Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 11, No. 2, 2023, pp. 332-344
DOI:10.22034/cmde.2022.51628.2153

To study existence of unique solution and numerically solving for a kind of three-point boundary fractional high-order problem subject to Robin condition

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Abstract

In this paper, we prove the existence and uniqueness of the solutions for a non-integer high order boundary value problem which is subject to the Caputo fractional derivative. The boundary condition is a non-local type. Analytically, we introduce the fractional Green's function. The main principle applied to simulate our results is the Banach contraction fixed point theorem. We deduce this paper by presenting some illustrative examples. Furthermore, it is presented a numerical based semi-analytical technique to approximate the unique solution to the desired order of precision.

Keywords. High order fractional differential equation, Caputo fractional derivative, Boundary value problem, Existence and uniqueness, Fixed point theorem.

2010 Mathematics Subject Classification. 34B10; 34B15; 34B27; 34B99.

1. Preliminaries

Fractional calculus while standing for differential and integral equations, due to non-local property in behind, have various applications in applied branches such as blood flow problems, anomalous diffusion, disease spread, control processing, population dynamics, etc. for instance see [10, 18, 22–24, 26, 29, 34, 36].

The theory of fixed point to study the solution of the boundary value problems is a consequential tool and it is played important role in both proving the existence of the solution and obtain it approximately, see [1–4, 7–9, 11–13, 15–17, 20, 27, 30, 32]. It can be seen a lot of interest in researching the subject of nonlocal nonlinear fractional order in boundary value problems in recent years.

In the article [35], the following result was established by considering the problem:

$$\begin{cases} y''' + \Lambda(t, y(t)) = 0, \quad t \in (0, 1), \\ y(0) = y'(0) = 0, \quad y(1) = \gamma y(\eta), \end{cases} \tag{1.1}$$

with $0 < \eta < 1$, $\gamma \in \mathbb{R}$, $\Lambda \in C([0, 1] \times \mathbb{R}, \mathbb{R})$ and $\Lambda(t, 0) \neq 0$.

Theorem 1.1. *Assume that $\Lambda : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous mapping and follows a uniform Lipschitz inequality with respect to y on $[0, 1] \times \mathbb{R}$, such that there is a constant L, where for every $(t, y), (t, z) \in [0, 1] \times \mathbb{R}$,*

$$|\Lambda(t, y) - \Lambda(t, z)| \leq L|y - z|.$$

If $\gamma\eta^2 \neq 1$ such that

$$1 + \left|\frac{\gamma}{1 - \gamma\eta^2}\right| < \frac{3}{L}, \tag{1.2}$$

then there exists unique solution for Eq. (1.1).

Received: 16 May 2022 ; Accepted: 20 September 2022.
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In reference [6], the author studied the fractional problem

$$\begin{cases} {}^{C}D_t^{\alpha}u + f(t, u(t), u'(t), u''(t), u'''(t)) = 0, \quad t \in (0,1), \\ u(0) = u''(0) = u'''(0) = 0, \quad \lambda u'(0) + (1-\lambda)u'(1) = \int_0^1 u(s)ds, \end{cases} \tag{1.3}$$

where $3 < \alpha \leq 4$, $0 \leq \lambda \leq 1$, ${}^{C}D_t^{\alpha}u$ is the Caputo fractional derivative and $f : (0,1) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. In the mentioned work, first, the existence and uniqueness of the solution have been proved and then a numerical technique is given to reach the solution.

In our work, we intend to develop the above result by considering a high order fractional derivative of Caputo sense (we consult the reader to see [21] for the definitions and basic consequences on non-integer order calculus) in the place of the classical operator y''', i.e., we prove the existence of unique solutions for the following non-integer high-order differential boundary value problem

$$\begin{cases} {}_{c}D_t^{\vartheta}w + g(t, w(t), w'(t)) = 0, \quad t \in (0,1), \\ w(0) = w'(0) = ... = w^{(m-2)}(0) = 0, \quad w(1) = \gamma_1 w(\eta) + \gamma_2 w'(\eta), \end{cases} \tag{1.4}$$

where $m-1 < \vartheta \leq m$, $m \geq 2$, $0 < \eta < 1$, $\gamma_1, \gamma_2 \in \mathbb{R}$, ${}_{c}D_t^{\vartheta}w$ is the Caputo fractional derivative and $g \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $g(t,0,0) \neq 0$.

The topic of multi-point non-local boundary value equations has been come up by different researchers, follow [5, 14, 25]. The multi-point boundary constraints arise in definite problems of physics, fluid mechanics, and wave propagation, refer [31, 37] for the interest. The multi-point boundary constraints may be found out such that the controllers at the end points spread or add energy with related sensors placed at middle-level positions. The third order differential equations in which differentiation of acceleration is involved are called jerk (time derivative of acceleration) equations and important for engineers and physicists, and they try to plan the vehicles in a way that jerk may be minimum. The third order differential equations are known to be the generalization of fractional differential equations with non-integer order. The fractional order goes through three, the considered equations possibly corresponds to jerk equations.

The existence and uniqueness of solutions issue for non-linear multi-point boundary value equations have been searched by number of researchers. For example, Rehman and Khan [33], Mehmood and Ahmad, [28] Haq et al. [19] and references therein. In [28], the authors have proved the existence of a solution for non-integer order boundary value equations with non-local multi- point boundary constraints by employing Shaefer type and Krasnoselskiis fixed point theorems. The remaining article is arranged as follows. In Section 2, we estimate fractional Green's function by implementing integral equation formula and a few additional assumptions. Section 3 is assigned to the establishment of the Green's function. In section 4, we justify with the proof of our main theorem on the existence of a unique solution of the considered problem. Also, we have given an example to utilize the adopted results. The conclusion is set out in section 5.

2. Construction of the Green's function

At the first instance, let us establish Green's function for the following two-point boundary value constraint

$$\begin{cases} {}_{c}D_t^{\vartheta}u(t) + h(t) = 0, \quad t \in (0,1), \\ u(0) = u'(0) = ... = u^{(m-2)}(0) = 0, \quad u(1) = 0, \end{cases} \tag{2.1}$$

and afterwards, supposing that the solution of the following three-point boundary value problem

$$\begin{cases} {}_{c}D_t^{\vartheta}w(t) + h(t) = 0, \quad t \in (0,1), \\ w(0) = w'(0) = ... = w^{(m-2)}(0) = 0, \quad w(1) = \gamma_1 w(\eta) + \gamma_2 w'(\eta), \end{cases} \tag{2.2}$$

can be stated as follows

$$w(t) = u(t) + u(\eta) \sum_{j=0}^{m-1} \lambda_j t^j, \quad t \in (0,1), \tag{2.3}$$

where λ_js are constants that will be specified, we will estimate Greens's function for the equations (2.2).

Lemma 2.1. *If $h : [0,1] \to \mathbb{R}$ is continuous mapping, then boundary value problem (2.1) admits unique solution*

$$u(t) = \int_0^1 R(t,s)h(s)ds, \tag{2.4}$$

where

$$R(t,s) = \begin{cases} \frac{t^{m-1}(1-s)^{\vartheta-1}-(t-s)^{\vartheta-1}}{\Gamma(\vartheta)}, & 0 \leq s \leq t \leq 1; \\ \frac{t^{m-1}(1-s)^{\vartheta-1}}{\Gamma(\vartheta)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.5}$$

Proof. It is well known that the problem (2.1) is similar to solving the integral equation

$$u(t) = \sum_{j=0}^{m-1} c_j t^j - \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} h(s)ds, \tag{2.6}$$

where c_js are some real constants. Using boundary conditions (2.1), we can obtain

$$c_0 = c_1 = ... = c_{m-2} = 0, c_{m-1} = \frac{1}{\Gamma(\vartheta)} \int_0^1 (1-s)^{\vartheta-1} h(s)ds. \tag{2.7}$$

Thus, we get

$$\begin{aligned} u(t) &= \frac{t^{m-1}}{\Gamma(\vartheta)} \int_0^1 (1-s)^{\vartheta-1} h(s)ds - \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} h(s)ds \\ &= \frac{t^{m-1}}{\Gamma(\vartheta)} \int_0^t (1-s)^{\vartheta-1} h(s)ds + \frac{t^{m-1}}{\Gamma(\vartheta)} \int_t^1 (1-s)^{\vartheta-1} h(s)ds - \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} h(s)ds \\ &= \int_0^t \frac{t^{m-1}(1-s)^{\vartheta-1} - (t-s)^{\vartheta-1}}{\Gamma(\vartheta)} h(s)ds + \int_t^1 \frac{t^{m-1}(1-s)^{\vartheta-1}}{\Gamma(\vartheta)} h(s)ds \\ &= \int_0^1 R(t,s)h(s)ds. \end{aligned} \tag{2.8}$$

The uniqueness follows from the assumption, that the completely homogeneous boundary value problem has only the trivial solution. Hence Lemma 2.1 has proved. □

Theorem 2.2. *Let $h : [0,1] \to \mathbb{R}$ is a continuous mapping, if $\gamma_1 \eta^{m-1} + (m-1)\gamma_2 \eta^{m-2} \neq 1$ then boundary value problem (2.2) admits unique solution*

$$w(t) = \int_0^1 G(t,s)h(s)ds, \tag{2.9}$$

where

$$G(t,s) = R(t,s) + \frac{t^{m-1}}{1 - \gamma_1 \eta^{m-1} - (m-1)\gamma_2 \eta^{m-2}} \left[\gamma_1 R(\eta, s) + \gamma_2 \frac{\partial R(\eta, s)}{\partial t} \right]. \tag{2.10}$$

Proof. Let

$$w(t) = u(t) + u(\eta) \sum_{j=0}^{m-1} \lambda_j t^j, \tag{2.11}$$

where λ_js are constants that will be identified using boundary conditions (2.2) and

$$u(t) = \int_0^1 R(t,s)h(s)ds, \quad u'(t) = \int_0^1 \frac{\partial R(t,s)}{\partial t} h(s)ds$$

So,

$$\left. \frac{d^k w}{dt^k} \right|_{t=0} = 0, \quad k = 0, 1, ..., m-2. \tag{2.12}$$

Then, we get $\lambda_0 = \lambda_1 = ... = \lambda_{m-2} = 0$, because $\left.\frac{d^k u}{dt^k}\right|_{t=0} = 0$, for $k = 0, 1, ..., m-2$. To obtain λ_{m-1}, use $w(1) = \gamma_1 w(\eta) + \gamma_2 w'(\eta)$, therefore we set

$$u(1) + u(\eta)\lambda_{m-1} = \gamma_1 \left(u(\eta) + u(\eta)\lambda_{m-1}\eta^{m-1}\right) + \gamma_2 \left(u'(\eta) + (m-1)u(\eta)\lambda_{m-1}\eta^{m-2}\right),$$

hence, from $u(1) = 0$,

$$u(\eta)\lambda_{m-1} = \frac{\gamma_1 u(\eta) + \gamma_2 u'(\eta)}{1 - \gamma_1\eta^{m-1} - (m-1)\gamma_2\eta^{m-2}}. \tag{2.13}$$

Therefore, from (2.11),

$$w(t) = u(t) + \frac{\left[\gamma_1 u(\eta) + \gamma_2 u'(\eta)\right] t^{m-1}}{1 - \gamma_1\eta^{m-1} - (m-1)\gamma_2\eta^{m-2}}, \tag{2.14}$$

and then

$$\begin{aligned} w(t) &= \int_0^1 R(t,s)h(s)ds + \frac{\gamma_1 t^{m-1}}{1 - \gamma_1\eta^{m-1} - (m-1)\gamma_2\eta^{m-2}} \int_0^1 R(\eta,s)h(s)ds \\ &\quad + \frac{\gamma_2 t^{m-1}}{1 - \gamma_1\eta^{m-1} - (m-1)\gamma_2\eta^{m-2}} \int_0^1 \frac{\partial R(\eta,s)}{\partial t} h(s)ds \\ &= \int_0^1 \left[R(t,s) + \frac{\gamma_1 t^{m-1}}{1 - \gamma_1\eta^{m-1} - (m-1)\gamma_2\eta^{m-2}} R(\eta,s)\right. \\ &\quad \left. + \frac{\gamma_2 t^{m-1}}{1 - \gamma_1\eta^{m-1} - (m-1)\gamma_2\eta^{m-2}} \frac{\partial R(\eta,s)}{\partial t}\right] h(s)ds \\ &= \int_0^1 G(t,s)h(s)ds. \end{aligned} \tag{2.15}$$

Let us derive the proof of uniqueness. Let $z(t)$ is also a solution of Eq. (2.2), that is

$$\begin{cases} {}_cD_t^{\vartheta} z(t) + h(t) = 0, \quad t \in (0,1), \\ z(0) = z'(0) = ... = z^{(m-2)}(0) = 0, \quad z(1) = \gamma_1 z(\eta) + \gamma_2 z'(\eta), \end{cases} \tag{2.16}$$

Let $\Omega(t) = w(t) - z(t)$, Due to linearity property of Caputo's non-integer order derivative, we have

$${}_cD_t^{\vartheta}\Omega(t) = {}_c D_t^{\vartheta} w(t) - {}_c D_t^{\vartheta} z(t) = -h(t) + h(t) = 0.$$

Therefore, $\Omega(t) = \sum_{j=0}^{m-1} c_j t^j$ where c_js are some real constants that we will specify. If we take derivative from $\Omega(t) = w(t) - z(t)$, $(m-2)$ times

$$\left.\frac{d^k\Omega}{dt^k}\right|_{t=0} = \left.\frac{d^k w}{dt^k}\right|_{t=0} - \left.\frac{d^k z}{dt^k}\right|_{t=0} = 0 - 0 = 0, \quad k = 0, 1, ..., m-2, \tag{2.17}$$

then they yield $c_0 = c_1 = ... = c_{m-2} = 0$, and hence

$$\Omega(t) = c_{m-1}t^{m-1}.$$

By definition and the last boundary condition, we have

$$\begin{aligned} c_{m-1} &= \Omega(1) \\ &= w(1) - z(1) \\ &= \gamma_1 w(\eta) + \gamma_2 w'(\eta) - \gamma_1 z(\eta) - \gamma_2 z'(\eta) \\ &= \gamma_1\Omega(\eta) + \gamma_2\Omega'(\eta) \\ &= \gamma_1 c_{m-1}\eta^{m-1} + (m-1)\gamma_2 c_{m-1}\eta^{m-2} \end{aligned}$$

which implies

$$c_{m-1}\left(1 - \gamma_1\eta^{m-1} - (m-1)\gamma_2\eta^{m-2}\right) = 0,$$

since $\gamma_1\eta^{m-1}+(m-1)\gamma_2\eta^{m-2}\neq 1$ from assumption then $c_{m-1}=0$ and the proof is complete. □

3. The estimation of Green's function

Lemma 3.1. *Let $R(t,s)$ be the Green's function given in Lemma 2.1. Then*

$$\int_0^1 |R(t,s)|ds \leq \frac{2}{\Gamma(\vartheta+1)}, \tag{3.1}$$

and

$$\int_0^1 \left|\frac{\partial R(t,s)}{\partial t}\right| ds \leq \frac{m+\vartheta-1}{\Gamma(\vartheta+1)}, \tag{3.2}$$

for $t\in[0,1]$.

Proof. It is clearly seen that

$$\begin{aligned}
\int_0^1 |R(t,s)|ds &= \int_0^t |R(t,s)|ds + \int_t^1 |R(t,s)|ds \\
&= \int_0^t \left|\frac{t^{m-1}(1-s)^{\vartheta-1}-(t-s)^{\vartheta-1}}{\Gamma(\vartheta)}\right|ds + \int_t^1 \left|\frac{t^{m-1}(1-s)^{\vartheta-1}}{\Gamma(\vartheta)}\right|ds \\
&\leq \int_0^t \frac{t^{m-1}(1-s)^{\vartheta-1}+(t-s)^{\vartheta-1}}{\Gamma(\vartheta)}ds + \int_t^1 \frac{t^{m-1}(1-s)^{\vartheta-1}}{\Gamma(\vartheta)}ds \\
&= \frac{t^{m-1}\left(1-(1-t)^{\vartheta}\right)+t^{\vartheta}}{\Gamma(\vartheta+1)} + \frac{t^{m-1}(1-t)^{\vartheta}}{\Gamma(\vartheta+1)} \\
&= \frac{t^{m-1}+t^{\vartheta}}{\Gamma(\vartheta+1)} \\
&\leq \frac{2}{\Gamma(\vartheta+1)}.
\end{aligned} \tag{3.3}$$

Moreover, after taking derivative from $R(t,s)$ with respect to t, we receive

$$\begin{aligned}
\int_0^1 \left|\frac{\partial R(t,s)}{\partial t}\right| ds &= \int_0^t \left|\frac{\partial R(t,s)}{\partial t}\right| ds + \int_t^1 \left|\frac{\partial R(t,s)}{\partial t}\right| ds \\
&= \int_0^t \left|\frac{(m-1)t^{m-2}(1-s)^{\vartheta-1}-(\vartheta-1)(t-s)^{\vartheta-2}}{\Gamma(\vartheta)}\right|ds \\
&\quad + \int_t^1 \left|\frac{(m-1)t^{m-2}(1-s)^{\vartheta-1}}{\Gamma(\vartheta)}\right|ds \\
&\leq \int_0^t \frac{(m-1)t^{m-2}(1-s)^{\vartheta-1}+(\vartheta-1)(t-s)^{\vartheta-2}}{\Gamma(\vartheta)}ds \\
&\quad + \int_t^1 \frac{(m-1)t^{m-2}(1-s)^{\vartheta-1}}{\Gamma(\vartheta)}ds \\
&= \frac{(m-1)t^{m-2}\left(1-(1-t)^{\vartheta}\right)}{\Gamma(\vartheta+1)} + \frac{t^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{(m-1)t^{m-2}(1-t)^{\vartheta}}{\Gamma(\vartheta+1)} \\
&= \frac{(m-1)t^{m-2}}{\Gamma(\vartheta+1)} + \frac{t^{\vartheta-1}}{\Gamma(\vartheta)} \\
&\leq \frac{m-1}{\Gamma(\vartheta+1)} + \frac{1}{\Gamma(\vartheta)} = \frac{m+\vartheta-1}{\Gamma(\vartheta+1)}.
\end{aligned} \tag{3.4}$$

□

Theorem 3.2. *Let $G(t,s)$ be the Green's function given in Theorem 2.2. Then it satisfies*

$$\int_0^1 |G(t,s)|ds \leq \frac{1}{\Gamma(\vartheta+1)}\left(2+\frac{2|\gamma_1|+(m+\vartheta-1)|\gamma_2|}{|1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2}|}\right), \tag{3.5}$$

for $t\in[0,1]$.

Proof. It is easily observed, by the help of Lemma 3.1, that

$$\begin{aligned}\int_0^1 |G(t,s)|ds &= \int_0^1 \left|R(t,s)+\frac{t^{m-1}}{1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2}}\left[\gamma_1 R(\eta,s)+\gamma_2\frac{\partial R(\eta,s)}{\partial t}\right]\right|ds\\
&\leq \int_0^1 |R(t,s)|ds+\left|\frac{\gamma_1 t^{m-1}}{1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2}}\right|\int_0^1 |R(\eta,s)|ds\\
&+\left|\frac{\gamma_2 t^{m-1}}{1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2}}\right|\int_0^1 |\frac{\partial R(\eta,s)}{\partial t}|ds\\
&\leq \frac{2}{\Gamma(\vartheta+1)}+\left|\frac{\gamma_1}{1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2}}\right|\frac{2}{\Gamma(\vartheta+1)}\\
&+\left|\frac{\gamma_2}{1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2}}\right|\frac{m+\vartheta-1}{\Gamma(\vartheta+1)}\\
&\leq \frac{1}{\Gamma(\vartheta+1)}\left(2+\frac{2|\gamma_1|+(m+\vartheta-1)|\gamma_2|}{|1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2}|}\right),\end{aligned} \tag{3.6}$$

and the proof is established. □

Theorem 3.3. *Let $G(t,s)$ be the Green's function given in Theorem 2.2. Then it satisfies*

$$\int_0^1 \left|\frac{\partial G(t,s)}{\partial t}\right|ds \leq \frac{m-1}{\Gamma(\vartheta+1)}\left(1+\frac{\vartheta}{m-1}+\frac{(2|\gamma_1|+(m+\vartheta-1)|\gamma_2|)}{|1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2}|}\right), \tag{3.7}$$

for $t\in[0,1]$.

Proof. Eq. (2.10) implies that

$$\frac{\partial G(t,s)}{\partial t}=\frac{\partial R(t,s)}{\partial t}+\frac{(m-1)t^{m-2}}{1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2}}\left[\gamma_1 R(\eta,s)+\gamma_2\frac{\partial R(\eta,s)}{\partial t}\right]. \tag{3.8}$$

Now, as the same as the proof of the previous theorem, it is easily observed, by the help of Lemma 3.1, that

$$\int_0^1 \left|\frac{\partial G(t,s)}{\partial t}\right|ds = \int_0^1 \left|\frac{\partial R(t,s)}{\partial t}+\frac{(m-1)t^{m-2}}{1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2}}\left[\gamma_1 R(\eta,s)+\gamma_2\frac{\partial R(\eta,s)}{\partial t}\right]\right|ds \tag{3.9}$$

$$\begin{aligned}&\leq \int_0^1 |\frac{\partial R(t,s)}{\partial t}|ds+\left|\frac{\gamma_1 (m-1)t^{m-2}}{1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2}}\right|\int_0^1 |R(\eta,s)|ds\\
&+\left|\frac{\gamma_2 (m-1)t^{m-2}}{1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2}}\right|\int_0^1 |\frac{\partial R(\eta,s)}{\partial t}|ds\\
&\leq \frac{m+\vartheta-1}{\Gamma(\vartheta+1)}+\left|\frac{\gamma_1(m-1)}{1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2}}\right|\frac{2}{\Gamma(\vartheta+1)}\\
&+\left|\frac{\gamma_2(m-1)}{1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2}}\right|\frac{m+\vartheta-1}{\Gamma(\vartheta+1)}\\
&\leq \frac{m-1}{\Gamma(\vartheta+1)}\left(1+\frac{\vartheta}{m-1}+\frac{(2|\gamma_1|+(m+\vartheta-1)|\gamma_2|)}{|1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2}|}\right),\end{aligned} \tag{3.10}$$

and the proof is established. $\square$

4. Existence and uniqueness of the solution

Theorem 4.1. *Assume that $g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and follows a uniform Lipschitz inequality with respect to (w, w') on $[0,1] \times \mathbb{R} \times \mathbb{R}$, such that there is a constant L, for every $(t, \omega_1, \omega_2), (t, \varpi_1, \varpi_2) \in [0,1] \times \mathbb{R} \times \mathbb{R}$,*

$$\left|g(t, \omega_1, \omega_2) - g(t, \varpi_1, \varpi_2)\right| \leq L\left(\left|\omega_1 - \varpi_1\right| + \left|\omega_2 - \varpi_2\right|\right). \tag{4.1}$$

If $\gamma_1\eta^{m-1} + (m-1)\gamma_2\eta^{m-2} \neq 1$ and

$$1 + m + \vartheta + \frac{2m|\gamma_1| + m(m+\vartheta-1)|\gamma_2|}{|1 - \gamma_1\eta^{m-1} - (m-1)\gamma_2\eta^{m-2}|} < \frac{\Gamma(\vartheta+1)}{L}, \tag{4.2}$$

then there exists unique solution for the boundary value problem (1.4).

Proof. Assume that X be the Banach space of continuously differentiable mappings on $[0,1]$ with standard norm

$$\|w\| = \|w\|_\infty + \|w'\|_\infty = max\{|w(t)| : 0 \leq t \leq 1\} + max\{|w'(t)| : 0 \leq t \leq 1\}. \tag{4.3}$$

Note that $w(t)$ is a solution of (1.4) if and only if it is a solution of (2.2) with $h(t) = g(t, w(t), w'(t))$. But Eq. (2.2) has the unique solution

$$w(t) = \int_0^1 G(t,\theta) g(\theta, w(\theta), w'(\theta)) d\theta$$

where $G(t,\theta)$ is specified in (2.10). Derive the operator $T : X \to X$ by

$$Tw(t) = \int_0^1 G(t,\theta) g(\theta, w(\theta), w'(\theta)) d\theta, \tag{4.4}$$

for $t \in [0,1]$. By taking derivative with respect to t, we have

$$(Tw)'(t) = \int_0^1 \frac{\partial G(t,\theta)}{\partial t} g(\theta, w(\theta), w'(\theta)) d\theta, \tag{4.5}$$

as well. We now apply Banach fixed point theorem to demonstrate the operator T has the unique fixed point. Assume $w, z \in X$, then

$$\begin{aligned}
|Tw(t) - Tz(t)| &= \left|\int_0^1 G(t,\theta)\left(g(\theta, w(\theta), w'(\theta)) - g(\theta, z(\theta), z'(\theta))\right) d\theta\right| \\
&\leq \int_0^1 |G(t,\theta)| \left|g(\theta, w(\theta), w'(\theta)) - g(\theta, z(\theta), z'(\theta))\right| d\theta \\
&\leq \int_0^1 |G(t,\theta)| L\left(|w(\theta) - z(\theta)| + |w'(\theta) - z'(\theta)|\right) d\theta \\
&\leq L(\|w - z\|_\infty + \|w' - z'\|_\infty) \int_0^1 |G(t,\theta)| d\theta \\
&\leq \|w - z\| \frac{L}{\Gamma(\vartheta+1)} \left(2 + \frac{2|\gamma_1| + (m+\vartheta-1)|\gamma_2|}{|1 - \gamma_1\eta^{m-1} - (m-1)\gamma_2\eta^{m-2}|}\right),
\end{aligned} \tag{4.6}$$

where we have utilized Theorem 3.2. In the same way, suppose $w, z \in X$, then

$$\begin{aligned} |(Tw)'(t) - (Tz)'(t)| &= \left| \int_0^1 \frac{\partial G(t,\theta)}{\partial t} \left(g(\theta, w(\theta), w'(\theta)) - g(\theta, z(\theta), z'(\theta))\right) d\theta \right| \qquad (4.7) \\ &\leq \int_0^1 \left| \frac{\partial G(t,\theta)}{\partial t} \right| |g(\theta, w(\theta), w'(\theta)) - g(\theta, z(\theta), z'(\theta))| \, d\theta \\ &\leq \int_0^1 \left| \frac{\partial G(t,\theta)}{\partial t} \right| L \left(|w(\theta) - z(\theta)| + |w'(\theta) - z'(\theta)|\right) d\theta \\ &\leq L(\|w - z\|_\infty + \|w' - z'\|_\infty) \int_0^1 \left| \frac{\partial G(t,\theta)}{\partial t} \right| d\theta \\ &\leq \|w - z\| \frac{(m-1)L}{\Gamma(\vartheta+1)} \left(1 + \frac{\vartheta}{m-1} + \frac{(2|\gamma_1| + (m+\vartheta-1)|\gamma_2|)}{|1 - \gamma_1\eta^{m-1} - (m-1)\gamma_2\eta^{m-2}|}\right), \end{aligned} \tag{4.8}$$

where we have utilized Theorem 3.3. To sum up inequalities (4.6) and (4.7) results in

$$\|Tw - Tz\| = \|Tw - Tz\|_\infty + \|(Tw)' - (Tz)'\| \leq \psi \|w - z\|, \tag{4.9}$$

where

$$\begin{aligned} \psi &= \frac{L}{\Gamma(\vartheta+1)} \left(2 + \frac{2|\gamma_1| + (m+\vartheta-1)|\gamma_2|}{|1 - \gamma_1\eta^{m-1} - (m-1)\gamma_2\eta^{m-2}|}\right) + \frac{(m-1)L}{\Gamma(\vartheta+1)} \left(1 + \frac{\vartheta}{m-1} + \frac{(2|\gamma_1| + (m+\vartheta-1)|\gamma_2|)}{|1 - \gamma_1\eta^{m-1} - (m-1)\gamma_2\eta^{m-2}|}\right) \\ &= \frac{L}{\Gamma(\vartheta+1)} \left(1 + m + \vartheta + \frac{2m|\gamma_1| + m(m+\vartheta-1)|\gamma_2|}{|1 - \gamma_1\eta^{m-1} - (m-1)\gamma_2\eta^{m-2}|}\right). \end{aligned} \tag{4.10}$$

By assumption (4.2), we conclude that $\psi < 1$, therefore we deduce that T is a contraction mapping on X, and by the Banach contraction mapping theorem we receive the asked result. □

5. Illustrative examples

Here we recall some examples to show the usefulness of our derived outputs.

Example 5.1. Consider the following high order fractional boundary value problem

$$\begin{cases} {}_cD_t^{3.5} w + \beta_1 \frac{2w(t)}{1+\cosh(w(t))} + \beta_2 \exp(-\cos^2(w'(t))) + \sin\left(t^2\right) = 0, \quad t \in (0,1), \ \beta_1, \beta_2 > 0, \\ w(0) = w'(0) = w''(0) = 0, \quad w(1) = \frac{3}{4} w(\frac{1}{2}) + \frac{1}{4} w'(\frac{1}{2}). \end{cases} \tag{5.1}$$

We have $m = 4$, $\vartheta = 3.5$, $\eta = \frac{1}{2}$, $\gamma_1 = \frac{3}{4}$, $\gamma_2 = \frac{1}{4}$ and

$$g(t, w(t), w'(t)) = \beta_1 \frac{2w(t)}{1 + \cosh(w(t))} + \beta_2 \exp(-\cos^2(w'(t))) + \sin\left(t^2\right). \tag{5.2}$$

It is obviously observed

$$\begin{aligned} |g(t, w(t), w'(t)) - g(t, z(t), z'(t))| &= \left| \beta_1 \frac{2w(t)}{1 + \cosh(w(t))} + \beta_2 \exp(-\cos^2(w'(t))) + \sin\left(t^2\right) \right. \qquad (5.3) \\ &\quad \left. - \beta_1 \frac{2z(t)}{1 + \cosh(z(t))} - \beta_2 \exp(-\cos^2(z'(t))) - \sin\left(t^2\right) \right| \\ &\leq \beta_1 \left| \frac{2w(t)}{1 + \cosh(w(t))} - \frac{2z(t)}{1 + \cosh(z(t))} \right| \\ &\quad + \beta_2 \left| \exp(-\cos^2(w'(t))) - \exp(-\cos^2(z'(t))) \right| \\ &\leq \beta_1 |w(t) - z(t)| + \beta_2 |w'(t) - z'(t)| \\ &\leq \max\{\beta_1, \beta_2\} \left(|w(t) - z(t)| + |w'(t) - z'(t)|\right), \end{aligned} \tag{5.4}$$

So, g is Lipschitz with respect to w on $[0,1]\times\mathbb{R}\times\mathbb{R}$, with Lipschitz constant $L=\max\{\beta_1,\beta_2\}$. Since $\gamma_1\eta^{m-1}+(m-1)\gamma_2\eta^{m-2}=\frac{3}{4}(\frac{1}{2})^3+3\frac{1}{4}(\frac{1}{2})^2=\frac{9}{32}\neq 1$, the assumption of Theorem 4.1 holds and more

$$1+m+\vartheta+\frac{2m|\gamma_1|+m(m+\vartheta-1)|\gamma_2|}{|1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2}|}=\frac{1191}{46}, \tag{5.5}$$

so, if we have $\frac{1191}{46}<\frac{\Gamma(4.5)}{L}$ or equivalently $L<\frac{805\sqrt{\pi}}{3176}\simeq 0.4492523142251072$ then the other assumption of Theorem 4.1 holds as well. Now, an application of Theorem 4.1 proves that the problem (5.1) has unique solution. For instance, consider the problem (5.1) with $\beta_1=\beta_2=0.4$ i.e.

$$\begin{cases} {}_cD_t^{3.5}w+\frac{0.8w(t)}{1+\cosh(w(t))}+0.4\exp(-\cos^2(w'(t)))+\sin\left(t^2\right)=0, \quad t\in(0,1),\\ w(0)=w'(0)=w''(0)=0, \quad w(1)=\frac{3}{4}w(\frac{1}{2})+\frac{1}{4}w'(\frac{1}{2}). \end{cases} \tag{5.6}$$

dose have unique solution.

Example 5.2. Consider another high order fractional boundary value problem as follows:

$$\begin{cases} {}_cD_t^{4.5}w+\beta\frac{w(t)}{1+w(t)^2}+\exp(t)=0, \quad t\in(0,1), \ \beta>0,\\ w(0)=w'(0)=w''(0)=w'''(0)=0, \quad w(1)=0. \end{cases} \tag{5.7}$$

We have $m=5$, $\vartheta=4.5$, $\gamma_1=\gamma_2=0$ and

$$g(t,w(t),w'(t))=\beta\frac{w(t)}{1+w(t)^2}+\exp(t)\,. \tag{5.8}$$

It is obviously seen

$$\begin{aligned} |g(t,w(t),w'(t))-g(t,z(t),z'(t))| &= \left|\beta\frac{w(t)}{1+w(t)^2}+\exp(t)-\beta\frac{z(t)}{1+z(t)^2}-\exp(t)\right| &(5.9)\\ &\le \beta\left|\frac{w(t)}{1+w(t)^2}-\frac{z(t)}{1+z(t)^2}\right| \\ &\le \beta\left(|w(t)-z(t)|+|w'(t)-z'(t)|\right), &(5.10) \end{aligned}$$

Therefore, g is Lipschitz with respect to w on $[0,1]\times\mathbb{R}\times\mathbb{R}$, with Lipschitz constant $L=\beta$. Since $\gamma_1\eta^{m-1}+(m-1)\gamma_2\eta^{m-2}=0\neq 1$, the assumption of Theorem 4.1 holds and more

$$1+m+\vartheta+\frac{2m|\gamma_1|+m(m+\vartheta-1)|\gamma_2|}{|1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2}|}=\frac{21}{2}, \tag{5.11}$$

hence, if we have $\frac{21}{2}<\frac{\Gamma(5.5)}{\beta}$ or equivalently $\beta<\frac{45\sqrt{\pi}}{16}\simeq 4.985026455671764$ then the other assumption of Theorem 4.1 holds as well. Now, an application of Theorem 4.1 proves that the problem (5.7) has unique solution. For instance, consider the problem (5.7) with $\beta=1$ i.e.

$$\begin{cases} {}_cD_t^{4.5}w+\frac{w(t)}{1+w(t)^2}+\exp(t)=0, \quad t\in(0,1),\\ w(0)=w'(0)=w''(0)=w'''(0)=0, \quad w(1)=0. \end{cases} \tag{5.12}$$

dose have unique solution.

6. Numerical procedure

In this section, we focus on the numerical solution of the problem (1.4) which we have already proved the existence and the uniqueness of the solution by some assumptions. The method is somewhat straightforward via remembering Lemma 2.1 and Theorem 2.2 and recurrence formula

$$w_{k+1}(t)=\int_0^1 G(t,\theta)g(\theta,w_k(\theta),w_k'(\theta))d\theta, \tag{6.1}$$

or equivalently

$$
\begin{aligned}
w_{k+1}(t) &= \int_0^1 R(t,\theta) g(\theta, w_k(\theta), w_k'(\theta)) d\theta + \frac{t^{m-1}}{1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2}} \\
&\times \int_0^1 \left[\gamma_1 R(\eta,s) + \gamma_2 \frac{\partial R(\eta,s)}{\partial t}\right] g(\theta, w_k(\theta), w_k'(\theta)) d\theta \qquad (6.2)\\
&= \frac{t^{m-1}}{\Gamma(\vartheta)} \int_0^1 (1-\theta)^{\vartheta-1} g(\theta, w_k(\theta), w_k'(\theta)) d\theta - \frac{1}{\Gamma(\vartheta)} \int_0^t (t-\theta)^{\vartheta-1} g(\theta, w_k(\theta), w_k'(\theta)) d\theta \\
&+ \frac{\eta^{m-1}}{\Gamma(\vartheta)} \frac{\gamma_1 t^{m-1}}{1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2}} \int_0^1 (1-\theta)^{\vartheta-1} g(\theta, w_k(\theta), w_k'(\theta)) d\theta \\
&- \frac{1}{\Gamma(\vartheta)} \frac{\gamma_1 t^{m-1}}{1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2}} \int_0^\eta (\eta-\theta)^{\vartheta-1} g(\theta, w_k(\theta), w_k'(\theta)) d\theta \qquad (6.3)\\
&+ \frac{(m-1)\eta^{m-2}}{\Gamma(\vartheta)} \times \frac{\gamma_2 t^{m-1}}{1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2}} \int_0^1 (1-\theta)^{\vartheta-1} g(\theta, w_k(\theta), w_k'(\theta)) d\theta \\
&- \frac{\vartheta-1}{\Gamma(\vartheta)} \frac{\gamma_2 t^{m-1}}{1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2}} \int_0^\eta (\eta-\theta)^{\vartheta-2} g(\theta, w_k(\theta), w_k'(\theta)) d\theta, \qquad (6.4)
\end{aligned}
$$

where we have used the proof of Lemma 2.1. In the other words, briefly

$$
\begin{aligned}
w_{k+1}(t) &= -\frac{1}{\Gamma(\vartheta)} \int_0^t (t-\theta)^{\vartheta-1} g(\theta, w_k(\theta), w_k'(\theta)) d\theta + \left[\frac{t^{m-1}}{\Gamma(\vartheta)} + \frac{\gamma_1(\eta t)^{m-1} + (m-1)\gamma_2\eta^{m-2}t^{m-1}}{\Gamma(\vartheta)(1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2})}\right] \qquad (6.5)\\
&\times \int_0^1 (1-\theta)^{\vartheta-1} g(\theta, w_k(\theta), w_k'(\theta)) d\theta \\
&- \frac{\int_0^\eta \left[\gamma_1 t^{m-1}(\eta-\theta)^{\vartheta-1} + \gamma_2(\vartheta-1)t^{m-1}(\eta-\theta)^{\vartheta-2}\right] g(\theta, w_k(\theta), w_k'(\theta)) d\theta}{\Gamma(\vartheta)(1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2})}.
\end{aligned}
$$

This recurrence formula can be easily applied by the initial guess belonging to the $C^1[0,1]$, say for example $u_0(t) \equiv 0$, and then it can be terminated by a criteria like for example $\|u_{k+1} - u_k\| \le \varepsilon$. In order to easily implementing the iteration process while avoiding the boring and slowly symbolic computations, we suggest the following resolution

$$
\begin{aligned}
w_{k+1}(t) &= -\frac{1}{\Gamma(\vartheta)} \sum_{j=1}^{m_t} s_j^t (t-\theta_j^t)^{\vartheta-1} g(\theta_j^t, w_k(\theta_j^t), w_k'(\theta_j^t)) + \left[\frac{t^{m-1}}{\Gamma(\vartheta)} + \frac{\gamma_1(\eta t)^{m-1} + (m-1)\gamma_2\eta^{m-2}t^{m-1}}{\Gamma(\vartheta)(1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2})}\right] \\
&\times \sum_{j=1}^{m} s_j (1-\theta_j)^{\vartheta-1} g(\theta_j, w_k(\theta_j), w_k'(\theta_j)) \\
&- \frac{\sum_{j=1}^{m_\eta} s_j \left[\gamma_1 t^{m-1}(\eta-\theta_j)^{\vartheta-1} + \gamma_2(\vartheta-1)t^{m-1}(\eta-\theta_j)^{\vartheta-2}\right] g(\theta_j, w_k(\theta_j), w_k'(\theta_j))}{\Gamma(\vartheta)(1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2})}, \qquad (6.6)
\end{aligned}
$$

where we have used the composite Simpson's integration rule in both integrals. Notice that s_j^t and θ_j^t in the first summation denote that the Simpson's weights and nodes depend on t because of the integration interval of the first integral i.e. $(0,t)$. Now, we apply central-difference formula for first derivative which is of order $O(h^2)$ in where h is the uniform distance between nodes.

$$
w'(t_q) \approx \frac{w(t_{q+1}) - w(t_{q-1})}{2h}
$$

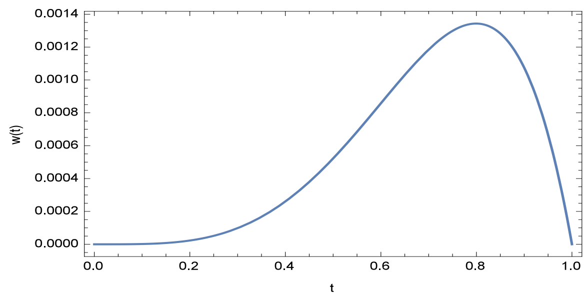

FIGURE 1. The diagram of numerical solution for Example 5.2.

Setting the above formulas in Eq. (6.6), we obtain

$$\begin{aligned} w_{k+1}(t) \quad &= -\frac{1}{\Gamma(\vartheta)}\sum_{j=1}^{m_t} s_j^t(t-\theta_j^t)^{\vartheta-1} g\left(\theta_j^t, w_k(\theta_j^t), \frac{w_k(\theta_{j+1}^t)-w_k(\theta_{j-1}^t)}{2h_t}\right) \\ &+\left[\frac{t^{m-1}}{\Gamma(\vartheta)}+\frac{\gamma_1(\eta t)^{m-1}+(m-1)\gamma_2\eta^{m-2}t^{m-1}}{\Gamma(\vartheta)(1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2})}\right] \\ &\times\sum_{j=1}^{m} s_j(1-\theta_j)^{\vartheta-1} g\left(\theta_j, w_k(\theta_j), \frac{w_k(\theta_{j+1})-w_k(\theta_{j-1})}{2h}\right) \\ &-\frac{\sum_{j=1}^{m_\eta} s_j\left[\gamma_1 t^{m-1}(\eta-\theta_j)^{\vartheta-1}+\gamma_2(\vartheta-1)t^{m-1}(\eta-\theta_j)^{\vartheta-2}\right] g\left(\theta_j, w_k(\theta_j), \frac{w_k(\theta_{j+1})-w_k(\theta_{j-1})}{2h}\right)}{\Gamma(\vartheta)(1-\gamma_1\eta^{m-1}-(m-1)\gamma_2\eta^{m-2})}, \end{aligned} \tag{6.7}$$

6.1. **Numerical experiment.** As an example, to verify the validity of the presented numerical method, consider that Example 5.2 In this example, consider $\beta = 1$, i.e. Eq. (5.12) for the simplicity but indeed β can be variety of numbers in feasible range, namely $0 \leq \beta < 4.985026455671764$. The explicit exact closed form solution is unknown indeed, but the criteria used for stopping the iteration process is set $\|u_{k+1} - u_k\|_\infty < 10^{-10}$, and then the numerical solution is obtained. The numerical solutions for different is shown graphically in the Figure 1.

7. CONCLUSION

We have applied the Banach contraction mapping theorem to prove the existence and the uniqueness of the solution to a kind of high-order nonlinear fractional differential equations involving Caputo fractional derivative whose boundary one of boundary condition is nonlocal. In conclusion, we have given some sufficient conditions which show the existence and uniqueness of solutions for a non-integer order boundary value problem. Some examples confirm if the derived results can be valid. Furthermore, we have given a stable and convergence numerical technique to approximate the solution and an illustrative example has been presented graphically.

ACKNOWLEDGMENTS

The authors are very grateful to two anonymous reviewers for carefully reading the paper and for their comments and suggestions which have improved the paper very much.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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