$$\varphi(x) = \begin{cases} 1, & 0 \leq x < 1, \\ 0, & elsewhere, \end{cases} \tag{2.1}$$

and using this function, the Haar wavelet will define as follows

$$\psi(x) = \varphi(2x) - \varphi(2x-1) = \begin{cases} 1, & 0 \leq x < \frac{1}{2}, \\ -1, & \frac{1}{2} \leq x < 1. \end{cases} \tag{2.2}$$

Let $j = 0, 1, 2, \cdots$, $m = 2^j$, $k = 0, 1, 2, \cdots, m-1$ and $i = m + k + 1$. Put $h_1(x) = \varphi(x)$ and

$$h_i(x) = \begin{cases} 1, & \alpha \leq x < \beta, \\ -1, & \beta \leq x < \gamma, \\ 0, & elsewhere, \end{cases} \tag{2.3}$$

where

$$\alpha = \frac{k}{m}, \ \beta = \frac{2k+1}{2m}, \ \gamma = \frac{k+1}{m}. \tag{2.4}$$

With the above definitions, the Haar wavelet family is formed.

Assuming that f is an square integrable function on the interval $[0.1)$, this function can be written as a linear combination of the Haar wavelet family as follows

$$f(x) = \sum_{i=1}^{\infty} a_i h_i(x), \tag{2.5}$$

which $a_i, i = 1, 2, \cdots$ are constants. Considering J as the maximum value for j and placing $M = 2^J$, we can get an approximation for the square integrable function f on the interval $[0, 1)$ as follows

$$f(x) \simeq \sum_{i=1}^{2M} a_i h_i(x). \tag{2.6}$$

We use the following notations to simplify the calculations in the following sections.

$$p_{i,1}(x) = \int_0^x h_i(t) d\tau, \tag{2.7}$$

$$p_{i,n+1} = \int_0^x p_{i,n}(t)d\tau, n = 1, 2, \cdots , \tag{2.8}$$

$$C_{i,n} = \int_0^1 p_{i,n}(x)dx, n = 1, 2, \cdots . \tag{2.9}$$

The following relationships can be easily obtained by the definition of the Haar wavelet and performing some preliminary calculations [20].

$$p_{i,n}(x) = \begin{cases} 0, & 0 \le x < \alpha, \\ \frac{1}{n!}(x-\alpha)^n, & \alpha \le x < \beta, \\ \frac{1}{n!}\left[(x-\alpha)^n - 2(x-\beta)^n\right], & \beta \le x < \gamma, \\ \frac{1}{n!}\left[(x-\alpha)^n - 2(x-\beta)^n + (x-\gamma)^n\right], & \gamma \le x < 1, \end{cases} \tag{2.10}$$

and

$$C_{1,n} = \frac{1}{(n+1)!}\left[(x-\alpha)^{n+1} - 2(x-\beta)^{n+1} + (x-\gamma)^{n+1}\right], \tag{2.11}$$

where $n = 1, 2, \cdots$ and $i = 2, 3, \cdots$. For the case $i = 1$ we have

$$p_{1,n} = \frac{x^n}{n!}, \quad n = 1, 2, \cdots , \tag{2.12}$$

and

$$C_{1,n} = \frac{1}{(n+1)!}, n = 1, 2, \cdots . \tag{2.13}$$

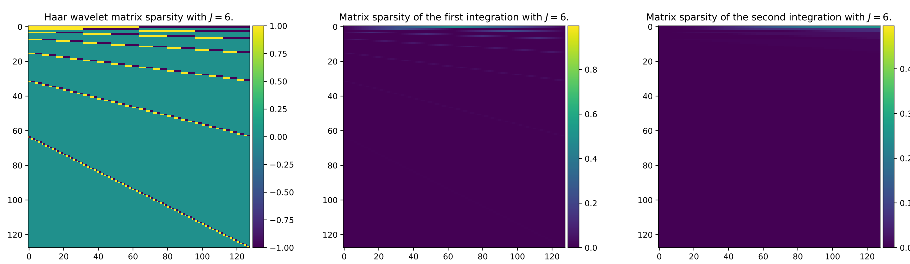

FIGURE 1. The matrix sparsity of Haar wavelets and their first and second integrations.

3. Black-Scholes-Merton Equation

Assume that the price of stock follows a geometric Brownian motion with the constant drift and volatility

$$dS(t) = \alpha S(t)d\tau + \sigma dW(t). \tag{3.1}$$

The first graph (from the top) in Figure (2) shows five random sample vectors from the normal standard distribution with size 128, the second one shows the five sample paths regarding to the given random sample vectors and the last graph shows five sample paths of the stock price, equation (3.1), with $S(0) = 3$, $\alpha = 0.1$ and $\sigma = 0.05$.

Consider the European option which its payoff is $f(S)$ at time T. Black, Scholes, and Merton argued that the value of this option at any time should depend on the five input variables: the strike price of an option, the current stock price, the time to expiration, the risk-free rate, and the volatility. Only two of these quantities, time and stock price are variable. Let $V(t, S)$ denote the value of the option at time t, if the stock price at that time is $S(t) = S$. The

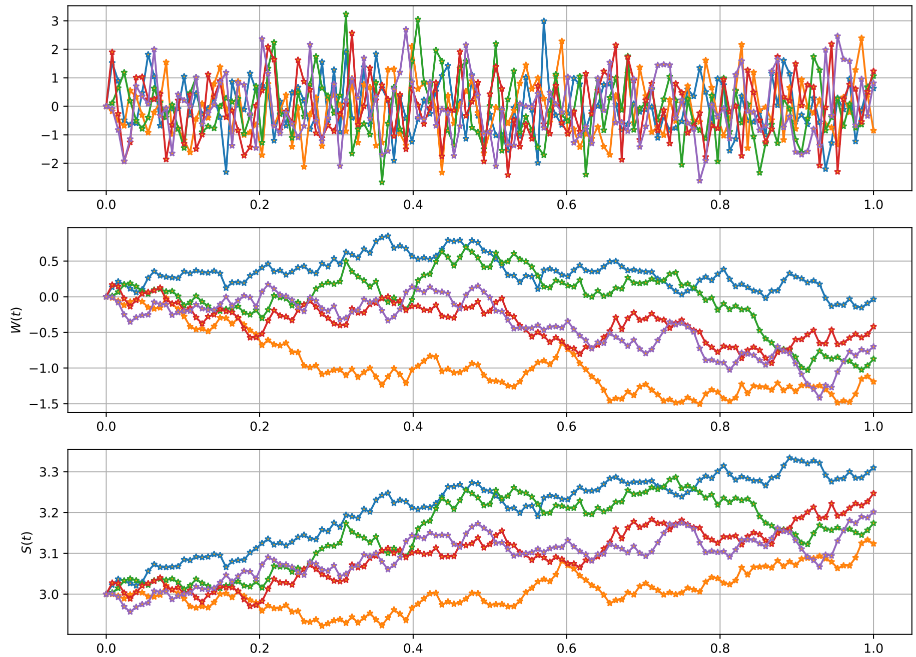

FIGURE 2. The top figure shows five random sample vectors from the normal standard distribution with size 128, the second one shows the five sample paths regarding to the given random sample vectors and the last figure shows five sample paths of the stock price with $S(0) = 3$, $\alpha = 0.1$ and $\sigma = 0.05$.

following equation can be deduced using Itö lemma and the Doeblin formula, as well as the application of the delta hedging rule[21]

$$V_t(t,S) + rSV_S(t,S) + \frac{1}{2}\sigma^2 S^2 V_{SS}(t,S) - rV(t,S) = 0,\ t \in [0,T],\ S \geq 0, \tag{3.2}$$

with the following terminal condition

$$V(T,S) = f(S), \tag{3.3}$$
$$V(t,0) = g(t), t \in [0,T], \tag{3.4}$$
$$\lim_{S\to\infty} V(t,S) = h(t), \tag{3.5}$$

which is a partial differential equation of the type called backward parabolic.

3.1. **Change of variables.** For the numerical solution, the infinite interval $0 \leq S$ must be replaced by a finite interval $0 \leq S \leq S_{\max}$. The end value $b = S_{\max} > 0$ must be chosen such that for the interval $0 \leq S \leq S_{\max}$ a sufficient quality of approximation is obtained.

Now, consider the equation (3.2) with the following change of variables

$$x = \frac{S}{b},\ \tau = T - t. \tag{3.6}$$

By applying the above change of variables on the equation (3.2), the following equation will obtained

$$-V_\tau(\tau,x)+rxV_x(\tau,x)+\frac{1}{2}\sigma^2x^2V_{xx}(\tau,x)-rV(\tau,x)=0,\ \tau\in[0,T],\ 0\le x\le 1. \tag{3.7}$$

The main reason of changing the spatial domain into the interval $[0,1]$ is because of the property of Haar wavelets which are defined on this interval.

4. Implementation of the method

Applying θ-weighted $(0\le\theta\le 1)$ scheme to spatial part and forward difference to temporal part of equation (3.7) yields

$$V^{k+1}(x)-\theta d\tau\left[\frac{\sigma^2}{2}x^2V^{k+1}_{xx}(x)+rxV^{k+1}_x(x)-rV^{k+1}(x)\right]=V^k(x)+(1-\theta)d\tau\left[\frac{\sigma^2}{2}x^2V^k_{xx}(x)+rxV^k_x(x)-rV^k(x)\right], \tag{4.1}$$

where $V^k(x)=V(x,\tau_k)$, $\tau_{k+1}=\tau_k+d\tau$ and $d\tau$ is time step. Now approximate mixed order derivative by Haar wavelets as follows:

$$V^{k+1}_{xx}(x)=\sum_{i=1}^{2M}\alpha_ih_i(x), \tag{4.2}$$

where α_j are wavelets coefficients to be determined and $h_i(x)$ are wavelets defined in equation (2.3). Integrating equation (4.2) from 0 to x, we obtain

$$V^{k+1}_x(x)=\sum_{i=1}^{2M}\alpha_ip_{i,1}(x)+V^{k+1}_x(0). \tag{4.3}$$

The unknown term $V^{k+1}_x(0)$ in equation (4.3) can be computed by integration of equation (4.3) w.r.t x from 0 to 1. By doing so we get following,

$$V^{k+1}_x(0)=V^{k+1}(1)-V^{k+1}(0)-\sum_{i=1}^{2M}\alpha_ip_{i,2}(1), \tag{4.4}$$

substituting equation (4.4) in equation (4.3) we have

$$V^{k+1}_x(x)=\sum_{i=1}^{2M}\alpha_i(p_{i,1}(x)-p_{i,2}(1))+V^{k+1}(1)-V^{k+1}(0). \tag{4.5}$$

Substituting $V^{k+1}_x(0)$ from equation (4.4) in equation (4.3) and integrating from 0 to x leads to

$$V^{k+1}(x)=\sum_{i=1}^{2M}\alpha_i\left(p_{i,2}(x)-xp_{i,2}(1)\right)+x\left(V^{k+1}(1)-V^{k+1}(0)\right)+V^{k+1}(0). \tag{4.6}$$

For the simplifying, we use the following notations

$$\begin{aligned} V^k_{xx}(\vec{\boldsymbol{x}}) &= \boldsymbol{\alpha}^k\boldsymbol{H}(\vec{\boldsymbol{x}}),\\ V^k_x(\vec{\boldsymbol{x}}) &= \boldsymbol{\alpha}^k\boldsymbol{P}(\vec{\boldsymbol{x}})+\boldsymbol{c}^k,\\ V^k(\vec{\boldsymbol{x}}) &= \boldsymbol{\alpha}^k\boldsymbol{Q}(\vec{\boldsymbol{x}})+\vec{\boldsymbol{x}}\,\boldsymbol{c}^k+\boldsymbol{d}^k, \end{aligned} \tag{4.7}$$

where

$$
\begin{aligned}
\overrightarrow{\boldsymbol{x}} &= (x_1, x_2, \cdots, x_{2M}), \\
\boldsymbol{\alpha}^k &= [\alpha_1 \ \alpha_2 \ \cdots \ \alpha_{2M}], \\
\boldsymbol{H} &= \left[h_1(\overrightarrow{\boldsymbol{x}}) \ h_2(\overrightarrow{\boldsymbol{x}}) \ \cdots \ h_{2M}(\overrightarrow{\boldsymbol{x}})\right]^T, \\
\boldsymbol{P} &= \left[p_{1,1}(\overrightarrow{\boldsymbol{x}}) - p_{1,2}(1) \ p_{2,1}(\overrightarrow{\boldsymbol{x}}) - p_{2,2}(1) \ \cdots \ p_{2M,1}(\overrightarrow{\boldsymbol{x}}) - p_{2M,2}(1)\right]^T, \\
\boldsymbol{Q} &= \left[p_{1,2}(\overrightarrow{\boldsymbol{x}}) - \overrightarrow{\boldsymbol{x}} p_{1,2}(1) \ p_{2,2}(\overrightarrow{\boldsymbol{x}}) - \overrightarrow{\boldsymbol{x}} p_{2,2}(1) \ \cdots \ p_{2M,2}(\overrightarrow{\boldsymbol{x}}) - \overrightarrow{\boldsymbol{x}} p_{2M,2}(1)\right]^T, \\
\boldsymbol{c}^k &= V^k(1) - V^k(0), \\
\boldsymbol{d}^k &= V^k(0).
\end{aligned}
\tag{4.8}
$$

Substituting the right hands of equations (4.2), (4.5) and (4.6) into equation (4.1), using the collocation points $x_j = \frac{j+0.5}{2M}$ and the notations introduced in (4.7), we will obtain the following system of algebraic equations

$$
\boldsymbol{\alpha}^{k+1} \boldsymbol{G} = \boldsymbol{\alpha}^k \boldsymbol{N} + \boldsymbol{R_1} - \boldsymbol{R_2}, \tag{4.9}
$$

where

$$
\begin{aligned}
\boldsymbol{G} &= (1 + \theta r d\tau) \boldsymbol{Q} - \theta r d\tau \overrightarrow{\boldsymbol{x}} \boldsymbol{P} - \frac{1}{2} \theta \sigma^2 \overrightarrow{\boldsymbol{x}}^2 \boldsymbol{H}, \\
\boldsymbol{N} &= [1 - (1-\theta) r d\tau] \, \boldsymbol{Q} + (1-\theta) r d\tau \overrightarrow{\boldsymbol{x}} \boldsymbol{P} + \frac{1}{2}(1-\theta)\sigma^2 \overrightarrow{\boldsymbol{x}}^2 \boldsymbol{H}, \\
\boldsymbol{R_1} &= \boldsymbol{c}^{k+1} \overrightarrow{\boldsymbol{x}} + r\theta d\tau \boldsymbol{d}^{k+1} + \boldsymbol{d}^{k+1}, \\
\boldsymbol{R_2} &= \boldsymbol{c}^k \overrightarrow{\boldsymbol{x}} - (1-\theta) r d\tau \boldsymbol{d}^k + \boldsymbol{d}^k,
\end{aligned}
\tag{4.10}
$$

and $\boldsymbol{\alpha}^1$ could be obtained easily from the below equation

$$
V(0, \overrightarrow{\boldsymbol{x}}) = \boldsymbol{\alpha}^1 \boldsymbol{Q} + \overrightarrow{\boldsymbol{x}}(V(0,1) - V(0,0)) + V(0,0). \tag{4.11}
$$

5. Numerical results

The proposed method is used to solve a set of test problems. We looked at two test examples: The European call option and the European put option. In each case, the generated results are compared to those found in the exact solutions. To determine the correctness of the scheme L_2, L_∞, (RMS), and relative error E_r were calculated. The following is a list of error norms:

$$
\begin{aligned}
L_2 &= \sqrt{\left(\sum_{i=1}^{L} \sum_{j=1}^{M} \left(V_{i,j}^{ext} - V_{i,j}^{app}\right)^2 \right)}, \\
L_\infty &= \max_{\substack{1 \le i \le L \\ 1 \le j \le M}} \| V_{i,j}^{ext} - V_{i,j}^{app} \|, \\
RMS &= \sqrt{\frac{\sum_{i=1}^{L} \sum_{j=1}^{M} \left(V_{i,j}^{ext} - V_{i,j}^{app}\right)^2}{LM}},
\end{aligned}
\tag{5.1}
$$

$$
E_r = \sqrt{\frac{\sum_{i=1}^{L} \sum_{j=1}^{M} \left(V_{i,j}^{ext} - V_{i,j}^{app}\right)^2}{\sum_{i=1}^{L} \sum_{j=1}^{M} \left(V_{i,j}^{ext}\right)^2}}. \tag{5.2}
$$

We offer the following examples and solve them using the strategy described in the previous section to demonstrate the efficacy of the presented method. The Python language programming was used to create and implement computer programs.

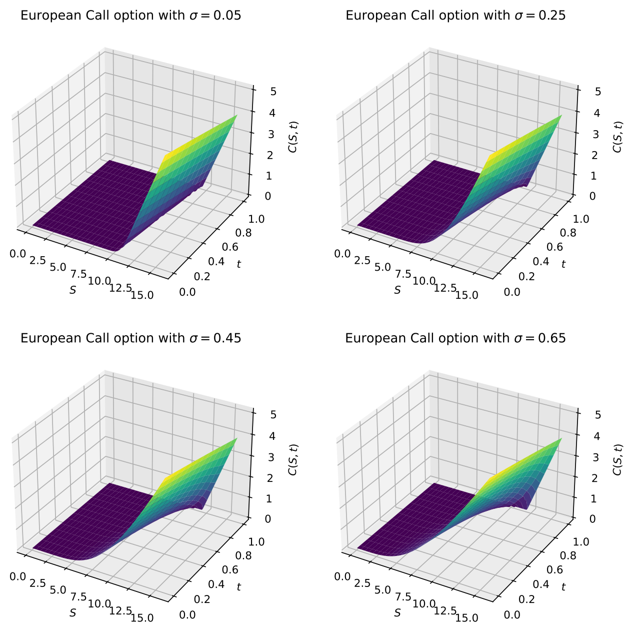

FIGURE 3. European call option prices with different values of volatility where $T = 1$, $r = 0.1$, $J = 5$.

5.1. **Call option.** Call options are financial contracts that allow the option buyer the right, but not the duty, to purchase a stock, bond, commodity, or other asset or instrument at a given price within a specified time period. The stock, bond, or commodity is called the underlying asset. When the price of the underlying asset rises, the call buyer profits. The following are the boundary conditions for a call option:

$$C(T, S) = (S - K)^+, \tag{5.3}$$
$$C(t, 0) = 0, \tag{5.4}$$
$$V(t, S_{\max}) = S_{\max} - Ke^{-r(T-t)}. \tag{5.5}$$

5.2. **Put option.** In finance, a put or put option is a financial market derivative instrument that gives the holder (i.e. the purchaser of the put option) the right to sell an asset (the underlying), at a specified price (the strike), by (or at) a specified date (the expiry or maturity) to the writer (i.e. seller) of the put. The purchase of a put option is interpreted as a negative sentiment about the future value of the underlying stock [11]. The term "put" comes from the fact that the owner has the right to "put up for sale" the stock or index. Put options are most commonly used in the stock market to protect against a fall in the price of a stock below a specified price. If the price of the stock declines below the strike price, the holder of the put has the right, but not the obligation, to sell the asset at the strike price, while the seller of the put has the obligation to purchase the asset at the strike price if the owner uses the right to do so (the holder is said to exercise the option). In this way, the buyer of the put will receive at least the strike

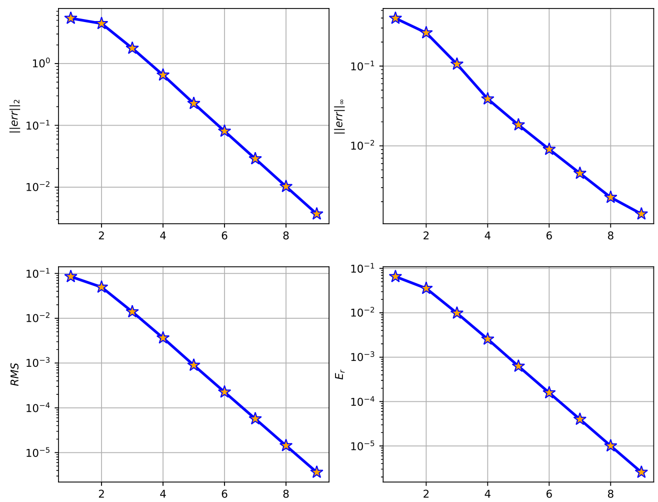

FIGURE 4. Errors plots of European call option prices with different values of J.

price specified, even if the asset is currently worthless. The put option's boundary requirements are as follows:

$$P(T,S) = (K-S)^+, \tag{5.6}$$

$$P(t,0) = Ke^{-r(T-t)}, \tag{5.7}$$

$$P(t,S_{\max}) = 0. \tag{5.8}$$

Figure (5) shows the solution of $P(t,S)$ in 3D forms. The values are obtained for $d\tau = 0.05$, $\theta = \frac{1}{2}$, $T = 1$ and $J = 5$.

6. Conclusion

An efficient algorithm based on Haar wavelets coupled with finite differences was proposed to determine the value of European options. For this purpose, first, we considered the Black-Scholes/Merton equation and truncated the spatial domain from $[0,\infty)$ to $[0,S_{\max}]$ then, with some linear changes of variables, the problem converted into a time forward problem which its spatial domain is $[0,1]$. Haar wavelets and finite differences methods with collocation points were used to change the problem into a system of linear equations. In order to demonstrate the algorithm's efficiency and precision, the algorithm was used to determine the price of European call and European put options. To show the convergence of the approach, four distinct kinds of error norms with some levels of multiscale approximation were generated and compared.

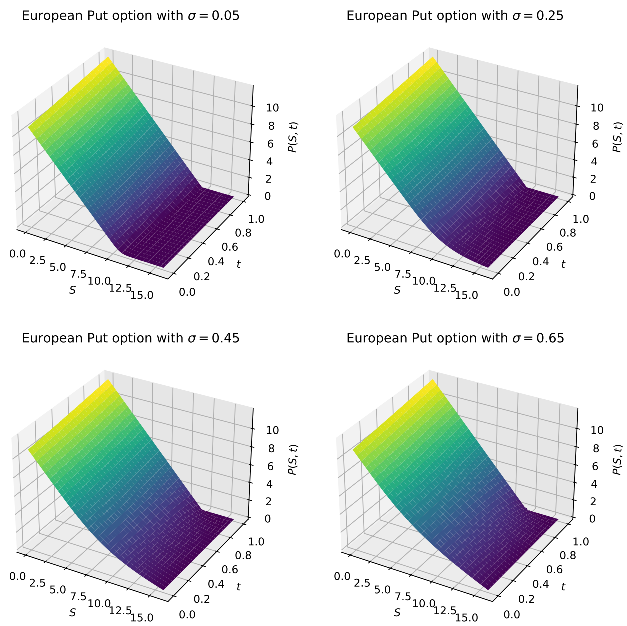

FIGURE 5. European put option prices with different values of volatilites where $T = 1$, $r = 0.1$, $J = 5$.

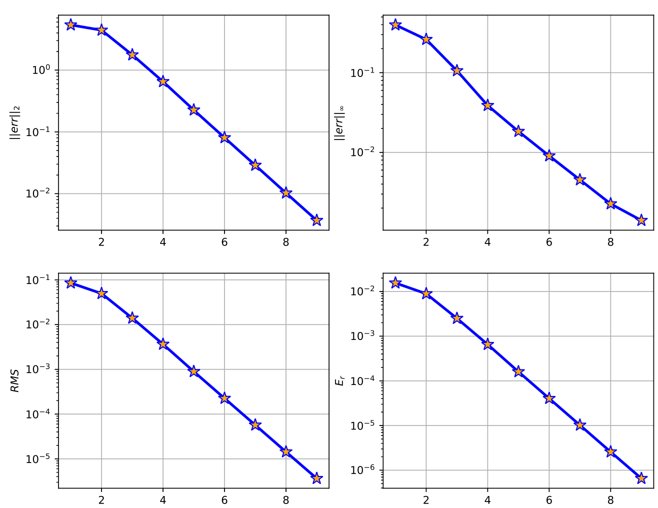

FIGURE 6. Errors plots of European put option prices with different values of J.

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