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On approximating eigenvalues and eigenfunctions of fractional order Sturm-Liouville problems

Arezu Aghazadeh and Yaghoub Mahmoudi*
Department of Mathematics, Tabriz Branch, Islamic Azad University, Tabriz, Iran.

Abstract

In this paper, the eigenvalues and corresponding eigenfunctions of a fractional order Sturm-Liouville problem (FSLP) are approximated by using the fractional differential transform method (FDTM), which is a generalization of the differential transform method (DTM). FDTM reduces the proposed fourth-order FSLP to a system of algebraic equations. The resulting coefficient matrix defines a characteristic polynomial which its roots correspond to the eigenvalues of FSLP. The obtained numerical results which are compared with the results of other papers confirm the efficiency of the method.

Keywords. Sturm-Liouville problem, Caputo fractional derivative, Eigenvalue, Eigenfunction.
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1. Introduction

Differential and integral equations are the common language of science for the mathematical expression of natural problems. In recent years, many attempts have been made to solve these equations analytically and numerically [12, 13, 23, 27, 31, 38]. One of the applicable problems in the theory of differential equations is finding the eigenvalues and eigenfunctions of the Sturm-Liouville problem, which has many applications in mathematics and physics. The description of the vibrations of a string or a quantum mechanical oscillator is modeled as Sturm-Liouville equations [2, 3, 10]. More information about the application of Sturm-Liouville problems can be found in [6, 7, 10, 14, 21], and references therein. With the advent of fractional calculus, the use of fractional derivatives in Sturm-Liouville equations led to a new class of equations, which are known as fractional Sturm-Liouville problems. The Caputo concept of fractional order derivatives is mostly used for Sturm-Liouville equations because it is more compatible with practical application in physics and engineering, [20, 28, 29, 40]. The Sturm-Liouville problem is usually not solvable analytically. This problem has limited the application of these equations in various fields. In simpler cases, the analytical solutions to this equation can be expressed in terms of specific functions such as Mittag-Leffler functions, which have their complexities in terms of calculations. So, in most cases, scientists seek to find numerical methods to solve Sturm-Liouville problems.

In this study, we consider a special class of these equations, which are known as fourth-order fractional Sturm-Liouville problems, as follows

$$D^{\alpha}u(t) + \sum_{j=1}^{3} q_j(t) D^{\beta_j} u(t) + q_0(t)u(t) = \lambda q_4(t)u(t), \qquad a \le t \le b, \tag{1.1}$$

corresponding to the boundary conditions

$$\begin{cases} u^{(i_r)}(a) = 0, & r = 0, 1, \dots, l-1, \\ u^{(i_r)}(b) = 0, & r = l, \dots, 3, \end{cases} \tag{1.2}$$

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* Corresponding author. Email: mahmoudi@iaut.ac.ir .

where, $q_j(t)$ $(j = 0, \dots, 4)$, are assumed to be continuous on $[a, b]$, and $3 < \alpha \leq 4$, $2 < \beta_3 = \alpha - 1 \leq 3$, $1 < \beta_2 = \alpha - 2 \leq 2$ and $0 < \beta_1 = \alpha - 3 \leq 1$. The notation D^α $(\alpha \in R^+)$ in (1.1) indicates the left-sided fractional derivative of order α in Caputo sense.

The authors in [2] established sufficient conditions for the existence and uniqueness of solutions for various classes of initial and boundary value problem for fractional differential equations involving the Caputo fractional derivative. Also, the existence and uniqueness of the solution for a fractional Sturm-Liouville boundary value problem based on Banach fixed point theorem was proved in [25]. We also refer to [8, 30, 45], for more studies on the existence and uniqueness of the solution for boundary value problems of type (1.1) and (1.2).

A wide range of numerical methods have been used to solve the problems (1.1) and (1.2) [1, 4, 6, 7, 26]. Laplace transform method is applied to convert (1.1)-(1.2) to the equivalent integral equation with weakly singular kernel in [41]. Then, the authors applied a piecewise Lagrange integration method to solve the corresponding integral equation numerically.

In this study, we applied FDTM to the FSLP (1.1)-(1.2). The differential transform method (DTM) is a reliable and effective method that was constructed by Zhou [46] for solving linear and nonlinear differential equations arising in electrical circuit problems. In this method, the analytical solution is constructed in terms of a power series that applies Taylor expansion. The proposed method is different from the traditional Taylor series method. DTM makes an iterative procedure to obtain Taylor series expansion, which needs less computational time compared to the Taylor series method. DTM was extended by Chen and Ho [19] into two-dimensional functions for solving partial differential equations. One and two-dimensional differential transform methods were generalized for ODEs and PDEs with fractional order (GDTM) by Odibat and Momani [35, 36]. DTM and GDTM were applied by many authors to find approximate solutions for different kinds of equations. Ray [39] used the modification of GDTM to find a numerical solution for KdV type equation [24]. More over, Soltanalizadeh, Sirvastava and Garg [22, 42, 43] applied GDTM and DTM to find exact and numerical solutions for telegraph equations in the cases of one-space dimensional, two and three dimensional and space-time fractional derivatives. Furthermore, GDTM was applied to find numerical approximations of the time-fractional diffusion equation and differential-difference equations [17, 32, 47], as well as DTM was used to obtain an approximate solution of fractional telegraph equation [16, 33, 34].

The rest of this paper is organized as follows: In section 2, we provide a brief description of fractional calculus and FDTM. In section 3, we introduce the new semi-analytic method for solving problem (1.1)-(1.2). Numerical illustrations are presented in section 4. Finally, a brief conclusion is drawn in section 5.

2. Preliminaries

In this part, some preliminary results which will be used in the next sections are briefly reviewed.

2.1. **Fractional Calculus.**

In this section, we mainly recall some definitions which will be used in this study.

Definition 2.1. Suppose $\nu \in R$ and $n \in N$. A function $f : R^+ \to R$ belongs to C_ν if there exists $p \in R$, $p > \nu$ and $f_1 \in C[0, \infty)$ such that $f(t) = t^p f_1(t)$, moreover, $f \in C_\nu^{(n)}$ if $f^{(n)} \in C_\nu$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f(t) \in C_\nu$, $\nu \geq -1$ is defined as [37]

$$\begin{cases} I_a^\alpha f(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad \alpha > 0, \\ I_a^0 f(t) = f(t), \end{cases} \tag{2.1}$$

where $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$, $z \in C$ and a is called the index of integration.

For $f \in C_\nu$, $\nu \geq -1$, $\alpha,\ \beta \geq 0$ and $\gamma \geq -1$, the operator I_a^α satisfies the following properties

(1) $I_a^\alpha I_a^\beta f(t) = I_a^{\alpha+\beta} f(t) = I_a^\beta I_a^\alpha f(t)$,
(2) $I_a^\alpha (t-a)^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)} (t-a)^{\alpha+\gamma}$.

Definition 2.3. The Caputo fractional differential operator ${}^{C}D_a^{\beta}$ is defined as

$$ {}^{C}D_a^{\beta} f(t) = \begin{cases} I_a^{n-\beta} \frac{d^n}{dt^n} f(t), & n-1<\beta<n, \\ \frac{d^n}{dt^n} f(t), & \beta = n, \end{cases} \tag{2.2}$$

Let $n-1<\beta\leq n$, $n\in N$ and $f\in C_\nu^n$, $\nu\geq -1$, $\gamma>\beta-1$, then the operator ${}^{C}D_a^{\beta}$ satisfies the following properties

(1) ${}^{C}D_a^{\beta} J^{\beta} f(t) = f(t)$,

(2) $I_a^{\beta}[{}^{C}D_a^{\beta} f(t)] = f(t) - \sum_{k=0}^{n-1} f^{(k)}(a)\frac{(t-a)^k}{k!},\ t>a$,

(3) ${}^{C}D_a^{\beta}(t-a)^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\beta)}(t-a)^{\gamma-\beta}$,

(4) ${}^{C}D_a^{\beta} c = 0$, where c is a constant.

2.2. **Fractional differential transform method.** In this section, the fractional differential transform method is recalled. A smooth function $u(t)$ can be expressed by a fractional power series as follows

$$ u(t) = \sum_{k=0}^{\infty} U(k)(t-a)^{k\beta}, \tag{2.3}$$

where $U(k)$ is considered as fractional differential transform of $u(t)$. In practical applications, the initial conditions of the problem are usually derived by integer-order derivatives, then, the initial conditions are transformed as follows [9].

$$ U(k) = \begin{cases} \frac{1}{(k\beta)!}\left[\frac{d^{(k\beta)}u(t)}{dt^{k\beta}}\right]_{t=a}, & k\beta\in Z^+, \\ \\ 0, & k\beta\notin Z^+, \end{cases} \tag{2.4}$$

where $k=0,1,2,\ldots,[\alpha/\beta]-1$. Also α is the order of differential equation, so β should be chosen such that $\alpha\beta\in N$. To see some basic properties of FDTM we refer to [9]. Also, more theorems on generalized Taylor formula, can be found in [35].

Suppose $u(t)$, $v(t)$ and $w(t)$ are functions of time t and U(k), V(k) and W(k) are their related fractional transforms of order β respectively. Then, the following relations hold [9],

(1) If $u(t)=v(t)+w(t)$ then $U(k)=V(k)+W(k)$.

(2) If $u(t)=cv(t)$ then $U(k)=cV(k)$ where c is a constant.

(3) If $u(t)=v(t)w(t)$ then $U(k)=\sum_{l=0}^{k} V(l)W(k-l)=\sum_{l_1+l_2=k} V(l_1)W(l_2)$.

(4) If $u(t)=(t-a)^p$ where p is a constant then $U(k)=\delta(k-p/\beta)$ while

$$ \delta(k) = \begin{cases} 1, & k=0, \\ 0, & k\neq 0, \end{cases} $$

here β should be chosen such that $p/\beta\in Z^+$.

(5) If $u(t)={}^{C}D_a^{\alpha} v(t)$ then $U(k)=\frac{\Gamma(k\beta+\alpha+1)}{\Gamma(k\beta+1)}V(k+\alpha/\beta)$, where β should be chosen such that $\alpha/\beta\in Z^+$.

Lemma 2.4. *Let $u(t)=y^j(t)$ $(j\in N)$, is function of t and $U(k)$ and $Y(k)$ are FDTs for $u(t)$ and $y(t)$ with order β, then*

$$ U(k) = \sum_{l_1+l_2+\ldots+l_j=k} Y(l_1)Y(l_2)\ldots Y(l_j), \qquad j=2,3,\ldots. \tag{2.5}$$

Proof. By (2.3) and the properties of FDTM, the proof is obvious. □

Lemma 2.5. *Let $u(t)=a(t)y^j(t)$ $(j\in N)$, and $U(k)$, $Y(k)$ and $A(k)$ are their FDTs of order β, then*

$$ U(k) = \sum_{l+l_1+l_2+\ldots+l_j=k} A(l)Y(l_1)Y(l_2)\ldots Y(l_j), \qquad k=0,1,2,\ldots. \tag{2.6}$$

Proof. By (2.3) and the properties of FDTM, the proof is obvious. □

3. Implementation of the method

To apply FDTM to the main problem (1.1), we chose an appropriate β such that $\frac{\alpha}{\beta} \in Z^+$, $\frac{\beta_i}{\beta} \in Z^+$, $i = 1, 2, 3, 4$ and $\frac{1}{\beta} \in Z^+$ and then we apply fractional transform method to (1.1) as we discussed in previous section as follows;

$$\begin{aligned} \frac{\Gamma(\beta k+\alpha+1)}{\Gamma(\beta k+1)} U(k+\frac{\alpha}{\beta}) = &- \sum_{j=1}^{3} \sum_{L=0}^{k} \frac{\Gamma(\beta L+\beta_j+1)}{\Gamma(\beta L+1)} U(L+\frac{\beta_j}{\beta}) Q_j(k-L) \\ &- \sum_{L=0}^{k} U(L) Q_0(k-L) + \lambda \sum_{L=0}^{k} U(L) Q_4(k-L). \end{aligned} \tag{3.1}$$

So,

$$\begin{aligned} U(k+\frac{\alpha}{\beta}) = \frac{\Gamma(\beta k+1)}{\Gamma(\beta k+\alpha+1)} \Bigg\{ &- \sum_{j=1}^{3} \sum_{L=0}^{k} \frac{\Gamma(\beta L+\beta_j+1)}{\Gamma(\beta L+1)} U(L+\frac{\beta_j}{\beta}) Q_j(k-L) \\ &- \sum_{L=0}^{k} U(L) Q_0(k-L) + \lambda \sum_{L=0}^{k} U(L) Q_4(k-L) \Bigg\}, \end{aligned} \tag{3.2}$$

where $Q_0(k)$, $Q_1(k)$, $Q_2(k)$, $Q_3(k)$ and $Q_4(k)$ are k-th fractional transform for $q_0(x)$, $q_1(x)$, $q_2(x)$, $q_3(x)$ and $q_4(x)$ respectively.

According to (3.2), the starting values $U(0), U(1), \dots, U(\frac{\alpha}{\beta} - 1)$ are needed to apply FDTM. By (2.4) we set $U(k) = 0$ if $k/\beta \notin Z^+$, while the other indexes $i \in \{0, 1, \dots \frac{\alpha}{\beta} - 1\}$ are divided into two subsets S_0 and S_1. We set the starting values $U(i)$ for $i \in S_0$ by boundary conditions (1.2) which defined at the left boundary $t = a$, while the other starting values $U(i)$, $i \in S_1$ which can not be initialized by boundary conditions are set equal to unknown parameters c_i, $i \in S_1$, which will be determined latter.

Now, the approximate solution (1.1) can be rewritten as follows:

$$u_N(t;\lambda) = \sum_{k=0}^{N} U(k)(t-a)^{k\beta}, \tag{3.3}$$

where the nonzero coefficients $U(k)$, include at least one parameter c_i, because the homogeneous boundary conditions (1.2), include at least one boundary condition at the right end of the domain ($t = b$).

Imposing the homogeneous boundary conditions (1.2) at point $t = b$ we get

$$u_N^{(i)}(b;\lambda) = 0, \qquad i \in S_1 = \{i_1, i_2, \dots, i_r\}. \tag{3.4}$$

Eq. (3.4) represents a homogeneous linear system of r equations with r unknown parameters $c_{i_1}, \dots, c_{i_r}$ in where $i_1, \dots, i_r \in S_1$. In order to have a nontrivial solution, the determinant of the coefficients matrix should be zero in (3.4). This determinant is a polynomial of λ whose zeros correspond to the eigenvalues of original problem (1.1) and (1.2). After approximating the eigenvalue λ_i, the corresponding eigenfunction $u_N(t, \lambda_i)$ can be evaluated by imposing some auxiliary conditions uniquely.

4. Illustrations

In this section, we illustrate some examples to present the accuracy and simplicity of the FDTM.

Example 4.1. *Consider the following eigenvalue problem*

$$u^{(4)}(t) - 0.02t^2u''(t) - 0.04tu'(t) + (0.0001t^4 - 0.02)u(t) = \lambda\ u(t), \qquad 0 \le t \le 5, \tag{4.1}$$

corresponding to the boundary conditions

$$\begin{cases} u(0) = 0, & u''(0) = 0, \\ u(5) = 0, & u''(5) = 0. \end{cases} \tag{4.2}$$

We use the proposed method to solve (4.1) and (4.2). Here we set $q_3(t) = 0$, $q_2(t) = -0.02t^2$, $q_1(t) = -0.04t$, $q_0(t) = 0.0001t^4 - 0.02$ and $q_4(t) = 1$. With $\alpha = 4$ we set $\beta = 1$,

$$Q_3(k) = 0, \qquad k = 0, 1, \dots,$$

$$Q_2(k) = \begin{cases} -0.02, & k = 2, \\ 0, & k \neq 2, \end{cases}$$

$$Q_1(k) = \begin{cases} -0.04, & k = 1, \\ 0, & k \neq 1, \end{cases}$$

$$Q_0(k) = \begin{cases} -0.02, & k = 0, \\ 0.0001, & k = 4, \\ 0, & k \neq 0, 4. \end{cases}$$

With $U(0) = 0$, $U(1) = c_1$, $U(2) = 0$ and $U(3) = c_3$, as initial values for FDTs, we obtain other FDTs by suing (3.2). For $N = 65$ some first eigenvalues are presented and compared with the results of [15], [11], CSCM [44] and [18] in Table 1. We use the auxiliary condition $u_i'(0) = 1$ to compute five first eigenfunctions of Example 4.1 which

Table 1. Eigenvalues of Example 4.1.

	Proposed Method	DTM [15]	ADM [11]	CSCM [44]	[18]
λ_1	0.215050864369716	0.2150508643697155	0.21505086436971596	0.2150508643160	0.21505086437
λ_2	2.754809934683034	2.7548099346830	2.754809934682985	2.7548099336169	2.75480993468
λ_3	13.21535154055818	13.2153515405582	13.215351540558824	13.215351540581	13.2153515406
λ_4	40.95081975916148	40.950819759160	40.95081975913761	40.95081978144	40.9508193487
λ_5	99.05347806345359	99.05347806671	99.05347813813881	99.0534780383535	-
λ_6	204.3557337519479	204-35573364024	204.35449348957832	204.355735479344	-

are plotted in the Figure 1.

Example 4.2. *For the second example we solve the following eigenvalue problem [5]*

$$D^\alpha u(t) = \lambda\ t\ u(t), \qquad 0 \le t \le 1, \tag{4.3}$$

corresponding to the boundary conditions

$$\begin{cases} u(0) = 0, & u'(0) = 0, \\ u(1) = 0, & u'(1) = 0. \end{cases} \tag{4.4}$$

The eigenvalues of (4.3) and (4.4) are approximated in [5] in terms of Mittag-Lefler functions. We apply the proposed method to solve (4.3) and (4.4).Here $q_3(t) = q_2(t) = q_1(t) = q_0(t) = 0$ and $q_4(t) = t$, then we set

$$Q_0(k) = Q_1(k) = Q_2(k) = Q_3(k) = 0, \qquad k = 0, 1, \dots,$$

$$Q_4(k) = \begin{cases} 1, & k\beta = 1, \\ 0, & k\beta \neq 1, \end{cases}$$

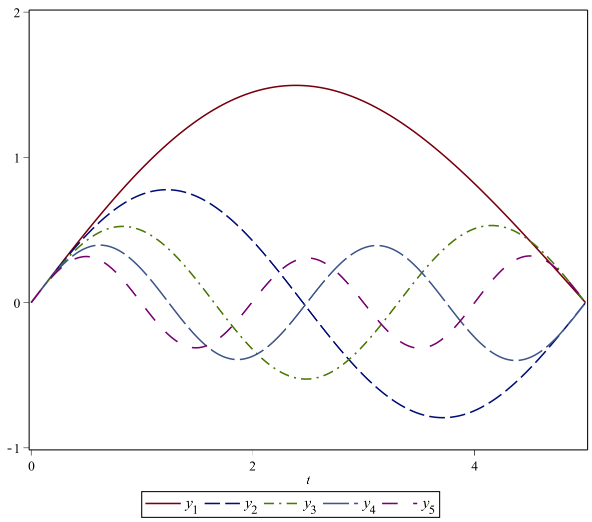

FIGURE 1. First five eigenfunctions for Example 4.1.

We choose β such that $\alpha/\beta \in Z^+$ and $1/\beta \in Z^+$ and

$$U(k) = \begin{cases} 0, & k\beta \notin Z^+, \\ 0, & k\beta = 0, 1, \\ c_k, & k\beta = 2, 3, \end{cases}$$

as initial values, then (3.2) is used to compute other values for $U(k)$. For example for $\alpha = 3.8$ we choose $\beta = 0.2$ and then

$$Q_0(k) = Q_1(k) = Q_2(k) = Q_3(k) = 0, \qquad k = 0, 1, \dots,$$

$$Q_4(k) = \begin{cases} 1, & k = 5, \\ 0, & k \neq 5, \end{cases}$$

and

$$U(k) = \begin{cases} c_{10}, & k = 10, \\ c_{15}, & k = 15, \\ 0, & k = 0, 1, \dots, 18, \text{and } k \neq 10, 15, \end{cases}$$

as initial values.

$N = 65 \times 5$ some first eigenvalues are compared with the results of [5] in Table 2 for different values of α. The results show that the results of this paper are in good agreement with [5].

We use the auxiliary condition $u_i''(0) = 1$ to compute five first eigenfunctions of Example 2 which are plotted in Figure 2 (for $\alpha = 3.8$).

Example 4.3. *For the next example we solve the following eigenvalue problem*

$$D_t^{\alpha} u(t) + t D_t^{\beta_3} u(t) = \lambda\, u(t), \qquad 0 \le t \le 1, \tag{4.5}$$

corresponding to the boundary conditions

$$\begin{cases} u(0) = 0, & u'(0) = 0, \\ u(1) = 0, & u'(1) = 0. \end{cases} \tag{4.6}$$

TABLE 2. Eigenvalues of Example 4.2.

	$\alpha = 3.4$		$\alpha = 3.6$	
	FDTM	[5]	FDTM	[5]
λ_1	820.10085729189607	820.101	757.3550395666089	757.355
λ_2	2163.4671692135933	2163.467	3755.320272974486	3755.320
λ_3	-	-	14984.79946057585	14984.799
λ_4	-	-	32031.97442091199	32031.974
λ_5	-	-	78352.46700307698	-
λ_6	-	-	112341.2635616008	-
	$\alpha = 3.8$		$\alpha = 4$	
	FDTM	[5]	FDTM	[5]
λ_1	833.1420212797868	833.142	988.2208595503757	988.221
λ_2	5561.542151196505	5561.542	8028.718709486515	8028.719
λ_3	21052.55082341833	21052.551	32056.95911264450	32056.959
λ_4	54446.22427668598	54446.224	89587.73202072882	89587.732
λ_5	119450.4203584932	-	202887.4072956590	-
λ_6	224133.4647527775	-	391872.9916555228	-

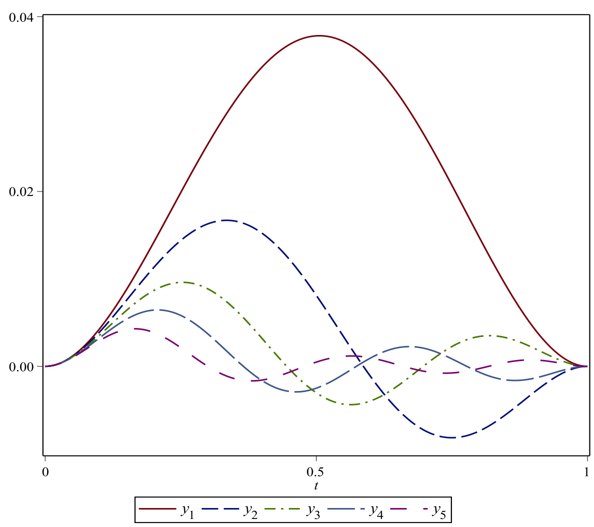

FIGURE 2. First five eigenfunctions for Example 4.2 ($\alpha = 3.8$).

The eigenvalues of (4.5) and (4.6) are approximated in [5] in terms of Mittag-Lefler functions but not reported as a Table. We use the proposed method to solve (4.5) and (4.6). We set $q_3(t) = t$, $q_2(t) = q_1(t) = q_0(t) = 0$ and $q_4(t) = 1$, then we can initiate the method as follows

$$Q_0(k) = Q_1(k) = Q_2(k) = 0, \qquad k = 0, 1, \ldots,$$

$$Q_4(k) = \begin{cases} 1, & k\beta = 0, \\ 0, & k\beta \neq 0, \end{cases}$$

$$Q_3(k) = \begin{cases} 1, & k\beta = 1, \\ 0, & k\beta \neq 1. \end{cases}$$

TABLE 3. Eigenvalues of Example 3.

	α			
	3.4	3.6	3.8	4
λ_1	-	491.7487079055093	471.4556483590656	520.30478703860847
λ_2	-	1682.913139285847	2695.930323203532	3877.7057743435724
λ_3	-	8720.806071093415	10079.18343046790	14777.308692490123
λ_4	-	12100.43538518033	24478.99763662057	40221.193937288526
λ_5	-	95558.75853846906	54314.51364939441	89562.581460487919
λ_6	-	395131.2097367171	97690.01796610059	174490.34491562140

We choose β such that $\alpha/\beta \in Z^+$ and $1/\beta \in Z^+$ and

$$U(k) = \begin{cases} 0, & k\beta \notin Z^+, \\ 0, & k\beta = 0, 1, \\ c_k, & k\beta = 2, 3. \end{cases}$$

With $N = 65 \times 5$ some first eigenvalues are listed in Table 3 for different values of α. As the table 3 shows, there is no specific value for $\alpha = 3.4$.

We use the auxiliary condition $u_i''(0) = 1$ to compute five few first eigenfunctions of Example 4.3 which are plotted in Figure 3 (for $\alpha = 4$).

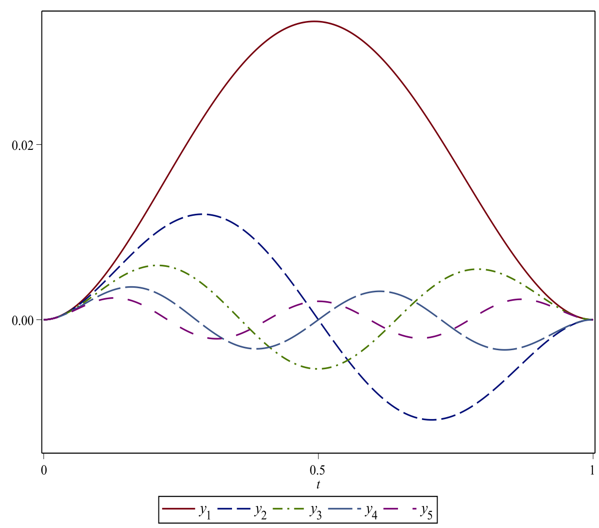

FIGURE 3. First five eigenfunctions for Example 4.3 ($\alpha = 4$).

5. Conclusion

In this paper, a special class of fourth-order fractional Sturm-Liouville problems with various types of homogeneous boundary conditions is considered. The fractional differentiation transform method (FDTM) is applied to approximate the eigenvalues and corresponding eigenfunctions of the problem. The unknown solution is expanded by the fractional-order Taylor series, while the unknown coefficients are determined by FDTM. Therefore, the proposed method leads to a homogeneous system of linear equations containing the unknown eigenvalue λ as the parameter in the coefficient matrix. The existence of nontrivial solutions for the Sturm-Liouville problem requires that the eigenvalues of the problem should be the roots of the determinant of the coefficient matrix. Newton's iteration method is used to approximate the eigenvalues which are influenced by the choice of boundary conditions. After calculating the eigenvalues, the eigenfunctions of the equation are approximated by adding some appropriate auxiliary conditions to the eigenfunction and the corresponding eigenfunction was determined uniquely. As reported in the references, unlike the integer order Sturm-Liouville problems in fractional order problems for some order of fractional derivative α, no eigenvalue found.

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