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Applying moving frames to finding conservation laws of the nonlinear Klein-Gordon equation

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Abstract

In this paper, we use a geometric approach based on the concepts of variational principle and moving frames to obtain the conservation laws related to the one-dimensional nonlinear Klein-Gordon equation. Noether's First Theorem guarantees conservation laws, provided that the Lagrangian is invariant under a Lie group action. So, for calculating conservation laws of the Klein-Gordon equation, we first present a Lagrangian whose Euler-Lagrange equation is the Klein-Gordon equation, and then according to Gonçalves and Mansfield's method, we obtain the space of conservation laws in terms of vectors of invariants and the adjoint representation of a moving frame, for that Lagrangian, which is invariant under a hyperbolic group action.

Keywords. Nonlinear Klein-Gordon equation, Conservation laws, Moving frame, Differential invariants, Syzygy.
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1. Introduction

Partial differential equations arise frequently in the modelling of many nonlinear phenomena, that emerge in many areas of scientific fields such as fluid dynamics, solid state physics, plasma physics, and mathematical biology. One of the most prominent equations is the nonlinear Klein-Gordon (NKG) equation with application in the area of theoretical physics, mathematical physics, and the context of relativistic quantum mechanics. The NKG equation appears in diverse physical events such as the motion of rigid pendula attached to a stretched wire, dislocations in crystals, solid state physics, plasma physics, nonlinear optics, and quantum field theory [18].

The one-dimensional nonlinear Klein-Gordon equation is given by

$$u_{tt} - c^2 u_{xx} + g(u) = 0, \tag{1.1}$$

where c is a known constant, $u = u(x,t)$ represents the wave displacement at position x and time t and $g(u)$ is the nonlinear force. When $g(u) = ku + \gamma u^2$ or $g(u) = ku + \gamma u^3$, where k and γ are known constants, equation (1.1) is called the nonlinear Klein-Gordon equation with quadratic or cubic nonlinearity, respectively. In special case, if the nonlinear Klein-Gordon equation is $u_{tt} - c^2 u_{xx} + V'(u) = 0$, the function $V'(u)$ is a nonlinear function of u, usually chosen as the derivative of the potential energy $V(u)$.

Several methods have been presented to solve the Klein-Gordon type equations, such as the auxiliary equation method [17], the decomposition method [19], the numerical method [1, 2, 7, 8, 13–15], and the variational iteration method [16].

The vast importance of the problem of finding conservation laws in a large number of applications in physics and mechanics is beyond any doubt. The existence of an adequate number of conservation laws leads to the complete integrability of the dynamical system, which is one of the most appealing questions for researchers. Conservation laws

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describe physically conserved quantities such as momentum, angular momentum, mass, and energy. In fact, in the study of differential equations, conservation laws have many considerable uses. They are important for establishing the existence and uniqueness of solutions. Also, they provide an essential starting point for finding potential variables and play a fundamental role in the development of numerical methods. In addition, the structure of conservation laws is coordinate-independent because all point transformations and contact transformations map a conservation law to a conservation law.

There are various methods for finding conservation laws of differential equations. In this paper, we are going to find the conservation laws of the equation

$$u_{tt} - c^2 u_{xx} + ku + \gamma u^3 = 0, \tag{1.2}$$

by applying Noether's Theorem and moving frames. In 1918, Emmy Noether in his pivotal paper [11], proved the substantial result that for systems arising from a variational principle, every conservation law of the system comes from a Lie group action that leaves the Lagrangian invariant (see [12], Theorem 4.29.).

Recently in [4, 6, 10], Mansfield and Gonçalves considered diverse Lagrangians, which are invariant under a Lie group action, where independent variables are invariant. They presented the mathematical structure behind both the Euler-Lagrange equations and the set of conservation laws, and they proved that Noether's conservation laws can be displayed as the product of adjoint representation of a right moving frame and vectors that come from invariants. These results were presented in [4] for the standard $SE(2)$ and $SE(3)$ actions, and in [6] for all three inequivalent $SL(2)$ actions in the complex plane. In a recent work [5], Mansfield and Gonçalves considered invariant Lagrangians under a Lie group action, where independent variables are no longer invariant.

In this paper, first, we give a Lagrangian that its Euler-Lagrange equation is the Klein-Gordon equation, then according to [5], we calculate the conservation laws of the Euler-Lagrange equation in which the two independent variables are not invariant.

In section 2, we will briefly give some background on moving frames, differential invariants of a group action, invariant differentiation operators, and invariant forms. Throughout section 2, we will use the group action of a hyperbolic group on the space $(x, t, u(x, t))$, that this group is a symmetry group of the nonlinear Klein-Gordon equation (1.2).

In section 3, we concentrate on the invariant calculus of variations and find the adjoint representation associated with the hyperbolic group action. Then, we end this section with the calculation of Noether's conservation laws associated with equation (1.2), in terms of vectors of invariants, the adjoint representation of the moving frame, and a matrix which represents the group action on the 1-forms.

2. Moving frames, differential invariants of a group action and invariant forms

In this section, we present some vital concepts regarding moving frames, differential invariants of a group action, invariant differential operators, and invariant forms as formulated by Fels and Olver [3], Mansfield [10], and Kogan and Olver [9].

The nonlinear Klein-Gordon equation (1.2) is the Euler-Lagrange equation for the variational problem

$$\Phi[u] = \iint \frac{1}{4}(-2u_t^2 + 2c^2 u_x^2 + 2ku^2 + \gamma u^4)dxdt, \tag{2.1}$$

in other words, the equation (1.2) is the Euler-Lagrange equation of the Lagrangian

$$L = \frac{1}{4}(-2u_t^2 + 2c^2 u_x^2 + 2ku^2 + \gamma u^4).$$

So, variational symmetry (hyperbolic) group G of the functional $\Phi[u]$ with infinitesimal generators

$$-c\partial_x + \partial_t, \qquad c\partial_x + \partial_t, \qquad c^2 t\partial_x + x\partial_t, \tag{2.2}$$

is a symmetry group of the nonlinear Klein-Gordon equation (1.2) (see [12], Theorem 4.14.).

Definition 2.1. A group action of G on M is a map

$$G \times M \to M, \qquad (g, z) \ \to \ \tilde{z} = g \cdot z,$$

which satisfies either $g\cdot(h\cdot z)=(gh)\cdot z$, called a *left action*, or $g\cdot(h\cdot z)=(hg)\cdot z$, called a *right action*.

The action of the Lie group G associated to vector field (2.2) on a 2-dimensional manifold M with coordinates (x,t), is given as follows

$$\tilde{x}=-c\alpha+c\beta+\frac{1}{2}(e^{c\theta}x+e^{-c\theta}x+ce^{c\theta}t-ce^{-c\theta}t)=-c\alpha+c\beta+x\cosh(c\theta)+ct\sinh(c\theta),$$

$$\tilde{t}=\frac{1}{2c}(2c\alpha+2c\beta+e^{c\theta}x-e^{-c\theta}x+ce^{c\theta}t+ce^{-c\theta}t)=\alpha+\beta+\frac{x}{c}.\sinh(c\theta)+t\cosh(c\theta), \tag{2.3}$$

where α, β, and θ are constants that parametrize the group action.

Definition 2.2. We say, two smooth surfaces $\mathcal{K}$ and $\mathcal{O}$ contained in $\mathbb{R}^n$, such that, $\dim(\mathcal{K})=\alpha$, $\dim(\mathcal{O})=\beta$, $0\le\alpha,\beta\le n$, $\alpha+\beta\ge n$, intersect *transversally* if for every $x\in\mathcal{K}\cap\mathcal{O}$, the tangent spaces $T_x\mathcal{K}$ and $T_x\mathcal{O}$, as subspaces of $T_x\mathbb{R}^n$, satisfy

$$T_x\mathcal{K}+T_x\mathcal{O}=T_x\mathbb{R}^n.$$

Consider a Lie group G acting smootlhy on M such that the action is *free* and regular in some domain $\mathcal{U}\subset M$. This declares that
- the group orbits have the same dimension of the group G and foliate $\mathcal{U}$,
- there is a surface $\mathcal{K}\subset\mathcal{U}$ which intersects the group orbits transversally at a single point, known as cross section,
- if $\mathcal{O}(z)$ represents the orbit through z, then the element $g\in G$ taking $z\in\mathcal{U}$ to $\{k\}=\mathcal{O}(z)\cap\mathcal{K}$ is unique.

Under these conditions, we define a *right moving frame* as the map $\rho:\mathcal{U}\to G$ which sends $z\in\mathcal{U}$ to the unique element $g=\rho(z)\in G$ which satisfies $\rho(z)\cdot z=k$. To obtain the right moving frame, which sends z to k, we define the cross section $\mathcal{K}$ as the locus of the set of equations $\psi_j(z)=0$, for $j=1,\cdots,r$, where r is the dimension of G. Then, solving the set of equations

$$\psi_j(\tilde{z})=\psi_j(g.z)=0, \qquad j=1,\cdots,r,$$

known as the *normalization equations*, for the r parameters describing G yields the frame in parametric form.

We now consider the hyperbolic group action G associated to the transformation (2.3) on the space $(x,t,u(x,t))$, where u is invariant.

Example 2.3. Consider the group action G on the space $(x,t,u(x,t))$ as follows

$$\begin{pmatrix}\tilde{x}\\ \tilde{t}\end{pmatrix}=\begin{pmatrix}\cosh(c\theta) & c\sinh(c\theta)\\ \frac{1}{c}\sinh(c\theta) & \cosh(c\theta)\end{pmatrix}\begin{pmatrix}x\\ t\end{pmatrix}+\begin{pmatrix}-c\alpha+c\beta\\ \alpha+\beta\end{pmatrix}, \qquad \tilde{u}=u\,, \tag{2.4}$$

where α, β, and θ are constants that parametrize the group action. The prolonged action on u_x and u_t is given explicitly by

$$g.u_x=\tilde{u}_x=\tilde{D}_x\tilde{u}, \qquad g\cdot u_t=\tilde{u}_t=\tilde{D}_t\tilde{u}.$$

The transformed total differentiation operators $\tilde{D}_i$ are defined by

$$\tilde{D}_i=\frac{d}{d\tilde{x}_i}=\sum_{k=1}^{p}\left((d\tilde{x}/dx)^{-T}\right)_{ik}D_k,$$

where $d\tilde{x}/dx$ is the Jacobian matrix. So,

$$\tilde{u}_x=\cosh(c\theta)u_x-\frac{1}{c}\sinh(c\theta)u_t, \qquad \tilde{u}_t=-c\sinh(c\theta)u_x+\cosh(c\theta)u_t.$$

If we take M to be the space with coordinates $(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt},\cdots)$, then the action is locally free near the identity of hyperbolic group G and regular. So, if we take the normalization equations to be $\tilde{x}=0$, $\tilde{t}=0$ and

$\tilde{u}_x = 0$, we obtain

$$\alpha = -\frac{1}{2c}.\frac{(ct-x)(cu_x - u_t)}{\sqrt{-c^2u_x^2+u_t^2}}, \quad \beta = \frac{1}{2c}.\frac{cxu_x + c^2tu_x + xu_t + ctu_t}{\sqrt{-c^2u_x^2+u_t^2}}, \quad \theta = \frac{1}{c}.\ln\left(\frac{\sqrt{-c^2u_x^2+u_t^2}}{cu_x - u_t}\right), \tag{2.5}$$

as the frame in parametric form.

Theorem 2.4. *Let $\rho(z)$ be a right moving frame. Then the quantity $I(z) = \rho(z)\cdot z$ is an invariant of the group action. (See [3], Theorem 4.5. in page 11)*

Consider $\mathbf{z} = (z_1, \cdots, z_n) \in M$, and let the normalization equations $\tilde{z}_i = c_i$ for $i = 1, \cdots, r$, then

$$\rho(\mathbf{z})\cdot \mathbf{z} = (c_1, \cdots, c_r, I(z_{r+1}), \cdots, I(z_n)),$$

where

$$I(z_k) = g\cdot z_k|_{g=\rho(z)}, \qquad k = r+1, \cdots, n.$$

Definition 2.5. For any prolonged action in the jet space $M = J^n(X \times U)$, the invariantized jet coordinates are denoted as

$$J^i = I(x_i) = \tilde{x}_i\,|_{g=\rho(z)}\,, \qquad I_k^\alpha = I(u_k^\alpha) = \tilde{u}_k^\alpha\,|_{g=\rho(z)}\,.$$

These are also known as the *normalized differential invariants*. According to Replacement Theorem [10], any invariant is a function of the $I(z_k)$. Particularly, the set $\{J^i, I_k^\alpha\}$ is a complete set of differential invariants for a prolonged action.

Now, we turn our attention to considering the invariants for the Example 2.3.

Example 2.3 (Continuing). The normalized differential invariants up to order two are as follows

$$\begin{aligned} g\cdot z &= (\tilde{x}, \tilde{t}, \tilde{u}, \tilde{u}_x, \tilde{u}_t, \tilde{u}_{xx}, \tilde{u}_{xt}, \tilde{u}_{tt})\,|_{g=\rho(z)} \\ &= (J^x, J^t, I^u, I_1^u, I_2^u, I_{11}^u, I_{12}^u, I_{22}^u) \\ &= \Big(0, 0, u, 0, -\sqrt{-c^2u_x^2+u_t^2}, -\frac{u_{xx}u_t^2 - 2u_{xt}u_xu_t + u_{tt}u_x^2}{c^2u_x^2 - u_t^2}, \\ &- \frac{-c^2u_{xx}u_xu_t + c^2u_{xt}u_x^2 + u_{xt}u_t^2 - u_{tt}u_xu_t}{c^2u_x^2 - u_t^2}, -\frac{c^4u_{xx}u_x^2 - 2c^2u_{xt}u_xu_t + u_{tt}u_t^2}{c^2u_x^2 - u_t^2}\Big). \end{aligned}$$

The first, second and fourth components correspond to the normalization equations and are known as the *phantom invariants*.

Definition 2.6. The *invariant differential operators* denoted as

$$\mathcal{D}_i = \tilde{D}_i\,|_{g=\rho(z)}\,, \qquad \tilde{D}_i = \frac{d}{d\tilde{x}_i} = \sum_{k=1}^{p}\left(\left({}^{d\tilde{x}}\!/_{dx}\right)^{-T}\right)_{ik} D_k,$$

where these invariant differentiation operators map differential invariants to differential invariants.

We know that $\partial u_k^\alpha/\partial x_i = u_{ki}^\alpha$, although the same is not true for their invariantized version, it means that in general

$$\mathcal{D}_i I_k^\alpha \neq I_{ki}^\alpha.$$

Definition 2.7. *Invariant differentiation* of the jet coordinates, J^i and I_k^α, are defined respectively, as

$$\mathcal{D}_j J^i = \delta_{ij} + N_{ij}\,, \qquad \mathcal{D}_j I_K^\alpha = I_{Kj}^\alpha + M_{Kj}^\alpha\,,$$

where δ_{ij} is the Kronecker delta, and N_{ij} and M_{Kj}^α are the correction terms. For more information on correction terms see page 133 in [10].

If we consider the two generating invariants I^{α}_{J} and I^{α}_{L}, and let $JK = LM$ so that $I^{\alpha}_{JK} = I^{\alpha}_{LM}$. This implies that

$$\mathcal{D}_K I^{\alpha}_{J} - \mathcal{D}_M I^{\alpha}_{L} = M^{\alpha}_{JK} - M^{\alpha}_{LM}.$$

These equations are called *syzygies* or *differential identities.* For more information on syzygies, see section 5 in [10]. To obtain the correction terms, we define the following notion of infinitesimals of a prolonged group action (for more details see [10]). Let G be a group parametrized by $a_1, \cdots, a_r$, where $r = \dim(G)$, in a neighbourhood of the identity element. The *infinitesimals of the prolonged group* action with respect to these parameters are

$$\xi^j_i = \frac{\partial \tilde{x}_j}{\partial a_i}|_{g=e}\ , \qquad \phi^{\alpha}_{K,j} = \frac{\partial \tilde{u}^{\alpha}_K}{\partial a_j}|_{g=e}\ .$$

Also, Let the normalisation equations be $\{\psi_\lambda(z) = 0,\ \lambda = 1, \cdots, r\}$ and suppose the n variables actually occurring in the $\psi_\lambda(z)$ are $\zeta_1, \cdots, \zeta_n$ such that m of these are independent variables and $n-m$ of them are dependent variables and their derivatives. Let $\mathbf{T}$ denote the invariant $p \times n$ total derivative matrix

$$\mathbf{T}_{ij} = I\left(\frac{D}{Dx_i}\zeta_j\right),$$

and define ϕ to be the $r \times n$ matrix as follows,

$$\phi_{ij} = \left(\frac{\partial(g\cdot\zeta_j)}{\partial g_i}|_{g=e}\right)(I),$$

and $\mathbf{J}$ to be the $n \times r$ matrix

$$\mathbf{J}_{ij} = \frac{\partial \psi_j(I)}{\partial I(\zeta_i)},$$

that is, transpose of the Jacobian matrix of the normalisation equations $\psi_1, \cdots, \psi_r$, with invariantised arguments.

Using the above defined matrices, the correction terms can be obtained as follows, which has been proved in [10].

Theorem 2.8. *The formulae for the correction terms are*

$$N_{ij} = \sum_{l=1}^{r} \mathbf{K}_{jl}\xi^i_l(I), \qquad M^{\alpha}_{Kj} = \sum_{l=1}^{r} \mathbf{K}_{jl}\phi^{\alpha}_{K,l}(I),$$

where l is the index for the group parameters, $r = \dim(G)$, and the $p \times r$ correction matrix $\mathbf{K}$, is given by

$$\mathbf{K} = -\mathbf{T}\mathbf{J}(\phi\mathbf{J})^{-1}.$$

Now, we calculate the invariant differentiation of the jet coordinates and the syzygies of the transformation (2.4) in Example 2.3.

Example 2.3 (Continuing). If we set $u = u(x,t,\tau)$ and $\tilde{\tau} = \tau$ and take the normalization equations as before, we obtain

$$\begin{aligned}
\tilde{u}_\tau\Big|_{g=\rho(z)} &= I^u_3 = u_\tau,\\
\tilde{u}_t\Big|_{g=\rho(z)} &= I^u_2 = -\sqrt{-c^2u_x^2 + u_t^2},\\
\tilde{u}_{xx}\Big|_{g=\rho(z)} &= I^u_{11} = -\frac{u_{xx}u_t^2 - 2u_{xt}u_xu_t + u_{tt}u_x^2}{c^2u_x^2 - u_t^2},\\
\tilde{u}_{xt}\Big|_{g=\rho(z)} &= I^u_{12} = -\frac{-c^2u_{xx}u_xu_t + c^2u_{xt}u_x^2 + u_{xt}u_t^2 - u_{tt}u_xu_t}{c^2u_x^2 - u_t^2},\\
\tilde{u}_{tt}\Big|_{g=\rho(z)} &= I^u_{22} = -\frac{c^4u_{xx}u_x^2 - 2c^2u_{xt}u_xu_t + u_{tt}u_t^2}{c^2u_x^2 - u_t^2}.
\end{aligned}$$

According to Theorem 2.8 we obtain the Invariant differentiation of the jet coordinates as follows

$$\mathcal{D}_x I_2^u = I_{12}^u, \qquad \mathcal{D}_t I_2^u = I_{22}^u, \qquad \mathcal{D}_\tau I_2^u = I_{23}^u,$$

$$\mathcal{D}_x I_{11}^u = I_{111}^u - \frac{2I_{11}^u I_{12}^u}{I_2^u}, \qquad \mathcal{D}_t I_{11}^u = I_{112}^u - \frac{2(I_{12}^u)^2}{I_2^u}, \qquad \mathcal{D}_\tau I_{11}^u = I_{113}^u - \frac{2I_{12}^u I_{13}^u}{I_2^u},$$

$$\mathcal{D}_x I_{22}^u = I_{122}^u - \frac{2c^2 I_{11}^u I_{12}^u}{I_2^u}, \qquad \mathcal{D}_t I_{22}^u = I_{222}^u - \frac{2c^2(I_{12}^u)^2}{I_2^u}, \qquad \mathcal{D}_\tau I_{22}^u = I_{223}^u - \frac{2c^2 I_{12}^u I_{13}^u}{I_2^u},$$

$$\mathcal{D}_x I_{12}^u = I_{112}^u - \frac{I_{11}^u}{I_2^u}(c^2 I_{11}^u + I_{22}^u), \qquad \mathcal{D}_t I_{12}^u = I_{122}^u - \frac{I_{12}^u}{I_2^u}(c^2 I_{11}^u + I_{22}^u), \qquad \mathcal{D}_\tau I_{12}^u = I_{123}^u - \frac{I_{13}^u}{I_2^u}(c^2 I_{11}^u + I_{22}^u).$$

We know that there are two ways to reach I_{112}^u and since both ways must be equal, we get the following syzygy between I^u and I_{11}^u:

$$\mathcal{D}_2 I^u \left((\mathcal{D}_1)^2 \mathcal{D}_2 I^u - \mathcal{D}_2 I_{11}^u \right) + (I_{11}^u)^2 + I_{11}^u (\mathcal{D}_2)^2 I^u - 2(\mathcal{D}_1 \mathcal{D}_2 I^u)^2 = 0.$$

Similarly, there are two possibilities to obtain I_{113}^u, so we get a syzygy between I_3^u and I_{11}^u and the syzygy is:

$$\mathcal{D}_3 I_{11}^u = \left((\mathcal{D}_1)^2 - \frac{2I_{12}^u \mathcal{D}_1}{I_2^u} + \frac{I_{11}^u \mathcal{D}_2}{I_2^u} \right) I_3^u, \tag{2.6}$$

and likewise, the syzygy between I_3^u and I_{22}^u is:

$$\mathcal{D}_3 I_{22}^u = \left((\mathcal{D}_2)^2 - \frac{c^2 I_{12}^u \mathcal{D}_1}{I_2^u} \right) I_3^u. \tag{2.7}$$

Finally, there are two syzygies between I_3^u and I_{12}^u, which are as follows:

$$\mathcal{D}_3 I_{12}^u = \left(\mathcal{D}_1 \mathcal{D}_2 - \frac{I_{22}^u \mathcal{D}_1}{I_2^u} \right) I_3^u, \tag{2.8}$$

$$\mathcal{D}_3 I_{12}^u = \left(\mathcal{D}_2 \mathcal{D}_1 + \frac{I_{12}^u \mathcal{D}_2}{I_2^u} - \frac{c^2 I_{11}^u \mathcal{D}_1}{I_2^u} - \frac{I_{22}^u \mathcal{D}_1}{I_2^u} \right) I_3^u. \tag{2.9}$$

From Equations (2.8) and (2.9), we can verify that the invariant operators $\mathcal{D}_x$ and $\mathcal{D}_t$ do not commute. In general, the invariant total differentiation operators do not commute. In fact, we have the following theorem.

Theorem 2.9. *[3] Denote the invariantized derivatives of the infinitesimals ξ_l^k, for $k, i = 1, \cdots, p$ and $l = 1, \cdots, r$, by*

$$\Xi_{li}^k = \tilde{D}_i \xi_l^k(\tilde{z}) \big|_{g=\rho(z)},$$

then the commutators are given by

$$[\mathcal{D}_i, \mathcal{D}_j] = \sum_{k=1}^{p} A_{ij}^k \mathcal{D}_k, \qquad A_{ij}^k = \sum_{l=1}^{r} K_{jl} \Xi_{li}^k - K_{il} \Xi_{lj}^k.$$

We now define invariant one-forms that will be required in the next section.

Definition 2.10. The invariant one-forms are denoted as

$$I(dx_i) = d\tilde{x}_i \big|_{g=\rho(z)} = \left(\sum_{j=1}^{p} D_j(\tilde{x}_i) dx_j \right) \big|_{g=\rho(z)}.$$

As for differential invariants, the invariant total differentiation operators send invariant differential forms to invariant differential forms.

Remark 2.11. Let the invariant differential operator $\mathcal{D}_i$ be associated to the vector field V_i as follows

$$\mathcal{D}_i = f_1(z)D_1 + \cdots + f_p(z)D_p \leftrightarrow V_i = (f_1(z), \cdots, f_p(z)).$$

Then $\mathcal{D}_i(\omega)$, denote as the *Lie derivative*

$$\mathcal{D}_i(\omega) = d(V_i \rfloor \omega) + V_i \rfloor d(\omega),$$

where d is the usual exterior derivative, and $\rfloor$ is the interior product of a vector field with a form.

Theorem 2.12. *[5] Consider the set of invariant total differentiation operators, $\{\mathcal{D}_i\}$, and the set of invariant one-forms, $\{I(dx_j)\}$. So if*

$$\mathcal{D}_i(I(dx_j)) = \sum_{k=1}^{p} B_{ij}^k I(dx_k),$$

then $B_{ki}^j = A_{jk}^i$.

Finally, in the end of this section, from the above theorem we obtain the Lie derivatives of $I(dx_j)$ with respect to $\mathcal{D}_i$ for the hyperbolic group action on (x, t, τ), that has been given in Example 2.3.

Example 2.3 (Continuing). Recall that $g \in G$ (the hyperbolic group) acts on (x, t, τ), where t is an invariant dummy independent variable introduced to effect variation. So the Lie derivatives of $I(dx_j)$ with respect to $\mathcal{D}_i$ are as shown in Table 1.

Table 1. Lie derivatives of the $I(dx_j)$ with respect to the $\mathcal{D}_i$

Lie derivative	$I(dx)$	$I(dt)$	$I(d\tau)$
$\mathcal{D}_x$	$c^2.\frac{I_{11}^u}{I_2^u}I(dt)$	$-\frac{I_{12}^u}{I_2^u}I(dt) - \frac{I_{13}^u}{I_2^u}I(d\tau)$	0
$\mathcal{D}_t$	$c^2\big(-\frac{I_{11}^u}{I_2^u}I(dx) - \frac{I_{13}^u}{I_2^u}I(d\tau)\big)$	$\frac{I_{12}^u}{I_2^u}I(dx)$	0
$\mathcal{D}_\tau$	$c^2.\frac{I_{13}^u}{I_2^u}I(dt)$	$\frac{I_{13}^u}{I_2^u}I(dx)$	0

3. Invariant calculus of variations and structure of Noether's conservation laws

We assume Lagrangians to be smooth functions of $\mathbf{x} = (x_1, \cdots, x_p)$, $\mathbf{u} = (u^1, \cdots, u^q)$ and finitely many derivatives of u^α and denote them as $\bar{\Phi}[\mathbf{u}] = \int \bar{L}[\mathbf{u}]\, d\mathbf{x}$, where $d\mathbf{x} = dx_1 \cdots dx_p$. Furthermore, suppose these are invariant under some group action and let the κ_j, for $j = 1, \cdots, N$, be the generating differential invariants of the group action. We can then rewrite $\bar{\Phi}(\mathbf{u})$ as $\Phi[\kappa] = \int L[\kappa]\, I(d\mathbf{x})$, where $I(d\mathbf{x}) = I(dx_1) \cdots I(dx_p)$ is the invariant volume form. According to [9], recall that if $\mathbf{x} \to (\mathbf{x}, \mathbf{u}(\mathbf{x}))$ extremizes the functional $\bar{\Phi}(\mathbf{u})$, then for a small perturbation of $\mathbf{u}$

$$0 = \frac{d}{d\varepsilon}\,|_{\varepsilon=0}\; \bar{\Phi}[\mathbf{u} + \varepsilon \mathbf{v}] = \int \sum_{\alpha=1}^{q} \Big[E^\alpha(\bar{L})v^\alpha + \sum_{i=1}^{p} \frac{d}{dx_i}\Big(\frac{\partial \bar{L}}{\partial u_i^\alpha}v^\alpha + \cdots\Big)\Big] d\mathbf{x},$$

where

$$E^\alpha = \sum_K (-1)^K \frac{D^{|K|}}{Dx_1^{k_1} Dx_2^{k_2} \cdots Dx_p^{k_p}} \frac{\partial}{\partial u_K^\alpha},$$

is the Euler operator with respect to the dependent variable u^α, and symbolically,

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \bar{\Phi}[\mathbf{u} + \varepsilon \mathbf{v}] = \frac{d}{d\tau}\,|_{u_\tau = v} \bar{\Phi}[\mathbf{u}]\ .$$

According to [5], we have

$$\begin{aligned} 0 &= \mathcal{D}_t \int L[\kappa]\, I(d\mathbf{x}) = \mathcal{D}_{p+1} \int L[\kappa]\, I(d\mathbf{x}) \\ &= \int \Big(\sum_{\alpha} E^{\alpha}(L) I^{\alpha}_{\tau} I(dx) + \sum_{i=1}^{p} \mathcal{D}_i \big[\sum_{j=1}^{p+1} F_{ij} I(dx_1) \cdots \widehat{I(dx_j)} \cdots I(dx_{p+1}) \big] \Big), \end{aligned}$$

where $E^{\alpha}(L)$ are the invariantized Euler-Lagrange equations, F_{ij} depends on $I^{\alpha}_{K,p+1}$ and I^{α}_{J} with K and J multi-indices of differentiation with respect to x_i, for $i = 1, \cdots, p$, and

$$I(dx_1) \cdots \widehat{I(dx_j)} \cdots I(dx_{p+1}) = I(dx_1) \cdots I(dx_{j-1}) I(dx_{j+1}) \cdots I(dx_{p+1}).$$

Theorem 3.1. *[5] The process of calculating the invariantized Euler-Lagrange equations produces boundary terms*

$$\int \sum_{i=1}^{p} \mathcal{D}_i \Big(\sum_{j=1}^{p+1} F_{ij} I(dx_1) \cdots \widehat{I(dx_j)} \cdots I(dx_{p+1}) \Big),$$

that can be written as

$$\int \sum_{i=1}^{p} d\Big((-1)^{i-1} \Big[\sum_{K,\alpha} I^{\alpha}_{K,\tau} C^{\alpha}_{K,i} \Big] I(dx_1) \cdots \widehat{I(dx_j)} \cdots I(dx_{p+1}) \Big),$$

where K is a multi-index of differentiation with respect to x_i, for $i = 1, \cdots, p$, and $C^{\alpha}_{K,i}$ are functions of I^{α}_{J}, with J a multi-index of differentiation with respect to x_i.

Now, in this section we consider the variational problem (2.1), that its Euler-Lagrange equation is the nonlinear Klein-Gordon equation (1.2).

Example 3.2. Consider the variational problem

$$\Phi[u] = \iint \frac{1}{4} \big(-2u_t^2 + 2c^2 u_x^2 + 2ku^2 + \gamma u^4 \big) dx dt, \tag{3.1}$$

which is invariant under the action (2.4). To find the invariantized Euler-Lagrange equation, introduce a dummy invariant independent variable τ to effect the variation, and set $u = u(x, t, \tau)$, therefore $\tilde{u}_{\tau} \big|_{g=\rho(z)} = I^u_3 = u_{\tau}$. Rewriting the above variational problem in terms of the invariants of the group action yields

$$\iint \frac{1}{4} \Big(-2(I^u_2)^2 + 2c^2 (I^u_1)^2 + 2k(I^u)^2 + \gamma (I^u)^4 \Big) I(dx) I(dt). \tag{3.2}$$

To obtain the invariantized Euler-Lagrange equation and boundary terms, after differentiating (3.2) under the integral sign we obtain

$$\begin{aligned} &\mathcal{D}_{\tau} \iint \frac{1}{4} \Big(-2(I^u_2)^2 + 2c^2 (I^u_1)^2 + 2k(I^u)^2 + \gamma (I^u)^4 \Big) I(dx) I(dt) \\ &= \iint \Big[\Big(-I^u_2 . \mathcal{D}_{\tau}(I^u_2) + c^2 I^u_1 . \mathcal{D}_{\tau}(I^u_1) + k I^u . \mathcal{D}_{\tau}(I^u) + \gamma (I^u)^3 . \mathcal{D}_{\tau}(I^u) \Big) I(dx) I(dt) \\ &+ \frac{1}{4} \Big(-2(I^u_2)^2 + 2c^2 (I^u_1)^2 + 2k(I^u)^2 + \gamma (I^u)^4 \Big) \mathcal{D}_{\tau} \left(I(dx) I(dt) \right) \Big]. \end{aligned}$$

Using Table 1 we see that $\mathcal{D}_{\tau} \left(I(dx) I(dt) \right) = 0$. Then substituting $\mathcal{D}_t I^u_{11}$ by (2.6), $\mathcal{D}_t I^u_{22}$ by (2.7), $\mathcal{D}_t I^u_{12}$ by (2.8), and substituting I^u_1 equal to zero in the second integral, and performing integration by parts yields

$$\iint \Big(I^u_{22} - c^2 I^u_{11} + k I^u + \gamma (I^u)^3 \Big) I(dx) I(dt) + \iint \mathcal{D}_t \Big(-I^u_2 I^u_3 I(dx) I(dt) \Big).$$

Thus, we obtain the invariantized Euler-Lagrange equation

$$E^u(L) = I^u_{22} - c^2 I^u_{11} + k I^u + \gamma (I^u)^3 = u_{tt} - c^2 u_{xx} + ku + \gamma u^3.$$

Therefore, according to the above theorem, the boundary terms can be written as

$$\iint d(I_2^u I_3^u I(dx)). \tag{3.3}$$

Theorem 3.3. *[5] Let $\int L(k_1, k_2, \cdots) I(d\mathbf{x})$ be invariant under $G \times M \to M$, where $M = J^n(X, U)$, with generating invariants κ_j, for $j = 1, \cdots .N$. Introduce a dummy invariant variable t to effect the variation and then integration by parts yields*

$$\mathcal{D}_t \int L(k_1, k_2, \cdots) I(d\mathbf{x}) = \int \Big[\sum_\alpha E^\alpha(L) I_t^\alpha I(d\mathbf{x}) + \sum_{k=1}^{p} d\Big((-1)^{k-1}\Big(\sum_{J,\alpha} I_{Jt}^\alpha C_{J,k}^\alpha\Big) I(dx_1)\cdots\widehat{I(dx_k)}\cdots I(dx_{p+1})\Big)\Big],$$

where this defines the vectors $\mathbf{C}_k^\alpha = (C_{J,k}^\alpha)$. Recall that $E^\alpha(L)$ are the invariantized Euler- Lagrange equations and $I_{Jt}^\alpha = I(u_{Jt}^\alpha)$, where J is a multi-index of differentiation with respect to the variables x_i, for $i = 1, \cdots, p$. Let $(a_1, \cdots, a_r)$ be the coordinates of G near the identity e, and $\mathbf{v}_i$, for $i = 1, \cdots, r$, the associated infinitesimal vector fields. Furthermore, let $Ad(g)$ be the Adjoint representation of G with respect to these vector fields. For each dependent variable, define the matrices of characteristics to be,

$$\mathcal{Q}^\alpha(\tilde{z}) = (\widetilde{D_K(Q_i^\alpha)}), \qquad \alpha = 1, \cdots, q,$$

where K is a multi-index of differentiation with respect to the x_k, and

$$Q_i^\alpha = \phi_i^\alpha - \sum_{k=1}^{p} \xi_i^k u_k^\alpha = \frac{\partial \widetilde{u^\alpha}}{\partial a_i}\Big|_{g=e} - \sum_{k=1}^{p} \frac{\partial \widetilde{x_k}}{\partial a_i}\Big|_{g=e} u_k^\alpha,$$

are the components of the q-tuple $\mathbf{Q}_i$ known as the characteristic of the vector field $\mathbf{v}_i$. Let $\mathcal{Q}^\alpha(J, I)$, for $\alpha = 1, \cdots, q$, be the invariantization of the above matrices. Then, the r conservation laws obtained via Noether's Theorem can be written in the form,

$$d(Ad(\rho)^{-1}(\upsilon_1, \cdots, \upsilon_p) M_{\mathcal{J}} d^{p-1}\hat{\mathbf{x}}) = 0\,,$$

where

$$\upsilon_k = \sum_\alpha (-1)^{k-1}(\mathcal{Q}^\alpha(J, I)\mathbf{C}_k^\alpha + L(\Xi(J, I))_k),$$

are the vectors of invariants, with $(\Xi(J, I))_k$ the k^{th} column of $\Xi(J, I)$, $M_{\mathcal{J}}$ is the matrix of first minors of the Jacobian matrix evaluated at the frame, $\mathcal{J} = {d\tilde{x}}/{dx}\,|_{g=\rho(z)}$, and

$$d^{p-1}\hat{x} = \begin{pmatrix} \widehat{dx_1} dx_2 \cdots dx_p \\ dx_1 \widehat{dx_2} dx_3 \cdots dx_p \\ \vdots \\ dx_1 \cdots dx_{p-1}\widehat{dx_p} \end{pmatrix} = \begin{pmatrix} dx_2 dx_3 \cdots dx_p \\ dx_1 dx_3 \cdots dx_p \\ \vdots \\ dx_1 dx_2 \cdots dx_{p-1} \end{pmatrix}.$$

Lemma 3.4. *The inverse of the Adjoint representation of the hyperbolic group G with respect to its generating vector fields evaluated at the frame (2.5) is*

$$Ad(\rho(z))^{-1} = \begin{bmatrix} \dfrac{cu_x - u_t}{\sqrt{-c^2u_x^2 + u_t^2}} & 0 & 0 \\ 0 & \dfrac{\sqrt{-c^2u_x^2 + u_t^2}}{cu_x - u_t} & 0 \\ -\dfrac{1}{2}\dfrac{(ct - x)(cu_x - u_t)}{\sqrt{-c^2u_x^2 + u_t^2}} & \dfrac{1}{2}\dfrac{(x + ct)\sqrt{-c^2u_x^2 + u_t^2}}{cu_x - u_t} & 1 \end{bmatrix}. \tag{3.4}$$

Proof. Consider the action (2.4) and let it act on the infinitesimal vector fields generating the hyperbolic group G,

$$\mathbf{v}_1 = -c\partial_x + \partial_t, \qquad \mathbf{v}_2 = c\partial_x + \partial_t, \qquad \mathbf{v}_3 = c^2 t\partial_x + x\partial_t,$$

as follow

$$\begin{aligned}
g.\big(\alpha'(-c\partial_x+\partial_t)+\beta'(c\partial_x+\partial_t)+\gamma'(c^2t\partial_x+x\partial_t)\big) &= \alpha'(-c\partial_{\tilde{x}}+\partial_{\tilde{t}})+\beta'(c\partial_{\tilde{x}}+\partial_{\tilde{t}})+\gamma'(c^2\tilde{t}\partial_{\tilde{x}}+\tilde{x}\partial_{\tilde{t}})\\
&= \alpha'(-c\cosh(c\theta)\partial_x+\sinh(c\theta)\partial_t-c\sinh(c\theta)\partial_x+\cosh(c\theta)\partial_t)\\
&+\beta'(c\cosh(c\theta)\partial_x-\sinh(c\theta)\partial_t-c\sinh(c\theta)\partial_x+\cosh(c\theta)\partial_t)\\
&+\gamma'\big[c^2(\alpha+\beta+\frac{x}{c}\sinh(c\theta)+t\cosh(c\theta))(\cosh(c\theta)\partial_x-\frac{1}{c}\sinh(c\theta)\partial_t)\\
&+(-c\alpha+c\beta+x\cosh(c\theta)+ct\sinh(c\theta))(-c\sinh(c\theta)\partial_x+\cosh(c\theta)\partial_t)\big]\\
&= \alpha'[(\cosh(c\theta)+\sinh(c\theta))(-c\partial_x+\partial_t)]\\
&+\beta'[(\cosh(c\theta)-\sinh(c\theta))(c\partial_x+\partial_t)]\\
&+\gamma'\big[-c\alpha\big(\cosh(c\theta)+\sinh(c\theta)\big)(-c\partial x+\partial t)\\
&+c\beta\big(\cosh(c\theta)-\sinh(c\theta)\big)(c\partial x+\partial t)+(c^2t\partial x+x\partial t)\big]
\end{aligned}$$

$$= \begin{pmatrix}\alpha' & \beta' & \gamma'\end{pmatrix}\begin{pmatrix}\cosh(c\theta)+\sinh(c\theta) & 0 & 0\\ 0 & \cosh(c\theta)-\sinh(c\theta) & 0\\ -c\alpha\,(\cosh(c\theta)+\sinh(c\theta)) & c\beta\,(\cosh(c\theta)-\sinh(c\theta)) & 1\end{pmatrix}\times\begin{pmatrix}-c\partial_x+\partial_t\\ c\partial_x+\partial_t\\ c^2t\partial_x+x\partial_t\end{pmatrix},$$

where the above 3×3 matrix, $Ad(g)$, is the Adjoint representation of G with respect to its generating infinitesimal vector fields. So $Ad(g)^{-1}$ is

$$Ad(g)^{-1} = \begin{pmatrix}\cosh(c\theta)-\sinh(c\theta) & 0 & 0\\ 0 & \cosh(c\theta)+\sinh(c\theta) & 0\\ c\alpha & -c\beta & 1\end{pmatrix}.$$

Now evaluating $Ad(g)^{-1}$ at the frame (2.5), we obtain

$$Ad(\rho(z))^{-1} = \begin{bmatrix}\frac{cu_x-u_t}{\sqrt{-c^2u_x^2+u_t^2}}\\ 0\\ 0\\ 0\\ \frac{\sqrt{-c^2u_x^2+u_t^2}}{cu_x-u_t}\\ 0\\ -\frac{1}{2}\frac{(ct-x)(cu_x-u_t)}{\sqrt{-c^2u_x^2+u_t^2}}\\ \frac{1}{2}\frac{(x+ct)\sqrt{-c^2u_x^2+u_t^2}}{cu_x-u_t}\\ 1\end{bmatrix}.$$

□

We now calculate the Noether's conservation laws of Euler-Lagrange equations for the variational problem (3.1), namely, the nonlinear Klein-Gordon equation (1.2).

Theorem 3.5. *The three Noether's conservation laws of Euler-Lagrange equations for the variational problem*

$$\iint \frac{1}{4}\big(-2u_t^2+2c^2u_x^2+2ku^2+\gamma u^4\big)dxdt,$$

are

$$d\left(\begin{bmatrix} \dfrac{cu_x-u_t}{\sqrt{-c^2u_x^2+u_t^2}} & 0 & 0 \\ 0 & \dfrac{\sqrt{-c^2u_x^2+u_t^2}}{cu_x-u_t} & 0 \\ -\dfrac{1}{2}\dfrac{(ct-x)(cu_x-u_t)}{\sqrt{-c^2u_x^2+u_t^2}} & \dfrac{1}{2}\dfrac{(x+ct)\sqrt{-c^2u_x^2+u_t^2}}{cu_x-u_t} & 1 \end{bmatrix}\right.$$
$$\times\begin{bmatrix} \frac{1}{4}\big(2c(I_2^u)^2-2ck(I^u)^2-c\gamma(I^u)^4\big) & \frac{1}{4}\big(-2(I_2^u)^2-2k(I^u)^2-\gamma(I^u)^4\big) \\ \frac{1}{4}\big(-2c(I_2^u)^2+2ck(I^u)^2+c\gamma(I^u)^4\big) & \frac{1}{4}\big(-2(I_2^u)^2-2k(I^u)^2-\gamma(I^u)^4\big) \\ 0 & 0 \end{bmatrix}$$
$$\left.\times\begin{bmatrix} \dfrac{-u_t}{\sqrt{-c^2u_x^2+u_t^2}} & \dfrac{-u_x}{\sqrt{-c^2u_x^2+u_t^2}} \\ \dfrac{-c^2u_x}{\sqrt{-c^2u_x^2+u_t^2}} & \dfrac{-u_t}{\sqrt{-c^2u_x^2+u_t^2}} \end{bmatrix}\begin{bmatrix} dt \\ dx \end{bmatrix}\right)=0.$$

Proof. According to Theorem 3.3 the elements of C_i^u correspond to the coefficients of the $I^{\alpha}_{J\tau}$ in (3.3), as follows:

$$C_1^u=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\qquad C_2^u=\begin{bmatrix} -I_2^u \\ 0 \\ 0 \end{bmatrix},$$

and the $(\Xi(J,I))_i$, for $i=1,2$, are

$$(\Xi(J,I))_1=\begin{array}{c} \\ \alpha \\ \beta \\ \theta \end{array}\begin{array}{c} \xi^x \\ \begin{pmatrix} -c \\ c \\ 0 \end{pmatrix} \end{array},\qquad (\Xi(J,I))_2=\begin{array}{c} \\ \alpha \\ \beta \\ \theta \end{array}\begin{array}{c} \xi^t \\ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \end{array}.$$

Since $I_1^u=0$, the invariantized matrix of characteristics is,

$$\mathcal{Q}^u(J,I)=\begin{array}{c} \\ \alpha \\ \beta \\ \theta \end{array}\begin{array}{c} \begin{array}{ccc} Q^u & D_x(Q^u) & D_t(Q^u) \end{array} \\ \begin{pmatrix} -I_2^u & cI_{11}^u-I_{12}^u & cI_{12}^u-I_{22}^u \\ -I_2^u & -cI_{11}^u-I_{12}^u & -cI_{12}^u-I_{22}^u \\ 0 & -I_2^u & 0 \end{pmatrix} \end{array},$$

thus, the vectors of invariants are

$$\upsilon_1=\begin{bmatrix} \frac{1}{4}\big(2c(I_2^u)^2-2ck(I^u)^2-c\gamma(I^u)^4\big) \\ \frac{1}{4}\big(-2c(I_2^u)^2+2ck(I^u)^2+c\gamma(I^u)^4\big) \\ 0 \end{bmatrix},\qquad \upsilon_2=\begin{bmatrix} \frac{1}{4}\big(-2(I_2^u)^2-2k(I^u)^2-\gamma(I^u)^4\big) \\ \frac{1}{4}\big(-2(I_2^u)^2-2k(I^u)^2-\gamma(I^u)^4\big) \\ 0 \end{bmatrix},$$

and according to Lemma 3.4 the inverse of the Adjoint representation $Ad(\rho)^{-1}$ is as (3.4). Finally, the Jacobian matrix $\mathcal{J}$ is

$$\begin{aligned}
\mathcal{J} &= \begin{bmatrix} \frac{\partial \tilde{x}}{\partial x}\big|_{g=\rho(z)} & \frac{\partial \tilde{x}}{\partial t}\big|_{g=\rho(z)} \\ \frac{\partial t}{\partial x}\big|_{g=\rho(z)} & \frac{\partial t}{\partial t}\big|_{g=\rho(z)} \end{bmatrix} \\
&= \begin{bmatrix} \cosh(c\theta) & c.\sinh(c\theta) \\ \\ \frac{1}{c}.\sinh(c\theta) & \cosh(c\theta) \end{bmatrix} \\
&= \begin{bmatrix} \frac{-u_t}{\sqrt{-c^2u_x^2+u_t^2}} & \frac{-c^2u_x}{\sqrt{-c^2u_x^2+u_t^2}} \\ \frac{-u_x}{\sqrt{-c^2u_x^2+u_t^2}} & \frac{-u_t}{\sqrt{-c^2u_x^2+u_t^2}} \end{bmatrix}
\end{aligned}$$

and its matrix of first minors, $M_{\mathcal{J}}$, is

$$M_{\mathcal{J}} = \begin{bmatrix} \frac{-u_t}{\sqrt{-c^2u_x^2+u_t^2}} & \frac{-u_x}{\sqrt{-c^2u_x^2+u_t^2}} \\ \frac{-c^2u_x}{\sqrt{-c^2u_x^2+u_t^2}} & \frac{-u_t}{\sqrt{-c^2u_x^2+u_t^2}} \end{bmatrix}.$$

Thus, the conservation laws are

$$d\Big(Ad(\rho)^{-1}.\big[\upsilon_1\ \upsilon_2\big].M_{\mathcal{J}}.d^1\hat{x}\Big) = 0,$$

where

$$d^1\hat{x} = \begin{bmatrix} dt \\ \\ dx \end{bmatrix}.$$

□

4. Concluding Remarks

We see that the three Noether's conservation laws of the nonlinear Klein-Gordon equation (1.2) are in terms of vectors of invariants, the adjoint representation of the moving frame and a matrix which represents the group action on the 1-forms. forms. Also, we notice that since equations (2.8) and (2.9) are equivalent, for calculation of boundary terms if we substitute $\mathcal{D}_\tau I^u_{12}$ by equation (2.9) instead of equation (2.8), or we use a combination of the two; in any case, the conservation laws are equivalent.

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References

[1] A. Bihlo, J. Jackaman, and F. Valiquette, *Invariant variational schemes for ordinary differential equations*, Studies in Applied Mathematics, *148* (2022), 991–1020.

[2] M. Dehghan and A. Shokri, *A numerical method for solution of the two-dimensional sine-Gordon equation using the radial basis functions*, Mathematics and Computers in Simulation, *79* (2008), 700–715.

[3] M. Fels and P. J. Olver, *Moving coframes: II. Regularization and theoretical foundations*, Acta Applicandae Mathematica, *55* (1999), 127–208.

[4] T. M. N. Gonçalves and E. L. Mansfield, *Moving frames and conservation laws for Euclidean invariant Lagrangians*, Studies in Applied Mathematics, *130* (2013), 134–166.

[5] T. M. N. Gonçalves and E. L. Mansfield, *Moving Frames and Noether's Conservation Laws - the General Case*, Vol. 4, Forum of Mathematics, Sigma, Cambridge University Press, 2016.

[6] T. M. N. Gonçalves and E. L. Mansfield, *On moving frames and Noether's conservation laws*, Studies in Applied Mathematics, *128* (2012), 1–29.

[7] D. Kaya and S. M. El-Sayed, *A numerical solution of the Klein-Gordon equation and convergence of the decomposition method*, Applied mathematics and computation, *156* (2004), 341–353.

[8] P. Kim, *Invariantization of numerical schemes using moving frames*, BIT Numerical Mathematics, *47* (2007), 525–546.

[9] I. A. Kogan and P. J. Olver, *Invariant Euler–Lagrange equations and the invariant variational bicomplex*, Acta Applicandae Mathematica, *76* (2003), 137–193.

[10] E. L. Mansfield, *A practical guide to the invariant calculus*, Vol. 26, Cambridge University Press, New York, 2010.

[11] E. Noether, *Invariant variation problems*, Transport theory and statistical physics, *1* (1971), 186–207.

[12] P. J. Olver, *Applications of Lie groups to differential equations*, Vol. 107, Springer Science and Business Media, 2000.

[13] E. Ozbenli and P. Vedula, *Construction of invariant compact finite-difference schemes*, Physical Review E, *101* (2020), 023303.

[14] E. Ozbenli and P. Vedula, *High order accurate finite difference schemes based on symmetry preservation*, Journal of Computational Physics, *349* (2017), 376–398.

[15] E. Ozbenli and P. Vedula, *Numerical solution of modified differential equations based on symmetry preservation*, Physical Review E, *96* (2017), 063304.

[16] F. Shakeri and M. Dehghan, *Numerical solution of the Klein-Gordon equation via He's variational iteration method*, Nonlinear Dynamics, *51* (2008), 89–97.

[17] Sirendaoreji, *Auxiliary equation method and new solutions of Klein-Gordon equations*, Chaos Solitons and Fractals, *31* (2007), 943–950.

[18] A. M. Wazwaz, *New travelling wave solutions to the Boussinesq and the Klein-Gordon equations*, Communications in Nonlinear Science and Numerical Simulation, *13* (2008), 889–901.

[19] A. M. Wazwaz, *The modified decomposition method for analytic treatment of differential equations*, Applied mathematics and computation, *173* (2006), 165–176.