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Modified Lucas polynomials for the numerical treatment of second-order boundary value problems

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Abstract

This paper is devoted to the construction of certain polynomials related to Lucas polynomials, namely, modified Lucas polynomials. The constructed modified Lucas polynomials are utilized as basis functions for the numerical treatment of the linear and non-linear second-order boundary value problems (BVPs) involving some specific important problems such as singular and Bratu-type equations. To derive our proposed algorithms, the operational matrix of derivatives of the modified Lucas polynomials is established by expressing the first-order derivative of these polynomials in terms of their original ones. The convergence analysis of the modified Lucas polynomials is deeply discussed by establishing some inequalities concerned with these modified polynomials. Some numerical experiments accompanied by comparisons with some other articles in the literature are presented to demonstrate the applicability and accuracy of the presented algorithms.

Keywords. Lucas polynomials; Boundary value problems; Bratu equations; Singular initial value problems; Spectral methods; Operational matrix; Convergence analysis.

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1. Introduction

Higher-order BVPs are important in physics, engineering, and applied mathematics. Because of this, a large number of authors are interested in developing numerical solutions for various BVPs. In this regard, for instance, recently, Abd-Elhameed and Alkenedri in [3] obtained spectral solutions of linear and non-linear even-order BVPs using certain Jacobi polynomials generalizing third-and fourth-kinds of Chebyshev polynomials. Other kinds of orthogonal polynomials, namely, generalized Jacobi polynomials were employed in [2, 21, 22] to treat some types of even-and odd-order BVPs.

BVPs of second-order occur in the mathematical modeling of cantilever beam displacements during concentrated load [16, 37], obstacle problems [41], thermal transport of the trapezoidal profile radiation fin [30, 37], and distortion of rays and plate perversion theory [25]. Several authors have used numerical methods to solve BVPs of second-order [12, 25, 30, 34, 37, 41, 50–52], such as the Walsh wavelets used in [25, 34] raising the numerical algorithm for the solution for BVPs of second-order under Neumann and Dirichlet boundary conditions.

Singular equations are among the most important higher-order differential equations found in many fields of applied mathematics, including elasticity, quantum mechanics, chemical reactor theory, and fluid dynamics.

The Bratu-type equation is the product of the simplification of the solid fuel fire model in the theory of thermal combustion. This form has various physical applications, including electrostatics, chemical reactor theory, radiation heat transfer, physical applications of fluid mechanics spanning from the chemical theory of reaction, and nanotechnology to expand the world [27]. Several numerical methods have been used to achieve an approximate solution to

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the problem of the Bratu-type such as the wavelet methods [40, 53, 56], Adomian decomposition method (ADM) [49], and spectral methods [4, 9, 17, 24].

Operational matrices of derivatives of different special functions are very important in solving different types of differential equations. Regarding some uses of the operational matrices in treating ordinary differential equations, Napoli and Abd-Elhameed [39] introduced an operational matrix of derivatives of a certain combination of Legendre polynomials. The nonzero elements of this matrix are given in terms of harmonic numbers. In addition, the authors employed this operational matrix to treat the even-order BVPs. Regarding the fractional differential equations, for example, Youssri [58] introduced an operational matrix of Caputo fractional derivatives of Fermat polynomials and employed it for the numerical solution of the Bagley-Torvik equation. The same equation was treated via wavelets operational matrix of derivatives [15].

Fibonacci and Lucas polynomial sequences, as well as their generalized sequences, play a vital role in a variety of fields. These sequences are used to approximate the solutions to various forms of DEs. For instance, in [5], Fibonacci polynomials were utilized to treat multi-term fractional DEs. For the numerical treatment of the sinh-Gordon equation, Lucas polynomials are used in [42]. The authors in [33] derived a matrix approach for treating generalized pantograph equations with functional arguments using Fibonacci polynomials. To get numerical solutions to the Sobolev problem in two dimensions, the authors in [26] follow a different approach based on mixed Fibonacci and Lucas polynomials. To achieve numerical solutions of multidimensional Burgers-type equations, the authors in [13] employed Lucas polynomials. To solve the fractional-order electro-hydrodynamics flow model, Lucas polynomials were also used in [45].

Many methods play critical roles in numerical analysis [28, 29, 43, 59]. Between these methods seem to be spectral methods, which play important roles in dealing with differential and partial differential equations. Spectral methods are extremely accurate and frequently yield good numerical solutions to differential equations. They are not only differentiated by the type of process used (collocation, Galerkin, or tau) but also by choosing the basis functions.

Spectral methods are one of the most extensively used numerical strategies for solving various types of differential equations. There are some advantages of these methods if compared with other methods. These methods are global methods, unlike finite element methods. The main characteristic of the spectral method is based on two kinds of basis functions, namely, trial functions and test functions. These two kinds of functions are often expressed in terms of suitable orthogonal polynomials or combinations of them. The choice of the trial and test functions depends on the method that we choose. Different special polynomials such as Jacobi polynomials and their special classes may be used as basis functions (see [19]). The spectral Galerkin method is based on choosing two coincident sets of polynomials that satisfy the underlying conditions of the problem, see for example [23]. The tau method has the flexibility in choosing the two basis sets unlike the Galerkin method, see for example, the two papers of Abd-Elhameed and Youssri in [7, 8] in which they utilized the tau method to treat some types of fractional differential equations. The collocation method is the most common method in application due to its capability to handle any type of differential equations, see [38, 54]. It is well-known that there are three main categories of spectral methods, namely Galerkin, collocation and tau methods.

This paper's structure is as follows: The next section gives an overview of Lucas polynomials and introduces some modified Lucas polynomials that will be used in the following sections. Numerical techniques for addressing linear and nonlinear second-order BVPs are presented in section 3. The convergence analysis is investigated in section 4. In section 5, numerical findings are presented along with some comparisons and comments. Finally, some conclusions are presented in section 6.

2. An Overview on Lucas polynomials and introducing modified ones

In this section, we give an overview of Lucas polynomials and some of their basic properties. In addition, we introduce a new set of polynomials, namely, modified Lucas polynomials that are given as a combination of Lucas polynomials. We will call these polynomials, modified Lucas polynomials "$MLPs$".

2.1. **Lucas polynomials.** Lucas polynomials $L_r(x)$ may be built using the following recurrence relation:

$$L_r(x) = x\, L_{r-1}(x) + L_{r-2}(x), \quad L_0(x) = 2,\; L_1(x) = x, \quad r \geq 2, \tag{2.1}$$

or via the Binet's form

$$L_r(x) = \frac{\left(x + \sqrt{4+x^2}\right)^r + \left(x - \sqrt{4+x^2}\right)^r}{2^r}, \quad r \geq 0.$$

Lucas polynomials $L_r(x)$ have the corresponding power form representation ([6]):

$$L_r(x) = r \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \frac{(r-k-1)!}{k!\,(r-2k)!}\, x^{r-2k}, \quad r \geq 1, \tag{2.2}$$

where $\lfloor Q \rfloor$ denotes the largest integer less than or equal to Q.
Here are the following first few Lucas polynomials:

$$\begin{aligned}
&L_0(x) = 2,\\
&L_1(x) = x,\\
&L_2(x) = x^2 + 2,\\
&L_3(x) = x^3 + 3\,x,\\
&L_4(x) = x^4 + 4\,x^2 + 2.
\end{aligned}$$

The inversion formula of Lucas polynomials is given as

$$x^{\iota} = \iota! \sum_{\substack{r=0\\(r+\iota)\ \text{even}}}^{\iota} \frac{(-1)^{\frac{\iota-r}{2}}\,\sigma_r}{\left(\frac{\iota-r}{2}\right)!\,\left(\frac{\iota+r}{2}\right)!}\, L_r(x), \tag{2.3}$$

where σ_r is defined by

$$\sigma_r = \begin{cases} \frac{1}{2}, & r = 0,\\ 1, & r > 0. \end{cases} \tag{2.4}$$

2.2. **Introducing certain modified Lucas polynomials.** In this section, we will propose a set of polynomials related to the Lucas polynomials. We will call them modified Lucas polynomials. These polynomials will be employed to treat the second-order BVPs. They are given by the following formula:

$$\varphi_r(x) = x(1-x)\, L_r(x). \tag{2.5}$$

In the following, we will state and prove two theorems concerned with modified Lucas polynomials. The first gives the inversion formula of these polynomials, whereas the second gives an expression for the first-order derivative of these polynomials in terms of their original ones.

Theorem 1. *The next inversion formula*

$$x^{\iota+1}\,(1-x) = \iota! \sum_{\substack{r=0\\(r+\iota)\ \text{even}}}^{\iota} \frac{(-1)^{\frac{\iota-r}{2}}\,\sigma_r}{\left(\frac{\iota-r}{2}\right)!\,\left(\frac{\iota+r}{2}\right)!}\, \varphi_r(x), \tag{2.6}$$

is true for each positive ι.

Proof. It is sufficient to demonstrate relation (2.6), by proving its alternative form:

$$(x)^{\iota+1}\,(1-x) = \sum_{s=0}^{\lfloor \frac{\iota}{2} \rfloor} (-1)^s \binom{\iota}{s}\, \sigma_{\iota-2s}\, \varphi_{\iota-2s}(x). \tag{2.7}$$

We will start with induction on ι. Identity (2.7) is obviously satisfied for $\iota = 0$. Suppose the validity of (2.7). To complete the proof, we should demonstrate the validity of the next identity:

$$(x)^{\iota+2}(1-x) = \sum_{s=0}^{\lfloor\frac{\iota+1}{2}\rfloor} (-1)^s \binom{\iota+1}{s} \sigma_{\iota-2s+1}\,\varphi_{\iota-2s+1}(x). \tag{2.8}$$

Multiplying both sides of (2.7) by x and applying the recurrence relation (2.1), we get

$$(x)^{\iota+2}(1-x) = \sum_{s=0}^{\lfloor\frac{\iota}{2}\rfloor} (-1)^s \binom{\iota}{s} \sigma_{\iota-2s}\,\varphi_{\iota-2s+1}(x) - \sum_{s=0}^{\lfloor\frac{\iota}{2}\rfloor} (-1)^s \binom{\iota}{s} \sigma_{\iota-2s}\,\varphi_{\iota-2s-1}(x). \tag{2.9}$$

After operating some manipulations, the last identity can be written as:

$$\begin{aligned}(x)^{\iota+2}(1-x) =& \sigma_\iota\,\varphi_{\iota+1}(x) + \sum_{s=1}^{\lfloor\frac{\iota}{2}\rfloor}\left[(-1)^s\binom{\iota}{s}\sigma_{\iota-2s} + (-1)^s\binom{\iota}{s-1}\sigma_{\iota-2s+2}\right]\varphi_{\iota-2s+1}(x)\\ &+ (-1)^{\lfloor\frac{\iota}{2}\rfloor+1}\binom{\iota}{\lfloor\frac{\iota}{2}\rfloor}\sigma_{\iota-2\lfloor\frac{\iota}{2}\rfloor}\,\varphi_{\iota-2\lfloor\frac{\iota}{2}\rfloor-1}(x).\end{aligned} \tag{2.10}$$

After some algebraic calculations, it is possible to show that equation (2.10) has the form

$$(x)^{\iota+2}(1-x) = \sum_{s=0}^{\lfloor\frac{\iota+1}{2}\rfloor} (-1)^s \binom{\iota+1}{s} \sigma_{\iota-2s+1}\,\varphi_{\iota-2s+1}(x). \tag{2.11}$$

Theorem 1 is now proved. □

Theorem 2. *Let $\varphi_r(x)$ be the modified Lucas polynomials given in (2.5). Then for all $r \geq 1$, one has*

$$D\varphi_r(x) = \sum_{s=0}^{r-1} A_{r,s}\,\varphi_s(x) + \theta_r(x), \tag{2.12}$$

where

$$A_{r,s} = \begin{cases} B\lfloor\frac{-s}{2}\rfloor F_{r-s}, & (r+s)\ \text{even},\\ B[\frac{1}{4}\left(-1+r-s-2\lfloor\frac{r}{2}\rfloor\right)]\,(1+r+F_{r-s}), & \frac{r-s-1}{4}\in\mathbb{Z},\\ B[\frac{1}{4}\left(-1+r-s-2\lfloor\frac{r}{2}\rfloor\right)]\,(1-r+F_{r-s}), & \frac{r-s-3}{4}\in\mathbb{Z},\end{cases} \tag{2.13}$$

$$B(r) = \begin{cases}\frac{1}{2}, & r\ \text{even},\\ 1, & r\ \text{odd},\end{cases} \tag{2.14}$$

$$\theta_r(x) = \begin{cases} 2-(L_r+2)\,x, & r\ \text{even},\\ -x\,L_r, & r\ \text{odd},\end{cases} \tag{2.15}$$

and F_r and L_r represent, respectively, the Fibonacci and Lucas numbers.

Proof. We prove the theorem by induction on r. For $r = 1$, the the left-hand and right-hand sides of (2.12) are equal, which both equal $(2x - 3x^2)$. We assume that relation (2.12) is true for $(r-1)$ and $(r-2)$, and we will prove the validity of (2.12) itself.

Now, we begin with the recurrence relation

$$\varphi_r(x) = x\,\varphi_{r-1}(x) + \varphi_{r-2}(x), \quad r = 1, 2, \ldots, \tag{2.16}$$

which instantly gives by differentiation with respect to x, the following relation

$$D\varphi_r(x) = \varphi_{r-1}(x) + x\,D\varphi_{r-1}(x) + D\varphi_{r-2}(x). \tag{2.17}$$

The substitution by $D\varphi_{r-1}(x)$ and $D\varphi_{r-2}(x)$ into (2.17) using relation (2.12), gives

$$D\varphi_r(x) = \varphi_{r-1}(x) + x\left[\sum_{s=0}^{r-2} A_{r-1,s}\,\varphi_s(x) + \theta_{r-1}(x)\right] + \sum_{s=0}^{r-3} A_{r-2,s}\,\varphi_s(x) + \theta_{r-2}(x), \tag{2.18}$$

which we can simplify into

$$D\varphi_r(x) = \varphi_{r-1}(x) + x\,\theta_{r-1}(x) + \theta_{r-2}(x) + x\sum_{s=0}^{r-2} A_{r-1,s}\,\varphi_s(x) + \sum_{s=0}^{r-3} A_{r-2,s}\,\varphi_s(x), \tag{2.19}$$

which can be again converted into

$$D\varphi_r(x) = \zeta_r + \varphi_{r-1}(x) + \sum_{s=0}^{r-2} A_{r-1,s}\,\left[\varphi_{s+1}(x) - \varphi_{s-1}(x)\right] + \sum_{s=0}^{r-3} A_{r-2,s}\,\varphi_s(x), \tag{2.20}$$

with

$$\zeta_r = \begin{cases} 2 - 2x - x^2\,L_{r-1} - x\,L_{r-2}, & r \text{ even},\\ -x^2 L_{r-1} - 2\,x^2 + 2x - x\,L_{r-2}, & r \text{ odd}. \end{cases} \tag{2.21}$$

Some simplifications lead to

$$\begin{aligned} D\varphi_r(x) &= \zeta_r + \varphi_{r-1}(x) + \sum_{s=0}^{r-2}\left[A_{r-1,s}\,\left(\varphi_{s+1}(x) - \varphi_{s-1}(x)\right) + A_{r-2,s}\,\varphi_s(x)\right] + A_{r-2,r-2}\,\varphi_{r-2}(x)\\ &= \zeta_r + \varphi_{r-1}(x) + A_{r-1,0}\,\left[\varphi_1(x) - \varphi_{-1}(x)\right] + A_{r-2,0}\,\varphi_0(x) + A_{r-1,r-2}\,\left[\varphi_{r-1}(x) - \varphi_{r-3}(x)\right]\\ &\quad + A_{r-2,r-2}\,\varphi_{r-2}(x) + \sum_{s=1}^{r-3}\left[A_{r-1,s}\,\left(\varphi_{s+1}(x) - \varphi_{s-1}(x)\right) + A_{r-2,s}\,\varphi_s(x)\right] + A_{r-2,r-2}\,\varphi_{r-2}(x).\\ &= \sum_{s=0}^{r-1} A_{r,s}\,\varphi_s(x) + \theta_r(x). \end{aligned}$$

This finalizes the proof of Theorem 2.

□

Now, if we consider the vector

$$\boldsymbol{\varphi}(x) = [\varphi_0(x), \varphi_1(x), \ldots, \varphi_M(x)]^T,$$

then, based on Theorem 2, the derivative of the vector $\boldsymbol{\varphi}(x)$ can be expressed as

$$\frac{d\,\boldsymbol{\varphi}(x)}{dx} = A\,\boldsymbol{\varphi}(x) + \boldsymbol{\theta}(x), \tag{2.22}$$

where the vector $\boldsymbol{\theta}(x)$ is given by

$$\boldsymbol{\theta}(x) = [\theta_0(x), \theta_1(x), \ldots, \theta_M(x)]^T, \tag{2.23}$$

with the components (2.15). Then the matrix A in (2.22) is the operational matrix of derivatives of the vector $\boldsymbol{\varphi}(x)$ with size $(M+1)\times(M+1)$ whose nonzero elements are given in (2.13).

For $M = 4$, the matrix can be written explicitly as

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ \frac{3}{2} & 0 & 0 & 0 & 0\\ \frac{1}{2} & 4 & 0 & 0 & 0\\ 0 & 1 & 5 & 0 & 0\\ \frac{3}{2} & -1 & 1 & 6 & 0 \end{pmatrix}. \tag{2.24}$$

Remark 1. *The second derivative of the vector $\boldsymbol{\varphi}(x)$ is given as*

$$\frac{d^2\,\boldsymbol{\varphi}(x)}{dx^2} = A^2\,\boldsymbol{\varphi}(x) + A\,\boldsymbol{\theta(x)} + \boldsymbol{\theta^{'}(x)}, \tag{2.25}$$

where

$$\begin{aligned} &\boldsymbol{\theta}^{'}(x) = [\theta^{'}_0(x), \theta^{'}_1(x), \dots, \theta^{'}_M(x)]^T, \\ &\quad \theta^{'}_r(x) = \begin{cases} -(L_r+2), & r \text{ even}, \\ -L_r, & r \text{ odd}. \end{cases} \end{aligned} \tag{2.26}$$

3. Numerical Solution of the second-order BVP

In this section, we concentrate on developing two numerical algorithms for treating the linear and nonlinear second-order BVPs.

3.1. **Linear second-order BVPs.** Consider the following linear second-order BVP:

$$v^{''}(x) + f_1(x)\,v^{'}(x) + f_2(x)\,v(x) = \varepsilon_1(x); \quad x \in (a,b), \tag{3.1}$$

governed by the homogeneous boundary conditions

$$v(a) = v(b) = 0. \tag{3.2}$$

Hence, it can be inferred that $v(x)$ has the following approximation

$$v(x) \simeq v_M(x) = \sum_{r=0}^{M} c_r\,\varphi_r(x) = \boldsymbol{C}^T\,\boldsymbol{\varphi}(x), \tag{3.3}$$

where

$$\boldsymbol{C}^T = [C_0, C_1, \dots, C_M]. \tag{3.4}$$

In virtue of relations (2.22) and (2.25), we can obtain the following approximations for $v(x)$, $v^{'}(x)$, and $v^{''}(x)$:

$$v(x) \simeq \boldsymbol{C}^T\,\boldsymbol{\varphi}(x), \tag{3.5}$$

$$v^{'}(x) \simeq \boldsymbol{C}^T\,A\,\boldsymbol{\varphi}(x) + \boldsymbol{C}^T\,\boldsymbol{\theta}(x), \tag{3.6}$$

$$v^{''}(x) \simeq \boldsymbol{C}^T\,A^2\,\boldsymbol{\varphi}(x) + \boldsymbol{C}^T\,A\,\boldsymbol{\theta}(x) + \boldsymbol{C}^T\,\boldsymbol{\theta}^{'}(x). \tag{3.7}$$

The residual $R(x)$ of equation (3.1) is given by:

$$R(x) = \boldsymbol{C}^T\,A^2\,\boldsymbol{\varphi}(x) + \boldsymbol{C}^T\,A\,\boldsymbol{\theta}(x) + \boldsymbol{C}^T\,\boldsymbol{\theta}^{'}(x) + f_1(x)\,\left(\boldsymbol{C}^T\,A\,\boldsymbol{\varphi}(x) + \boldsymbol{C}^T\,\boldsymbol{\theta}(x)\right) + f_2(x)\left(\boldsymbol{C}^T\,\boldsymbol{\varphi}(x)\right) - \varepsilon_1(x). \tag{3.8}$$

The Petrov Galerkin technique is used (see [19]) to give the following equation:

$$\int_0^1 R(x)\,\varphi_r(x)\,dx = 0, \quad r = 0, 1, 2, \dots, M. \tag{3.9}$$

As a consequence, Equation (3.9) generates a series of $(M+1)$ linear equations for the unknown components of the vector $\boldsymbol{C}$. As a result, we can get the approximate spectral solution $v_M(x)$ given in (3.3).

3.2. **Linear second-order BVPs with non-homogeneous boundary conditions.** Consider the next one-dimensional second-order BVP:

$$v^{''}(x) + f_1(x)\, v^{'}(x) + f_2(x)\, v(x) = \varepsilon_1(x); \quad x \in (a, b), \tag{3.10}$$

governed by the non-homogeneous boundary conditions:

$$v(a) = \alpha, \quad v(b) = \beta. \tag{3.11}$$

Based on the following transformation ([6]):

$$v(x) = y(x) + \frac{\alpha(b-x) + \beta(x-a)}{b-a}, \tag{3.12}$$

it can be easily shown that (3.10)-(3.11) can be transformed into the following modified one:

$$y^{''}(x) + f_1(x)\, y^{'}(x) + f_2(x)\, y(x) = \varepsilon(x); \quad x \in (a, b), \tag{3.13}$$

with

$$\varepsilon(x) = \varepsilon_1(x) - \frac{\beta-\alpha}{b-a}\, f_1(x) - \frac{\alpha(b-x) + \beta(x-a)}{b-a}\, f_2(x), \tag{3.14}$$

governed by the following homogeneous boundary conditions

$$y(a) = y(b) = 0. \tag{3.15}$$

3.3. **Nonlinear second-order BVPs with homogeneous conditions.** Consider the following nonlinear differential equation:

$$v^{''}(x) = F\left(x,\, v(x),\, v^{'}(x)\right), \tag{3.16}$$

with the homogeneous conditions

$$v(a) = v(b) = 0. \tag{3.17}$$

With the aid of the approximations in (3.5), (3.6), and (3.7), for $v(x)$, $v^{'}(x)$, *and* $v^{''}(x)$, we get

$$\boldsymbol{C}^T A^2\, \boldsymbol{\varphi}(x) + \boldsymbol{C}^T A\, \boldsymbol{\theta}(x) + \boldsymbol{C}^T\, \boldsymbol{\theta}^{'}(x) = F\left(x, \boldsymbol{C}^T\, \boldsymbol{\varphi}(x), \boldsymbol{C}^T A\, \boldsymbol{\varphi}(x) + \boldsymbol{C}^T\, \boldsymbol{\theta}(x)\right). \tag{3.18}$$

To find the approximate solution $v_M(x)$, we enforce equation (3.18) to be satisfied exactly at the first $(M+1)$ roots of the polynomial $\varphi_{M+1}(x)$. So, the set of $(M+1)$ nonlinear equations is produced in the expansion coefficients, c_r. This nonlinear system may be solved using the well-known Newton's iterative method, and thus an approximate solution $v_M(x)$ can be obtained.

Remark 2. *The nonlinear second-order BVP given in (3.16) with the non-homogeneous boundary conditions can be solved using a method similar to that given in Subsection 3.2.*

Remark 3. *The linear and nonlinear second-order IVPs can be solved using similar algorithms that were followed to solve the linear and nonlinear second-order BVPs, so the details are omitted.*

4. Investigation of the convergence analysis

In this section, we illustrate the convergence of the proposed expansion used for the numerical approximation of a solution of the BVPs.

Lemma 1. *At the origin, consider $g(x)$ to be an infinitely differentiable function. Then it has the following Lucas expansion*

$$g(x) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m\, \sigma_k\, g^{(k+2m)}(0)}{m!\,(k+m)!}\, \varphi_k(x). \tag{4.1}$$

Proof. We expand $g(x)$ as

$$g(x)=\sum_{\iota=0}^{\infty} a_{\iota}\, x^{\iota}; \quad a_{\iota}=\frac{g^{(\iota)}(0)}{\iota!}. \tag{4.2}$$

The inversion formula (2.6) along with into (4.2) lead to

$$g(x)=\sum_{\iota=0}^{\infty} a_{\iota} \sum_{\substack{k=0\\(k+\iota)\text{ even}}}^{\iota} \frac{(-1)^{\frac{\iota-k}{2}}\, \iota!\, \sigma_k}{\left(\frac{\iota-k}{2}\right)!\, \left(\frac{\iota+k}{2}\right)!}\, \varphi_k(x). \tag{4.3}$$

Reordering similar terms, the next expansion is acquired

$$g(x)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} a_{k+2m}\, \frac{(-1)^{m}\,(k+2m)!\, \sigma_k}{(m)!\,(k+m)!}\, \varphi_k(x). \tag{4.4}$$

□

Lemma 2. *[10] Let $I_{\ell}(x)$ indicate the first kind of modified Bessel function of the order ℓ. The next identity shall be valid*

$$I_{\ell}(2\,x)=\sum_{i=0}^{\infty} \frac{x^{\ell+2i}}{(\ell+i)!\, i!}, \quad \forall\, i>0. \tag{4.5}$$

Lemma 3. *[36] For all $x>0$. The next inequality is satisfied by the first kind of the modified Bessel function $I_{\omega}(x)$*

$$|I_{\omega}(x)| \le \frac{x^{\omega}\, \cosh(x)}{2^{\omega}\, \omega!}. \tag{4.6}$$

Lemma 4. *For $x\in[0,1]$, for Lucas polynomials, the following inequality holds:*

$$|L_r(x)| \le (\sqrt{3})^{r}. \tag{4.7}$$

Proof. The above inequality can be proved using mathematical induction, for $r=1$

$$|L_1(x)|=|x|\le \sqrt{3}. \tag{4.8}$$

Now, we suppose the inequality remains true for $r=k$

$$|L_k(x)|=|x\, L_{k-1}(x)+L_{k-2}(x)| \le (\sqrt{3})^{k}. \tag{4.9}$$

Finally, we show that the inequality stands true for $r=k+1$. We have

$$\begin{aligned}|L_{k+1}(x)| &= |x\, L_k(x)+L_{k-1}(x)|\\ &\le |x|\,|L_k(x)|+|L_{k-1}(x)|\\ &\le \sqrt{3}\,|L_k(x)|+|L_{k-1}(x)|\\ &\le (\sqrt{3})^{k+1}.\end{aligned} \tag{4.10}$$

Lemma 4 is now proved. □

Lemma 5. *For all $x\in[0,1]$, the next inequality holds for $\varphi_r(x)$*

$$|\varphi_r(x)| \le \frac{(\sqrt{3})^{r}}{2}. \tag{4.11}$$

Proof. We employ mathematical induction when the inequality holds for $r=1$

$$|\varphi_1(x)|=|x^2\,(1-x)| \le \frac{\sqrt{3}}{2}. \tag{4.12}$$

We now assume that the inequality holds for $r=k$

$$|\varphi_k(x)|=|x\,(1-x)\, L_k(x)| \le \frac{(\sqrt{3})^{k}}{2}. \tag{4.13}$$

Finally, we demonstrate that the inequality holds for $r = k+1$. We have

$$|\varphi_{k+1}(x)| = |x\,(1-x)\,L_{k+1}(x)| \leq |x\,(1-x)|\,(\sqrt{3})^{k+1} \leq \frac{(\sqrt{3})^{k}}{2\,L_k(x)}\,(\sqrt{3})^{k+1} \leq \frac{(\sqrt{3})^{k+1}}{2}. \tag{4.14}$$

□

Theorem 3. *If $g(x)$ is defined on $[0,1]$ and $|g^{(n)}(0)| \leq f^n$; $n \geq 0, f^n$ are positive constants, and if $g(x)$ has the expansion: $g(x) = \sum_{r=0}^{\infty} c_r\,\varphi_r(x)$, then we have*

(1) $|c_r| \leq \dfrac{f^r\,\cosh(2\,f)}{r!}$.
(2) *The series is absolutely convergent.*

Proof. From Lemma 1, we have

$$|c_r| \leq \left|\sum_{k=0}^{\infty} \frac{(-1)^k\,\sigma_r\,g^{(r+2k)}(0)}{k!\,(k+r)!}\right|, \tag{4.15}$$

and based on the assumption $|g^{(n)}(0)| \leq f^n$; $n \geq 0$, the next inequality holds

$$|c_r| \leq \sum_{k=0}^{\infty} \frac{f^{r+2k}}{k!\,(k+r)!}. \tag{4.16}$$

Following the application of Lemma 2, the following inequality arises

$$|c_r| \leq I_r(2f). \tag{4.17}$$

In virtue of the inequality (4.17) along with Lemma 3, then the next estimate for the expansion coefficients is acquired

$$|c_r| \leq \frac{(2f)^r\,\cosh(2\,f)}{2^r\,r!} = \frac{f^r\,\cosh(2\,f)}{r!}. \tag{4.18}$$

This proves the first part of Theorem 3.
Now, we are going to prove the second part of the theorem. Starting with the inequality (4.18)

$$|c_r\,\varphi_r(x)| \leq \left|\frac{f^r\,\cosh(2\,f)}{r!}\,\varphi_r(x)\right|, \tag{4.19}$$

and accordingly, the application of Lemma 5 produces

$$|c_r\,\varphi_r(x)| \leq \left|\frac{f^r\,\cosh(2\,f)}{r!}\,\frac{(\sqrt{3})^r}{2}\right|. \tag{4.20}$$

Now, since

$$e^{|\sqrt{3}\,f|} = \sum_{r=0}^{\infty}\left|\frac{\left(\sqrt{3}\,f\right)^r}{r!}\right|, \tag{4.21}$$

then the series converges absolutely. □

Theorem 4. *Let $g(x)$ follow the assumptions of Theorem 3. In addition, let $e_M(x) = \sum_{r=M+1}^{\infty} c_r\,\varphi_r(x)$, be the global error. The next inequality holds for $|e_M(x)|$:*

$$|e_M(x)| \leq \frac{e^{(\sqrt{3}\,f)}\,\cosh\left(2\,f\right)\,\left(\sqrt{3}\right)^{M+1}}{2\,(M+1)!}. \tag{4.22}$$

Proof. The first part of Theorem 4 allows one to write

$$|e_M(x)| \leq \frac{\cosh(2\,f)}{2} \sum_{r=M+1}^{\infty} \frac{\left(\sqrt{3}\,f\right)^r}{r!}, \tag{4.23}$$

and hence, we get

$$|e_M(x)| \leq \frac{e^{(\sqrt{3}\,f)}\,\cosh(2\,f)}{2} \left(1 - \frac{\Gamma(M+1, \sqrt{3}\,f)}{\Gamma(M+1)}\right). \tag{4.24}$$

For all $x \geq 0$, the integral forms of gamma and upper incomplete gamma function together along with inequality $e^{-x} < 1$, yields the following inequality:

$$|e_M(x)| \leq \frac{e^{(\sqrt{3}\,f)}\,\cosh(2\,f)\,\left(\sqrt{3}\right)^{M+1}}{2\,(M+1)!}. \tag{4.25}$$

□

5. Numerical Outcomes

In this section, some numerical examples are presented to show the effectiveness and accuracy of the current algorithms. More precisely, we present the results obtained by the application of the modified Lucas tau method $(MLTM)$ for treating linear problems, whereas the nonlinear ones are treated by the modified Lucas collocation method $(MLCM)$.

Example 1. *[1, 11, 18, 20, 32] We consider the following BVP of the non-linear Bratu-type:*

$$v^{''}(x) + \lambda\,e^{v(x)} = 0, \quad x \in [0,1], \tag{5.1}$$

with the boundary conditions

$$v(0) = v(1) = 0. \tag{5.2}$$

The analytic solution is

$$v(x) = -2\left[\ln \frac{\cosh\left(\frac{\theta}{4}(2x-1)\right)}{\cosh\left(\frac{\theta}{4}\right)}\right], \tag{5.3}$$

where θ is a solution to the equation

$$\sqrt{2\lambda}\,\cosh\left(\frac{\theta}{4}\right) - \theta = 0. \tag{5.4}$$

We solve this example for various values of M and λ=1, 2, and 3.51 by the $MLCM$. The resulted maximum absolute errors (AEs) are listed in Table 1. In Table 2, we compare the best errors resulted from the application of $MLCM$ for $\lambda = 1$ with those resulted from the following methods:

- *The Lie-group shooting method (LGSM) in [1].*
- *The B-Spline method (BSM) in [18].*
- *Laplace method (LM) in [32].*
- *The decomposition method (DM) in [20].*
- *Optimal spline method (OSM) in [11].*
- *Perturbation-iteration algorithm (PIA) in [11].*

In Figure 1, the log of the approximate solutions for $M = 16$ and λ=1, 2, and 3.51 are plotted. Figure 2 represents the AE for Example 1 with $M = 20$ and $\lambda = 1$.

TABLE 1. Maximum MAE for Example 1

M	λ	AE	λ	AE	λ	AE
2		5×10^{-5}		6×10^{-4}		4×10^{-2}
4		5×10^{-7}		1.6×10^{-5}		1.5×10^{-2}
6		8×10^{-9}		5.5×10^{-7}		3.5×10^{-3}
8		1.6×10^{-10}		2.6×10^{-8}		5×10^{-4}
10	1	3.2×10^{-12}	2	1.4×10^{-9}	3.51	1×10^{-4}
12		7×10^{-14}		7.5×10^{-11}		2×10^{-5}
14		2.2×10^{-15}		4×10^{-12}		3×10^{-6}
16		5×10^{-16}		2.2×10^{-13}		7×10^{-7}

TABLE 2. Comparison of the best errors for Example 1

x	$MLCM$	$LGSM$ [1]	BSM [18]	LM [32]	DM [20]	OSM [11]	PIA (1,2) [11]
0.1	5.97×10^{-16}	7.50×10^{-7}	2.97×10^{-6}	1.97×10^{-6}	2.68×10^{-3}	4.63×10^{-8}	1.68×10^{-5}
0.2	7.49×10^{-16}	1.01×10^{-6}	5.46×10^{-6}	3.93×10^{-6}	2.02×10^{-3}	1.02×10^{-7}	3.99×10^{-5}
0.3	8.19×10^{-16}	9.04×10^{-7}	7.33×10^{-6}	5.85×10^{-6}	1.52×10^{-4}	1.44×10^{-7}	4.91×10^{-5}
0.4	7.49×10^{-16}	5.23×10^{-7}	8.49×10^{-6}	7.70×10^{-6}	2.20×10^{-3}	1.71×10^{-7}	6.03×10^{-5}
0.5	8.60×10^{-16}	5.06×10^{-9}	8.89×10^{-6}	9.46×10^{-6}	3.01×10^{-3}	1.81×10^{-7}	5.92×10^{-5}
0.6	6.66×10^{-16}	5.13×10^{-7}	8.49×10^{-6}	1.11×10^{-5}	2.20×10^{-3}	1.71×10^{-7}	6.03×10^{-5}
0.7	5.69×10^{-16}	8.94×10^{-7}	7.33×10^{-6}	1.25×10^{-5}	1.52×10^{-4}	1.44×10^{-7}	4.91×10^{-5}
0.8	4.86×10^{-16}	1.00×10^{-6}	5.46×10^{-6}	1.34×10^{-5}	2.02×10^{-3}	1.02×10^{-7}	3.99×10^{-5}
0.9	2.71×10^{-16}	7.41×10^{-7}	2.97×10^{-6}	1.19×10^{-5}	2.68×10^{-3}	4.63×10^{-8}	1.68×10^{-5}

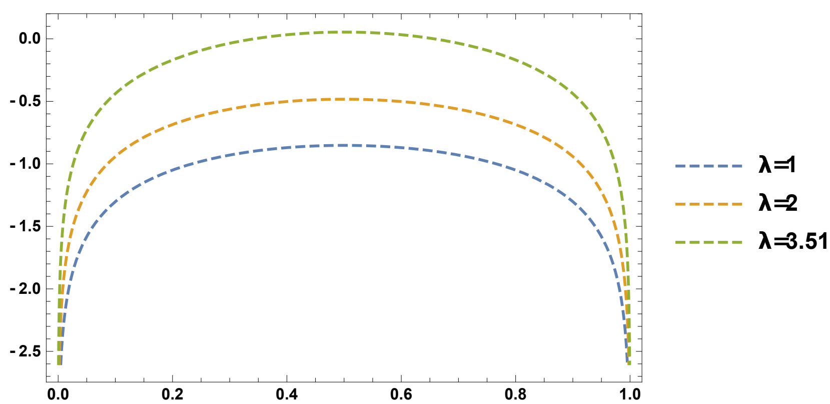

FIGURE 1. Log of approximate solution with $M = 16$ and $\lambda = 1,\ 2,$ and 3.51 for Example 1

Example 2. *[47] Consider the following is a non-linear singular initial value problem:*

$$v''(x) + \frac{1}{x} v'(x) - v^3(x) + 3\, v^5(x) = 0, \quad x \in [0, 1], \tag{5.5}$$

with non-homogeneous conditions:

$$v'(0) = 0,\ v(0) = 1. \tag{5.6}$$

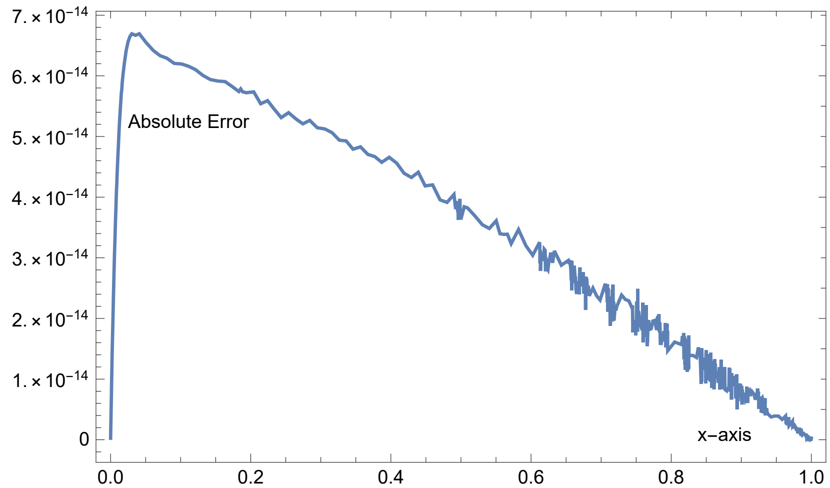

FIGURE 2. AE with $\lambda = 1$ and $M = 20$ for Example 1

The exact solution is $v(x) = \frac{1}{\sqrt{1+x^2}}$. *For* $M = 6$, *we conclude that* $MLCM$ *is more effective than the Hermite wavelets operational matrix method (HWOMM) in [47] and listed in Table 3. Furthermore, the AEs of Example 2 for* $M = 6$ *and* $M = 7$ *is depicted in Figure 3. In Figure 4, we show the AE for Example 2 for* $M = 7$.

TABLE 3. Comparison between $MLCM$ and $HWOMM$ in [47] with $M = 6$ for Example 2

x	$MLCM$	$HWOMM$ [47]
0.1	7.80626×10^{-18}	1.8706×10^{-5}
0.2	2.42861×10^{-17}	1.9982×10^{-5}
0.3	8.67362×10^{-18}	1.0719×10^{-5}
0.4	1.38778×10^{-17}	6.5087×10^{-6}
0.5	4.16334×10^{-17}	3.8619×10^{-6}
0.6	9.02056×10^{-17}	9.9962×10^{-7}
0.7	8.32667×10^{-17}	4.1085×10^{-6}
0.8	4.16334×10^{-17}	4.2547×10^{-6}
0.9	2.77556×10^{-17}	9.8647×10^{-6}

Example 3. *[47] Consider the non-linear initial value problem:*

$$x\, v^{''}(x) + 2\, v^{'}(x) + x\, v^5(x) = 0, \tag{5.7}$$

with the initial conditions:

$$v(0) = 1,\; v^{'}(0) = 0. \tag{5.8}$$

The actual solution of Example 3 is $v(x) = (1 + \frac{x^2}{3})^{\frac{-1}{2}}$. *We apply* $MLCM$ *with* $M = 6$ *and* $M = 7$, *then the* $MAEs$ *of* $MLCM$ *and* $HWOMM$ *developed in [47] are listed Table 4. The comparison between the approximate and exact solutions for Example 3 with* $M = 6$ *shown in Figure 5.*

Example 4. *[14] Consider the linear equation of the second order with homogeneous conditions:*

$$v^{''}(x) + 3\, x\, e^{x}\, (x+3) = 0, \quad x \in [0, 1], \tag{5.9}$$

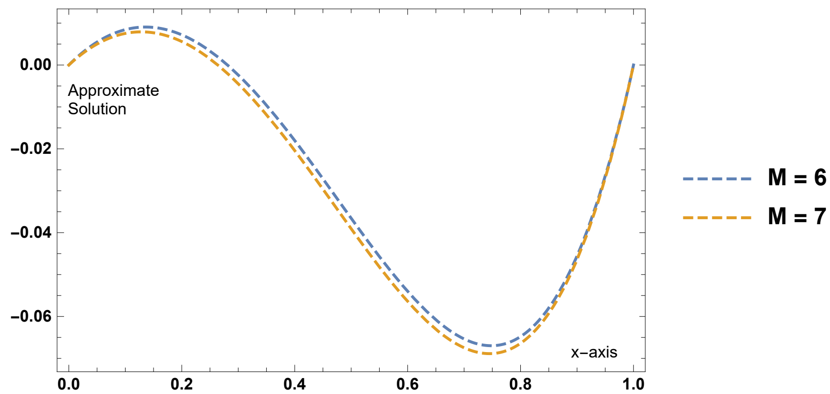

FIGURE 3. Approximate solution of Example 2

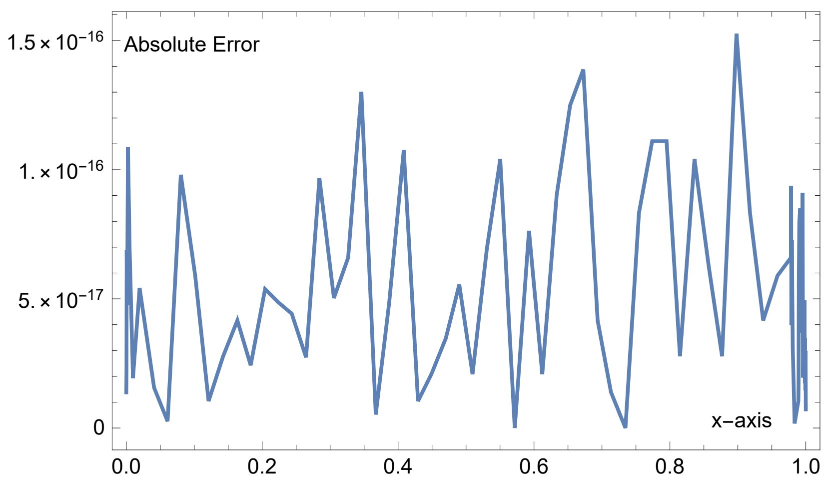

FIGURE 4. AE of Example 2 with $M = 7$

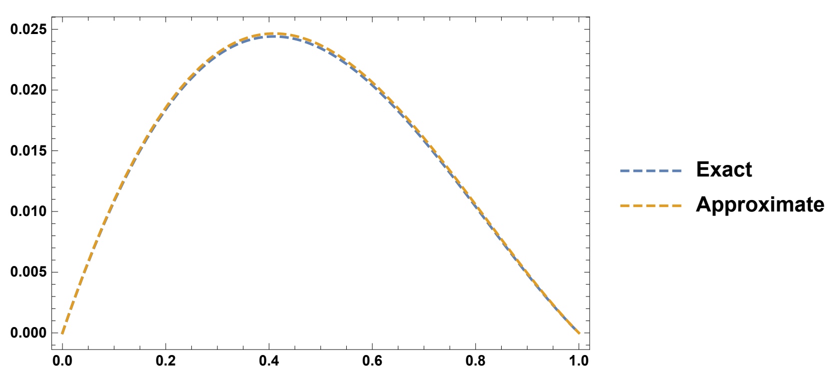

FIGURE 5. Comparison between approximate and exact solutions of Example 3 for $M = 6$

with the boundary conditions

$$v(0) = v(1) = 0. \tag{5.10}$$

TABLE 4. Comparison of the *AE* for Example 3

	$M=6$		$M=7$	
x	*MLCM*	*HWOMM* [47]	*MLCM*	*HWOMM* [47]
0.1	1.21431×10^{-17}	9.9165×10^{-8}	8.67362×10^{-18}	3.6989×10^{-8}
0.2	4.85723×10^{-17}	4.5437×10^{-8}	4.51028×10^{-17}	1.4473×10^{-8}
0.3	2.08167×10^{-17}	5.1208×10^{-8}	1.73472×10^{-17}	2.5323×10^{-9}
0.4	4.16334×10^{-17}	5.0684×10^{-8}	4.16334×10^{-17}	2.5781×10^{-9}
0.5	3.46945×10^{-18}	3.5157×10^{-8}	6.93889×10^{-18}	6.6328×10^{-9}
0.6	7.63278×10^{-17}	7.0668×10^{-8}	8.32667×10^{-17}	8.9403×10^{-9}
0.7	4.85723×10^{-17}	6.7098×10^{-8}	4.85723×10^{-17}	6.9532×10^{-9}
0.8	8.67362×10^{-18}	2.0163×10^{-9}	5.20417×10^{-18}	1.1810×10^{-8}
0.9	1.30104×10^{-17}	1.3919×10^{-7}	1.04083×10^{-17}	6.5729×10^{-9}

The actual solution for Example 4 is $v(x) = 3\,x\,(1-x)\,e^{x}$. *If we apply MLTM for various values M, then the maximum MAEs of the Sinc–Galerkin method developed in [14] and MLTM are listed in Table 5. MLTM produces more accurate results than method in [14]. We compare the behavior of the approximate solution for various M values as shown in Figure 6.*

TABLE 5. Comparison between the maximum *AE* for Example 4

M	*MLTM*	*Sinc–Galerkin* [14]
2	2.153×10^{-1}	1.264×10^{-1}
4	3.159×10^{-4}	3.698×10^{-2}
8	1.709×10^{-7}	4.543×10^{-3}
16	2.109×10^{-15}	1.804×10^{-4}
32	1.3064×10^{-12}	1.598×10^{-6}

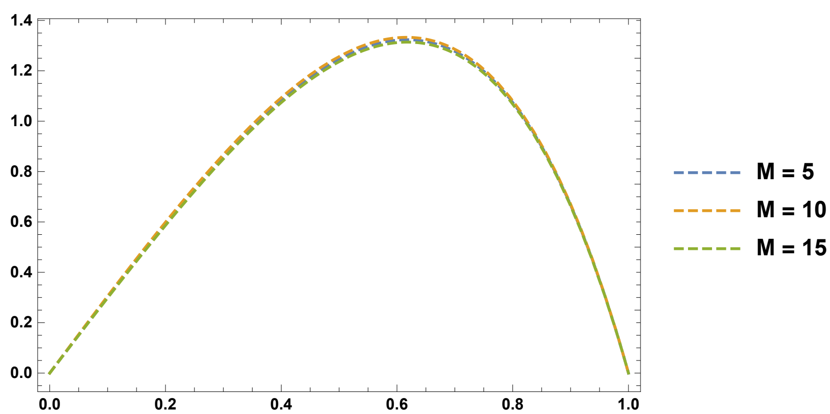

FIGURE 6. Approximate solution of Example 4

Example 5. *[48] Consider the following non-linear boundary value problem:*

$$v^{''}(x) + \left(1 + \frac{\mu}{x}\right)\, v^{'}(x) - \frac{5\,x^{3}\,(5\,x^{5}\,e^{v(x)} - x - \mu - 4)}{4 + x^{5}} = 0, \quad x \in [0,1], \tag{5.11}$$

with the initial conditions

$$v'(0) = 0,\ v(1) = \ln\left(\frac{1}{5}\right). \tag{5.12}$$

The exact solution for Example 5 is $v(x) = \ln\left(\frac{1}{4+x^5}\right)$. *We compare the AE for* $\mu = 0.25$ *and* $\mu = 0.75$ *with* $M = 10$ *if MLCM is applied with those obtained by the Hermite wavelets method (HWM) in [48]. The results are listed in Table 6. The displayed results show that our results are more precise than those obtained by the HWM. We compare the behavior of the exact and approximate solutions for* $M = 10$ *and* $\mu = 0.75$ *as shown in Figure 7.*

TABLE 6. Comparison of the AE with $M = 10$ for Example 5

	$\mu = 0.25$		$\mu = 0.75$	
x	$MLCM$	HWM [48]	$MLCM$	HWM [48]
0.1	1.38778×10^{-17}	3.9995×10^{-5}	1.04083×10^{-16}	3.9936×10^{-5}
0.2	1.59595×10^{-16}	4.0302×10^{-5}	7.63278×10^{-17}	4.0452×10^{-5}
0.3	1.80411×10^{-16}	3.8897×10^{-5}	8.32667×10^{-17}	3.9245×10^{-5}
0.4	1.38778×10^{-16}	3.9408×10^{-5}	2.22045×10^{-16}	4.0067×10^{-5}
0.5	5.55112×10^{-17}	4.0281×10^{-5}	0	4.1077×10^{-5}
0.6	1.38778×10^{-16}	3.8905×10^{-5}	8.32667×10^{-17}	3.9730×10^{-5}
0.7	0	3.8339×10^{-5}	8.32667×10^{-17}	3.9319×10^{-5}
0.8	8.32667×10^{-17}	3.9700×10^{-5}	9.71445×10^{-17}	4.0857×10^{-5}
0.9	8.32667×10^{-17}	3.8995×10^{-5}	1.249×10^{-16}	4.0162×10^{-5}

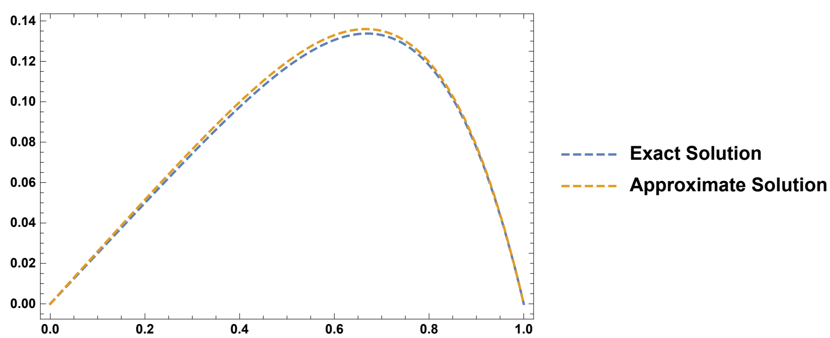

FIGURE 7. Comparison between the exact and approximate solutions of Example 5 for $M = 10$

Example 6. *[31, 46, 55, 57] Consider the following singular initial value problem*

$$v''(x) + \frac{2}{x}\,v'(x) = (4\,x^2 + 6)\,v(x), \qquad 0 < x \le 1, \tag{5.13}$$

with the initial conditions

$$v(0) = 1,\ v'(0) = 0. \tag{5.14}$$

The exact solution of Example 6 is $v(x) = e^{x^2}$. *The comparison of the maximum absolute error (MAE) of our method and the methods (Adomian decomposition method (ADM), variational iteration method (VIM), Haar wavelet collocation method (HWCM), and Haar wavelet adaptive grid method (HWAGM)) in [31, 46] is given in Table 7. We show the behavior of the approximate solution for various values of M in Figure 8 and the absolute error in Figure 9 with* $M = 12$.

Table 7. Comparison of MAE of Example 6

M	$MLTM$	ADM [46]	VIM [46]	$HWCM$ [46]	$HWAGM$ [31]
8	6.1062×10^{-16}	6.3098×10^{-4}	7.3037×10^{-4}	3.7226×10^{-4}	1.1798×10^{-4}
16	1.8319×10^{-15}	9.4542×10^{-4}	1.0927×10^{-3}	1.1858×10^{-4}	3.2938×10^{-5}
32	1.1944×10^{-11}	1.1519×10^{-3}	1.3304×10^{-3}	3.1412×10^{-5}	7.9978×10^{-6}
64	7.0756×10^{-7}	1.2701×10^{-3}	1.4664×10^{-3}	7.9072×10^{-6}	1.9995×10^{-6}

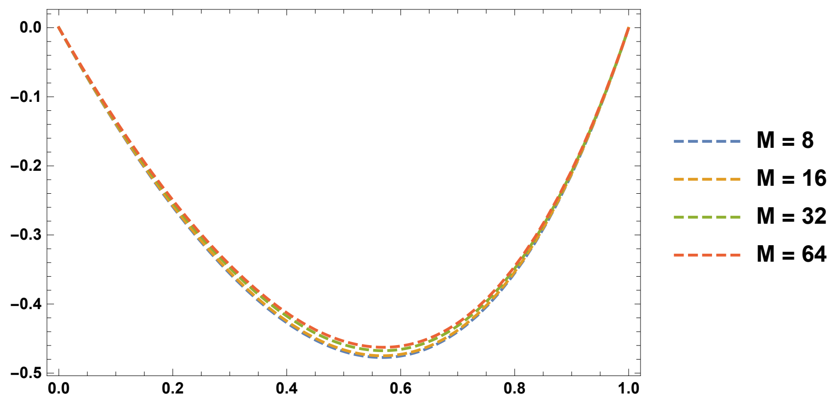

Figure 8. Comparison of approximate solutions of Example 6

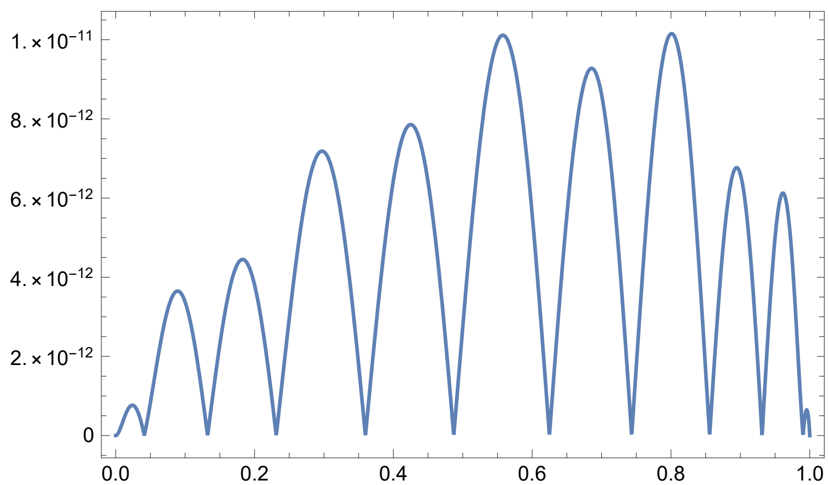

Figure 9. AE of Example 6 with $M = 12$

Example 7. *[35, 44] Consider the following linear boundary value problem:*

$$v^{''}(x) + x\,v(x) = (3 - x - x^2 + x^3)\,\sin x + 4\,x\,\cos x, \qquad 0 \le x \le 1, \tag{5.15}$$

with homogeneous boundary conditions:

$$v(0) = v(1) = 0. \tag{5.16}$$

The actual solution of Example 7 is $v(x) = (x^2 - 1)\,\sin x$. In Table 8, we compare the resulted MAE for various values of M for our proposed method MLTM and the methods (Quadratic spline method (QSM) in [44], Cubic spline method

(CSM) in [44], Non-polynomial spline method (NPSM) in [44], and the method in [35]. In Figure 10, we compare the exact and approximate solutions' behavior for $M = 16$.

Table 8. Comparison of the MAE for Example 7

Methods	*M=8*	*M=16*	*M=32*	*M=64*
QSM [44]	4.495×10^{-2}	3.081×10^{-3}	7.704×10^{-4}	1.926×10^{-4}
CSM [44]	1.154×10^{-2}	2.883×10^{-3}	7.207×10^{-4}	1.802×10^{-4}
NPSM [44]	2.669×10^{-3}	3.241×10^{-4}	3.988×10^{-5}	4.944×10^{-6}
Method in [35]	2.224×10^{-4}	5.045×10^{-6}	1.625×10^{-7}	5.577×10^{-9}
MLTM	1.349×10^{-7}	1.110×10^{-16}	1.083×10^{-12}	2.911×10^{-8}

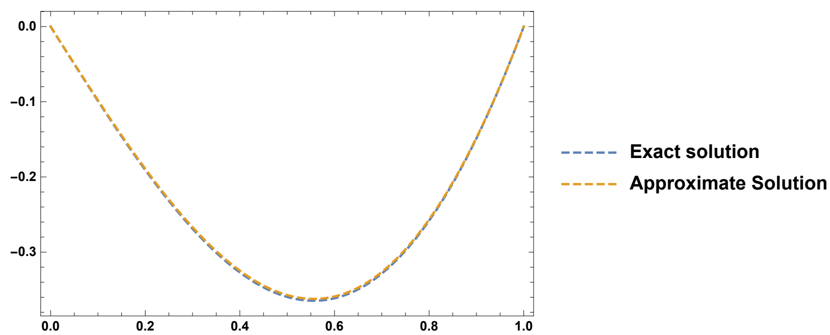

Figure 10. Comparison between the exact and approximate solutions of Example 7 with $M = 16$

6. Conclusions

In this paper, we have introduced a new type of polynomials related to Lucas polynomials, namely, modified Lucas polynomials. Some important formulas concerned with these polynomials were derived. The operational matrix of derivatives of these polynomials was also established and employed in the implementation of the proposed algorithms which were presented to treat both linear and non-linear second-order boundary value problems. The convergence analysis of the proposed modified Lucas polynomials was carefully investigated by proving some important inequalities. In order to demonstrate our algorithms using the new type of modified Lucas polynomials, some numerical examples were presented. These examples ensured the proposed algorithms' applicability, efficiency, and accuracy if compared to some other existing numerical methods. We do believe that other modified Lucas polynomials can be constructed to be capable of handling other higher-order initial and boundary value problems. This, of course, will enrich the field of solving differential equations of several forms.

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