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A new application for numerical computations of the modified equal width equation (MEW) based on Lumped Galerkin method with the cubic B-spline

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Abstract

In this paper, numerical computation of the modified equal width equation (MEW), which is one of the equations used to model nonlinear events, will be carried out. For this equation, numerical computations have been obtained by many researchers using different methods. The goal of the new approach is to check how well it performs with respect to the numerical calculations the researchers found. For this, the proposed study presents a Lie-Trotter splitting algorithm in accordance with the time-splitting technical rules combined with Lumped Galerkin FEM based on the basis function of the cubic B-spline. Two valid test examples are given to determine the validity and effectiveness of the current technique. The results obtained in a new way with the Matlab computational software are compared with the studies of other authors in the literature and are shown graphically. Based on these new results, it can clearly be stated that the benefit of the proposed approach is to demonstrate that reliability is achieved in obtaining approximate computations.

Keywords. The MEW equation; B-spline; Collocation method; Lumped Galerkin method; Lie-Trotter splitting.
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1. Introduction

The main theme of this article is to obtain the approximate solutions of the modified equal width (MEW) equation

$$U_t + \epsilon U^2 U_x - \mu U_{xxt} = 0, \tag{1.1}$$

with the initial and boundary conditions presented in form

$$U(x,0) = f(x), \qquad x_L \leq x \leq x_R, \tag{1.2}$$

and

$$\begin{aligned} U(x_L,t) &= U(x_R,t) = 0, \\ U_x(x_L,t) &= U_x(x_R,t) = 0, \\ U_{xx}(x_L,t) &= U_{xx}(x_R,t) = 0, \end{aligned} \tag{1.3}$$

which has the solitary wave solution given as follows

$$U(x,t) = csech[k(x - x_0 - vt)], \tag{1.4}$$

in which $v = \dfrac{c^2}{2}, k = \sqrt{\dfrac{1}{\mu}}$ and t is time and x is space dimensions. ϵ and μ are non-negative constants and also $f(x)$ is a smooth function and U is related to the vertical displacement of the water surface. The MEW equation can also be

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given under physical boundary conditions $U \to 0$ when $x \to \pm\infty$, due to its close relationship with the RLW equation given as follows

$$U_t + U_x - UU_x - U_{xxt} = 0,$$

and the solitary wave-type solution of the equation has not only in the unlimited region, but also a solitary wave solution in the closed range $[x_L, x_R]$. When the literature is examined, solitary ones, defined as traveling waves, maintain their shape and velocity due to the sensitive balance between nonlinearity and dispersal effects, whereas a soliton is a very private type of solitary waves, maintaining its shape and speed even after colliding with another wave [14]. While the amplitudes of these solitary waves can be both positive and negative, their velocity is commensurate to the square of their amplitudes and is also only positive. It has become very important recently to investigate the traveling wave solutions of nonlinear wave equations in relation to sciences such as optics, fluid mechanics, solid state physics, plasma physics, kinetics and geology, and make numerical calculations for natural systems in the field of mathematical modeling. When the literature is examined, it is seen that the MEW equation is used quite a lot in modeling nonlinear events and this equation is closely related to the EW equation proposed by Morrison *et al.* [25] given as follows

$$U_t - U_{xxt} + UU_x = 0.$$

In the literature, it can be seen that both analytical and numerical solutions have been obtained by many authors about the MEW equation. Some studies can be given as [17, 19, 23, 31–33] for analytical solutions of the MEW equation. Hamdi *et al.*[17] derived exact solitary wave solutions for GEW and GEW-Burgers equations. Wazwaz [32] examined the equation and two of its variants with a sine-cosine and tanh methods. Jin [19] solved the equation via the homotopy perturbation method. Lu [23] suggested a variational iteration method. Wang *et al.* [33] investigated by utilizing the method of a dynamical system for traveling wave solutions. Taghizadeh *et al.* [31] used the modified simple equation method. And some studies [1, 5–8, 10–16, 20–22, 29, 30, 34] can be given as numerically for solutions of the MEW equation. Gardner and Gardner [14] solved with Galerkin's method to the EW equation. Cheng and Liew [6] derived an improved element-free Galerkin (IEFG) method for an equation. Esen [11] and Karakoç and Geyikli [20] applied a Lumped Galerkin method with the quadratic B-spline. Çelikkaya [7] solved the equation with Strang splitting scheme implementing the cubic B-spline. Esen and Kutluay [12] and Raslan *et al.* [29] used the finite difference method for the equation. Essa [13] implemented the multigrid method. Geyikli and Karakoç [15] and Karakoç and Geyikli [21] obtained Subdomain finite element method with the help of quartic and sextic B-splines, respectively. Geyikli and Karakoç [16] and Roshan [30] utilized a Petrov Galerkin method for the equation MEW and GEW, respectively. At the same time, Evans and Raslan [10] presented a collocation method with quadratic B-splines for the GEW equation. Dereli[8] sought by utilizing the meshless method with radial basis functions collocation method. Karakoç *et al.* [22] used different linearization techniques via cubic B-spline collocation FEM. Başhan *et al.* [5] have worked on the finite difference method combined with the differential quadrature method. Additionally, in the last years, Başhan *et al.* [1] have submitted a new perspective for an equation. Yağmurlu and Karakaş [34] proposed using trigonometric cubic B-spline for the equation and also we can include [2–4] references in the article so that the reader is aware of the latest published articles on the subject. They explored numerical solutions for Regularized Long Wave (RLW) equation.
The error norms L_2 and L_∞ and the conservation constants I_1, I_2, and I_3 found by Olver [26] are computed in calculating solutions of solitary waves throughout the present study and the new findings are compared with some existing studies in the literature. The formulas of these calculated values are given as follows

$$L_2 = ||U - U_N||_2 = \sqrt{h \sum_{j=0}^{N} (U - U_N)^2},$$

$$L_\infty = ||U - U_N||_\infty = \max_j |U - U_N|,$$

and

$$I_1 = \int_{x_L}^{x_R} U(x,t)dx,$$

$$I_2 = \int_{x_L}^{x_R} [U^2(x,t) + \mu U_x^2(x,t)]dx,$$
$$I_3 = \int_{x_L}^{x_R} [U^4(x,t)]dx.$$

2. Cubic B-Splines

For approximate solutions, let the space and time domains of the problem be limited to the intervals $x_L \leq x \leq x_R$ and $0 \leq t \leq T$, respectively. The space region are divided into N finite elements uniformly as $x_L = x_0 < x_1 < ... < x_N = x_R$ with $h = x_{j+1} - x_j$, $j = 0(1)N-1$. Likewise, the time domain are divided into M finite elements uniformly as $0 = t_0 < t_1 < ... < t_M = T$ with $k = t_{n+1} - t_n$, $n = 0(1)M-1$. The cubic B-spline shape functions $\varphi_j(x)$ for $j = -1(1)N+1$ are given as [27]

$$\varphi_j(x) = \frac{1}{h^3}\begin{cases} (x-x_{j-2})^3, & x \in [x_{j-2}, x_{j-1}), \\ h^3 + 3h^2(x-x_{j-1}) + 3h(x-x_{j-1})^2 - 3(x-x_{j-1})^3, & x \in [x_{j-1}, x_j), \\ h^3 + 3h^2(x_{j+1}-x) + 3h(x_{j+1}-x)^2 - 3(x_{j+1}-x)^3, & x \in [x_j, x_{j+1}), \\ (x_{j+2}-x)^3, & x \in [x_{j+1}, x_{j+2}], \\ 0, & \text{otherwise}. \end{cases} \tag{2.1}$$

The whole of cubic B-spline bases functions are zero except $\varphi_{j-1}, \varphi_j, \varphi_{j+1}, \varphi_{j+2}$. Hence, for a typical element $[x_j, x_{j+1}]$ with the help of the local coordinate transformation described as $h = x - x_j$, $0 \leq \zeta \leq h$, the cubic B-spline bases functions on $[0, h]$ for variable ζ can be expressed as follows

$$\begin{aligned} \varphi_{j-1} &= (1-\zeta)^3, \\ \varphi_j &= 1 + 3(1-\zeta) + 3(1-\zeta)^2 - 3(1-\zeta)^3, \\ \varphi_{j+1} &= 1 + 3\zeta + 3\xi^2 - 3\zeta^3, \\ \varphi_{j+2} &= \zeta^3. \end{aligned}$$

3. B-spline Lie-Trotter splitting Lumped Galerkin Method

The numerical solutions of the MEW equation are obtained by the Lie-Trotter splitting algorithm combined with the cubic B-spline Lumped Galerkin method. For this reason, equation (1.1) is converted into two partial differential equations, each of which is solved according to the time intervals $[t_n, t_{n+1}]$, linear and non-linear given as below.

$$U_t - \mu U_{xxt} = 0, \tag{3.1}$$

$$U_t - \mu U_{xxt} + \epsilon U^2 U_x = 0. \tag{3.2}$$

In the equations (3.1) and (3.2), u and y are written instead of U, respectively, and the Lie-Trotter splitting algorithm has been applied as the equations given in the following form

$$u_t - \mu u_{xxt} = 0,$$

$$u(x, t_n) = U(x, t_n), \quad t \in [t_n, t_{n+1}], \tag{3.3}$$

$$y_t - \mu y_{xxt} + \epsilon y^2 y_x = 0,$$

$$y(x, t_n) = u(x, t_{n+1}), \quad t \in [t_n, t_{n+1}], \tag{3.4}$$

in which $t_{n+1} = (n+1)k$. Approximate solutions of the MEW equation presented with appropriate initial-boundary conditions (1.2) and (1.3) are possible by solving equations (3.3) and (3.4) based on initial conditions. To solve these problems, cubic B-spline Lumped Galerkin method is applied and $U(x_L,t)=U(x_R,t)=0, U_x(x_L,t)=U_x(x_R,t)=0$ is used as boundary conditions. For this purpose, the problems (3.3) and (3.4) are multiplied by the weight function W and integrated from x_L to x_R. For obtaining weak forms, the partial integrals of both problems are gotten by taking $Z_m=y^2$ for the equation (3.4). Let $u_N(x,t)$ and $y_N(x,t)$ be taken as approximate solutions corresponding to exact solutions $u(x,t)$ and $y(x,t)$ of problems (3.3) and (3.4), respectively and these approximate ones with the selection of the time dependent parameters δ_j and Ψ_j are given in the form below

$$u_N(x,t)=\sum_{j=-1}^{N+1}\varphi_j(x)\delta_j(t), \quad y_N(x,t)=\sum_{j=-1}^{N+1}\varphi_j(x)\Psi_j(t).$$

By applying the local coordinate transformation, the non-zero B-splines on e are $\varphi_{j-1},\varphi_j,\varphi_{j+1},\varphi_{j+2}$. So the approximations given above on [0,h] are presented on the typical element σ as follows

$$u_N(\sigma,t)=\sum_{j=m-1}^{m+2}\varphi_j(\sigma)\delta_j^e(t), \tag{3.5}$$

$$y_N(\sigma,t)=\sum_{j=m-1}^{m+2}\varphi_j(\sigma)\Psi_j^e(t). \tag{3.6}$$

For a typical element e, the following equations are formed as a result of applying the local coordinate transformation

$$\int_0^h \Big[Wu_t+\mu W_\sigma u_{\sigma t}\Big]d\sigma=\mu W u_{\sigma t}\mid_0^h, \tag{3.7}$$

$$\int_0^h \Big[Wy_t+\mu W_\sigma y_{\sigma t}+\epsilon W Z_m y_\sigma\Big]d\sigma=\mu W y_{\sigma t}\mid_0^h . \tag{3.8}$$

As it is known, W is taken as cubic B-splines in the Galerkin method. If the approximate functions of U_N and Y_N given in equations (3.5) and (3.6) are written in place of u and y in (3.7) and (3.8), and also cubic B-splines are written instead of W in (3.7) and (3.8), and the following equations are obtained

$$\sum_{j=m-1}^{m+2}\Big[\Big(\int_0^h \varphi_i\varphi_j+\mu\varphi_i^{'}\varphi_j^{'}\Big)d\sigma-\mu\varphi_i\varphi_j^{'}\mid_0^h\Big]\dot{\delta}_j=0, \tag{3.9}$$

$$\sum_{j=m-1}^{m+2}\Big[\Big(\int_0^h \varphi_i\varphi_j+\mu\varphi_i^{'}\varphi_j^{'}\Big)d\sigma-\mu\varphi_i\varphi_j^{'}\mid_0^h\Big]\dot{\Psi}_j+\sum_{j=m-1}^{m+2}\Big[\Big(\int_0^h \epsilon Z_m\varphi_i\varphi_j^{'}\Big]\Psi_j=0. \tag{3.10}$$

Taking $\delta^e=(\delta^e_{m-1},\delta^e_m,\delta^e_{m+1},\delta^e_{m+2})$, the equations (3.9) and (3.10) given above are obtained as follows

$$(A^e+\mu B^e-\mu E^e)\dot{\delta}^e=0, \tag{3.11}$$

$$(A^e+\mu B^e-\mu E^e)\dot{\Psi}^e+\epsilon(C_1^e)\Psi^e=0. \tag{3.12}$$

Here C_1^e is Z_mC^e matrix. For e, the matrices A^e, B^e, C^e and E^e are calculated as follows

$$A^e=\int_0^h\varphi_i\varphi_j d\sigma, B^e=\int_0^h\varphi_i^{'}\varphi_j^{'}d\sigma, C^e=\int_0^h\varphi_i\varphi_j^{'}d\sigma,, E^e=\varphi_i\varphi_j^{'}\mid_h^0 .$$

If the local equations obtained just above for e are used, the following global equations are obtained for $[x_L, x_R]$

$$(A + \mu B - \mu E)\dot{\delta} = 0, \tag{3.13}$$

$$(A + \mu B - \mu E)\dot{\Psi} + \epsilon(C_1)\Psi = 0. \tag{3.14}$$

In equations (3.13) and (3.14), the unknowns with $N + 3$ dimensional are $\delta = (\delta_{-1}, \delta_0, \delta_N, \delta_{N+1})^T$, and $\Psi = (\Psi_{-1}, \Psi_0, \Psi_N, \Psi_{N+1})^T$, The matrices A, B, C, and E with general rows presented in the following form are square matrices and

$$A = \frac{1}{140}(1, 120, 1191, 2416, 1191, 120, 1),$$

$$B = \frac{1}{10h}(-3, -72, -45, 240, -45, -72, -3),$$

$$E = (0, 0, 0, 0, 0, 0, 0),$$

$$C = \frac{1}{20}(-1, -56, -245, 0, 245, 56, 1),$$

$$Z_mC = \frac{1}{20}(Z_1, -18Z_1 - 38Z_2, 9Z_1 - 183Z_2 - 71Z_3, 10Z_1 + 150Z_2 - 150Z_3 - 10Z_4, 71Z_2 + 183Z_3 - 9Z_4, 38Z_3 + 18Z_4, Z_4),$$

where $Z_m = (\frac{U_m + U_{m+1}}{2})^2$. Now let's express that, for convenience, the discretized forms of the derivatives of the dependent variables u and y in Eqs.(3.3) and (3.4) are given as follows

$$(.)_t = \frac{(.)^{**} - (.)^*}{\Delta t}, \quad (.)_{xxt} = \frac{(.)^{**}_{xx} - (.)^*_{xx}}{\Delta t}, \tag{3.15}$$

$$(.)_x = \frac{(.)^{**}_x + (.)^*_x}{2}, \quad (.)_{xx} = \frac{(.)^{**}_{xx} + (.)^*_{xx}}{2}. \tag{3.16}$$

Substituting discretized forms (3.15) and (3.16) in Eqs.(3.13) and (3.14), the following matrix systems are acquired

$$(A + \mu B - \mu E)\delta^{n+1} = (A + \mu B - \mu E)\delta^n, \tag{3.17}$$

$$(A + \mu B - \mu E + (\epsilon C_1)\Delta t/2)\Psi^{n+1} = (A + \mu B - \mu E - (\epsilon C_1)\Delta t/2)\Psi^n. \tag{3.18}$$

By the use of the boundary condition $U(x_L, t) = U(x_R, t) = 0$ given in (1.3), the parameters $(\delta_{-3}, \delta_{-2}, \delta_{-1}, \delta_{N+1}, \delta_{N+2}, \delta_{N+3})$, $(\Psi_{-3}, \Psi_{-2}, \Psi_{-1}, \Psi_{N+1}, \Psi_{N+2}, \Psi_{N+3})$ are eliminated from the systems (3.17) and (3.18) and unknown parameters $\delta = (\delta_0, \delta_1, ..., \delta_N)^T$, $\Psi = (\Psi_0, \Psi_1, ..., \Psi_N)^T$ with $N + 1$ dimension are obtained. Therefore, $(N + 1)$ x $(N + 1)$ matrix system is acquired. To improve the nonlinear term existing in the system (3.18), an inner iteration given in the form $(\Psi^*)^n = \Psi^n + \frac{1}{2}(\Psi^n - \Psi^{n-1})$ is needed and it is applied 3-5 times at each time level. At the point $(x_m, t), (m = 0(1)N)$ for any time level, both the knot values and derivatives of the u_N and y_N can be expressed as

$$\begin{aligned}
(.)_m &= ()_{m+1} + 4()_m + ()_{m-1}, \\
(.)'_m &= \frac{3}{h}(()_{m+1} - ()_{m-1}), \\
(.)''_m &= \frac{6}{h^2}(()_{m+1} - 2()_m + ()_{m-1}).
\end{aligned} \tag{3.19}$$

Let's also state here that unknown time-parameters $\delta_j(t), \Psi_j(t)$ are used instead of () on the right side of Eq.(3.19). By substituting the nodal values in (3.19) in (3.17) and (3.18), the system of equations presented as below is obtained.

$$\begin{aligned}\mu_1\delta^{n+1}_{m-3} + \mu_2\delta^{n+1}_{m-2} + \mu_3\delta^{n+1}_{m-1} + \mu_4\delta^{n+1}_{m} + \mu_5\delta^{n+1}_{m+1} + \mu_6\delta^{n+1}_{m+2} + \mu_7\delta^{n+1}_{m+3} = \\ \mu_1\delta^{n}_{m-3} + \mu_2\delta^{n}_{m-2} + \mu_3\delta^{n}_{m-1} + \mu_4\delta^{n}_{m} + \mu_5\delta^{n}_{m+1} + \mu_6\delta^{n}_{m+2} + \mu_7\delta^{n}_{m+3},\end{aligned} \tag{3.20}$$

$$\begin{aligned}\rho_1\Psi^{n+1}_{m-3} + \rho_2\Psi^{n+1}_{m-2} + \rho_3\Psi^{n+1}_{m-1} + \rho_4\Psi^{n+1}_{m} + \rho_5\Psi^{n+1}_{m+1} + \rho_6\Psi^{n+1}_{m+2} + \rho_7\Psi^{n}_{m+3} = \\ \rho_1\Psi^{n}_{m-3} + \rho_2\Psi^{n}_{m-2} + \rho_3\Psi^{n}_{m-1} + \rho_4\Psi^{n}_{m} + \rho_5\Psi^{n}_{m+1} + \rho_6\Psi^{n}_{m+2} + \rho_7\Psi^{n}_{m+3},\end{aligned} \tag{3.21}$$

in which

$$\mu_1 = \frac{1}{140} - \frac{3\mu}{10h}, \quad \mu_2 = \frac{120}{140} - \frac{72\mu}{10h}, \quad \mu_3 = \frac{1191}{140} - \frac{45\mu}{10h}, \quad \mu_4 = \frac{2416}{140} + \frac{240\mu}{10h},$$
$$\mu_5 = \frac{1191}{140} - \frac{45\mu}{10h}, \quad \mu_6 = \frac{120}{140} - \frac{72\mu}{10h}, \quad \mu_7 = \frac{1}{140} - \frac{3\mu}{10h},$$

and

$$\rho_1 = \frac{1}{140} - \frac{3\mu}{10h} - \epsilon(\frac{\kappa}{20})\frac{\Delta t}{2}, \qquad \rho_2 = \frac{120}{140} - \frac{72\mu}{10h} - \epsilon(\frac{56\kappa}{20})\frac{\Delta t}{2},$$
$$\rho_3 = \frac{1191}{140} - \frac{45\mu}{10h} - \epsilon(\frac{245\kappa}{20})\frac{\Delta t}{2}, \qquad \rho_4 = \frac{2416}{140} + \frac{240\mu}{10h},$$
$$\rho_5 = \frac{1191}{140} - \frac{45\mu}{10h} + \epsilon(\frac{245\kappa}{20})\frac{\Delta t}{2}, \qquad \rho_6 = \frac{120}{140} - \frac{72\mu}{10h} + \epsilon(\frac{56\kappa}{20})\frac{\Delta t}{2},$$
$$\rho_7 = \frac{1}{140} - \frac{3\mu}{10h} + \epsilon(\frac{\kappa}{20})\frac{\Delta t}{2}.$$

where $\kappa = Z_m$. By using the systems (3.20) and (3.21), the calculation process is started up to the desired time level. First of all, it is necessary to know the unknown parameter δ^0. For this reason, the parameter δ^0 is found from the initial condition and its first derivative in Eq.(1.2). Firstly, system (3.20) is computed for δ^{n+1}. Then the earned value is written in place of Ψ^n in system (3.21). Now parameter δ^1 needs to be calculated. This is calculated with the parameter δ^0 found by Eq.(3.19). Thus, the process is completed at the desired time level.

3.1. **Stability Analysis.** To examine the stability analysis of systems (3.20) and (3.21) with the cubic B-spline Galerkin finite element method, the Fourier method [28] is utilized. For stability analysis in the nonlinear term y^2y_x in Equation (3.4), the local constant Z will be utilized instead of y^2. Then, $\kappa = Z_m$ in system (3.21) be going to be a constant number. When the Fourier modes $\delta^n_j = \varrho^n_1 e^{ij\Phi}, \Psi^n_j = \varrho^n_2 e^{ij\Phi}$ are substituted in systems (3.20),(3.21), respectively and also the Euler formula $e^{i\Phi} = cos\Phi + isin\Phi$, is used, Growth factors are obtained as follows

$$\varrho_1 = \frac{A_1 - iB_1}{A_1 + iB_1}, \quad \varrho_2 = \frac{A_1 - iC_1}{A_1 + iC_1}, \tag{3.22}$$

$$\begin{aligned}&A_1 = \frac{1}{70}a_1 - \frac{3\mu}{5h}a_2, \quad B_1 = 0, \quad C_1 = \frac{\epsilon z_m \Delta t}{20}c_3,\\ &a_1 = \frac{1}{70}(cos3\Phi + 120cos2\Phi + 1191cos\Phi + 1208),\\ &a_2 = -\frac{3\mu}{5h}(cos3\Phi + 24cos2\Phi + 15cos\Phi + 40),\\ &a_3 = cos3\Phi + cos2\Phi + cos\Phi.\end{aligned}$$

It is $|\varrho_1| = |\varrho_2| = 1$ from Equation (3.22) and therefore $|\varrho_1|.|\varrho_2| = 1$. It can be clearly observed that systems (3.20) and (3.21) are unconditionally stable because of the conditions $|\varrho_1| \leq 1$ and $|\varrho_2| \leq 1$ are satisfied.

Table 1. Analytical values of the invariants I_1, I_2, I_3 for different values of c of Example 1.

method	c	I_1	I_2	I_3
analytic	0.25	0.7853981633974	0.1666666666667	0.00520833333333
	0.50	1.5707963267949	0.6666666666667	0.08333333333333
	0.75	2.3561944901923	1.5	0.421875
	1	3.1415926535898	2.6666666666667	1.33333333333333

4. Numerical experiments and comparisons

In this section, well-known two test examples,that is, the motion of single solitary wave and the Maxwellian initial condition pulse, are investigated. New solutions are compared with those previously found in the literature. For this reason, error norms L_2 and L_∞ and invariant values I_1, I_2 and I_3 are calculated. Thus, it is checked how accurate results the new method produces and how reliable it is.

Example 1: The movement of a single solitary wave

The solitary wave solution of the MEW equation (1.1) with physical boundary conditions $U(\pm\infty) \to 0$ is presented as

$$U(x,t) = csech[k(x - x_0 - vt)],$$

in which $v = \dfrac{c^2}{2}, k = \sqrt{\dfrac{1}{\mu}}$. Invariants I_1, I_2 and I_3 of the Eq.(1.1) are computed as numerically and analytically respectively as follows

$$I_1 = h\sum_{j=0}^{N} U_j, \quad I_2 = h\sum_{j=0}^{N}[U_j^2 + \mu(U_j')^2], \quad I_3 = h\sum_{j=0}^{N} U_j^4,$$

and

$$I_1 = \frac{c\pi}{k}, I_2 = \frac{2c^2}{k} + \frac{2\mu kc^2}{3}, I_3 = \frac{4c^4}{3k}.$$

For the amplitude value $c = 1$, the analytical solutions of these invariants are given as $I_1 = 3.1415926535898, I_2 = 2.6666666666667, I_3 = 1.3333333333333$ respectively. The computed analytical values of the invariants for different amplitudes are presented in Table 1. In order to measure the effectiveness of the method studied, a comparison of the current results with the same parameter values of the existing previous studies in the literature is made. That's why, $\mu = 1, x_0 = 30$ and $c = 0.25$ with the parameter $h = 0.1$ are chosen. As can be seen from the Table 2-5, the acquired results are obtained at different time increments, different end time values T and regions $[0, 80], [0, 70]$ and we can easily state from the Table 2-5 that the results obtained with new approach are much better than other methods except refs.[5, 15, 21] given in Tablo 3 and the invariants are compatible and in addition, to further emphasize the importance of the proposed approach, we can be say that it is remarkable that the invariant values presented in Table 6 are the same as the analytical ones. For the parameters $h = 0.1, \Delta t = 0.05$ and different values of amplitudes $1, 0.75, 0.50, 0.25$, the motion of solitary wave is given in Figure 1. It is clear from this figure that the bigger wave with amplitude $c = 1$ goes a long way because it is faster than the waves with other smaller amplitudes.

Example 2: The Maxwellian initial condition

The movement of the solitary wave is presented with the Maxwellian initial condition

$$U(x,0) = e^{-x^2},$$

and boundary conditions $U(x_L,t) = U(x_R,t) = 0$. Considering the different values $0.5, 0.1, 0.05, 0.02, 0.005, 0.0025$ of μ and the parameters $h = 0.05, \Delta t = 0.01$ at time $T = 12.5$ throughout this example, a comparison of the invariants I_1, I_2 and I_3 is given in Table 7 taking into account [7] and [34] and also various solitary wave movements are obtained for the same values Δt and h at the time $T = 12.5$ on the region $[-20, 20]$ and these ones are plotted in Figure 2.

TABLE 2. According to the solutions of related studies, comparison of invariants and error norms at T=20 of Example 1.

method	t	I_1	I_2	I_3	L_2 x 10^4	L_∞ x 10^4
$\Delta t = 0.2$,$h = 0.1$, $T = 20$, $[0, 80]$						
LTS.LGall.	5	0.7853982	0.1666665	0.0052083	0.204391	0.114657
	10	0.7853982	0.1666665	0.0052083	0.406971	0.230054
	15	0.7853983	0.1666666	0.0052083	0.606022	0.345027
	20	0.7853983	0.1666666	0.0052083	0.800015	0.457917
TColl.[34]	20	0.7850300	0.1666259	0.0052058	1.471099	0.897036
Coll.[10]	20	0.7852864	0.1665818	0.0052061	2.021476	1.569539
FD.[12]	10	0.7853977	0.1664736	0.0052083	2.701647	2.576377
D.Quad.[5]	20	0.7854013	0.1666670	0.0052084	0.011493	0.007664
LGall.[11]	20	–	–	–	0.81006	0.45968

TABLE 3. For existing schemes, comparison of invariants and error norms at T=20 of Example 1.

method	I_1	I_2	I_3	L_2 x 10^4	L_∞ x 10^4
$\Delta t = 0.05$,$h = 0.1$, $T = 20$, $[0, 80]$					
LTS.LGall.	0.7853983	0.1666666	0.0052083	0.80284	0.46094
SS.Coll.[7]	0.7853982	0.1666666	0.0052083	1.75081	1.76288
T-Coll.[34]	0.7850300	0.1666259	0.0052058	1.46806	0.89667
FD.[12]	0.7853977	0.1664735	0.0052083	2.69281	2.56997
Multgrd.[13]	0.7853965	0.1666638	0.0052081	0.05208	0.05456
SD.[15]	0.7853967	0.1666664	0.0052083	0.51873	0.32113
SD.[21]	0.7853967	0.1666663	0.0052083	0.51774	0.32114
P.Gal.[16]	0.7853967	0.1666663	0.0052083	0.80146	0.46121
L.Gal.[20]	0.7853967	0.1666663	0.0052083	0.80098	0.46061
DL.Coll. [22]$\hat{1}$	0.7853966	0.1666662	0.0052083	1.75277	1.76465
DL.Coll.[22]$\hat{2}$	0.7853966	0.1666662	0.0052083	1.75270	1.76459
Coll.[10]	0.7849545	0.1664765	0.0051995	2.90516	2.49892
D.Quad.[5]	0.7853979	0.1666671	0.0052084	0.01653	0.01194
LGal.[11]	0.7853898	0.1667614	0.0052082	0.79694	0.46553
$\Delta t = 0.05$,$h = 0.1$, $T = 20$, $[0, 70]$					
LTSL.Gall.	0.7853983	0.1666666	0.0052083	0.80284	0.46094
LGal.[11]	0.7853970	0.1667636	0.0052083	0.80145	0.46009

TABLE 4. According to the solutions of related studies, comparison of invariants and error norms at T=1 of Example 1.

method	I_1	I_2	I_3	L_2 x 10^4	L_∞ x 10^4
$\Delta t = 0.1$,$h = 0.1$, $T = 1$, $[0, 80]$					
LTS.LGal.	0.785398	0.166666	0.0052083	0.04104	0.02297
FD.[29]$\hat{1}$	0.785341	0.166453	0.0052071	0.29610	0.23507
FD.[29]$\hat{2}$	0.787173	0.167079	0.0052460	7.27192	4.11720
FD.[29]$\hat{3}$	0.785398	0.166473	0.0052083	0.17711	0.10834

TABLE 5. According to the solutions of related studies, comparison of invariants and error norms at T=20 of Example 1.

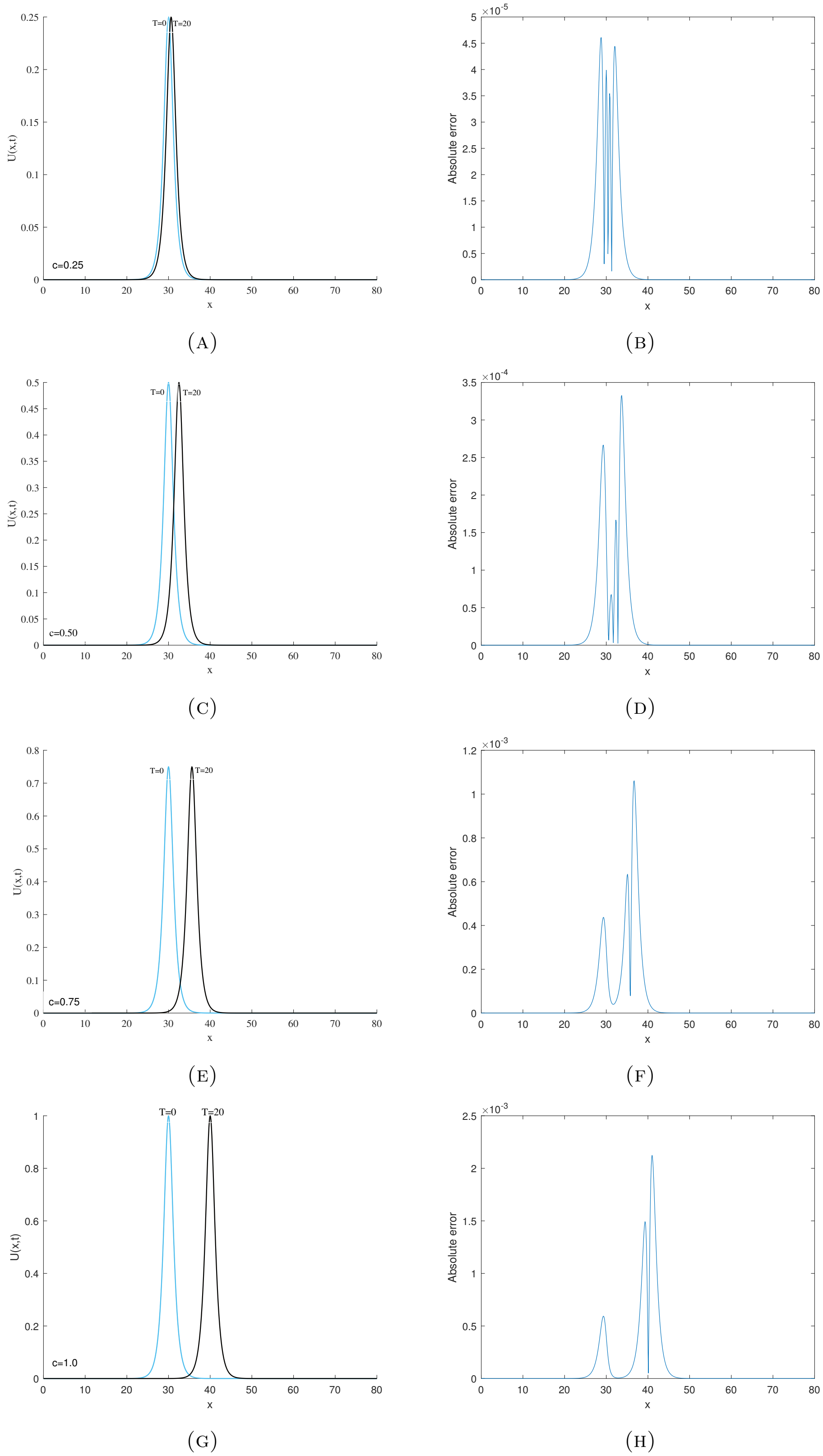
method	I_1	I_2	I_3	L_2 x 10^4	L_∞ x 10^4
$\Delta t = 0.01$,$h = 0.1, T = 20$,[0,80]					
LTS.LGal.	0.7853983	0.1666666	0.0052083	0.80302	0.46113
DL.Coll.[22]	0.7853967	0.1666662	0.0052083	1.75233	1.76422

TABLE 6. According to the solutions of related studies, comparison of invariants and error norms at T=20 for different amplitude values c of Example 1.

c		I_1	I_2	I_3	L_2 x 10^4	L_∞ x 10^4
$\Delta t = h = 0.01, T = 20$, $[0, 80]$						
1.0	LTS.LGall.	3.1415927	2.6666667	1.3333333	0.116354	0.065329
	TColl.[34]	3.1415779	2.6666660	1.3333267	1.010366	0.626081
	FD.[12]	3.1415790	2.6666350	1.3333310	1.494558	0.987068
	Analytical	3.1415927	2.6666667	1.3333333	–	–
0.75	LTS.LGall.	2.3561945	1.5000000	0.4218750	0.117483	0.082401
	TColl.[34]	2.3561834	1.4999963	0.4218729	0.229900	0.149503
	FD.[12]	2.3561860	1.4999790	0.4218745	0.519345	0.366739
	Analytical	2.3561945	1.5000000	0.4218750	–	–
0.50	LTS.LGall.	1.5707963	0.6666667	0.0833333	0.050773	0.032100
	TColl.[34]	1.5707889	0.6666650	0.0833329	0.057187	0.038677
	FD.[12]	1.5707920	0.6666588	0.0833333	0.186465	0.150972
	Analytical	1.5707963	0.6666667	0.0833333	–	–
0.25	LTS.LGall.	0.7853982	0.1666667	0.0052083	0.008053	0.004634
	TColl.[34]	0.7853945	0.1666663	0.0052083	0.014686	0.009014
	FD.[12]	0.7853963	0.1666644	0.0052083	0.026985	0.026867
	Analytical	0.7853982	0.1666667	0.0052083	–	–

TABLE 7. According to the solutions of related studies, The invariants of Example 2 for different values of µ at T = 12.5.

method	I_1	I_2	I_3	I_1	I_2	I_3
	$\mu = 0.5$			$\mu = 0.1$		
LTS.LGal.	1.77273	1.88048	0.88664	1.77107	1.37451	0.88132
SS.coll.[7]	1.77245	1.88008	0.88623	1.77249	1.37774	0.88627
TColl.[34]	1.77235	1.87971	0.88597	1.77244	1.37783	0.88619
	$\mu = 0.05$			$\mu = 0.02$		
LTS.LGal.	1.76970	1.30753	0.87524	1.76524	1.25697	0.85785
SS.coll.[7]	1.77254	1.31431	0.88639	1.77275	1.27458	0.88717
TColl.[34]	1.77246	1.31444	0.88644	1.77256	1.27424	0.88660
	$\mu = 0.005$			$\mu = 0.0025$		
LTS.LGal.	1.74495	1.180638	0.78338	1.72161	1.11702	0.71084
SS.coll.[7]	1.77465	1.25032	0.89902	1.77868	1.24930	0.92893
TColl.[34]	1.77311	1.23603	0.86783	1.76963	1.19626	0.81240



(A) (B)

(C) (D)

(E) (F)

(G) (H)

FIGURE 1. Movement of a single solitary wave at T=20 for different amplitude values c.

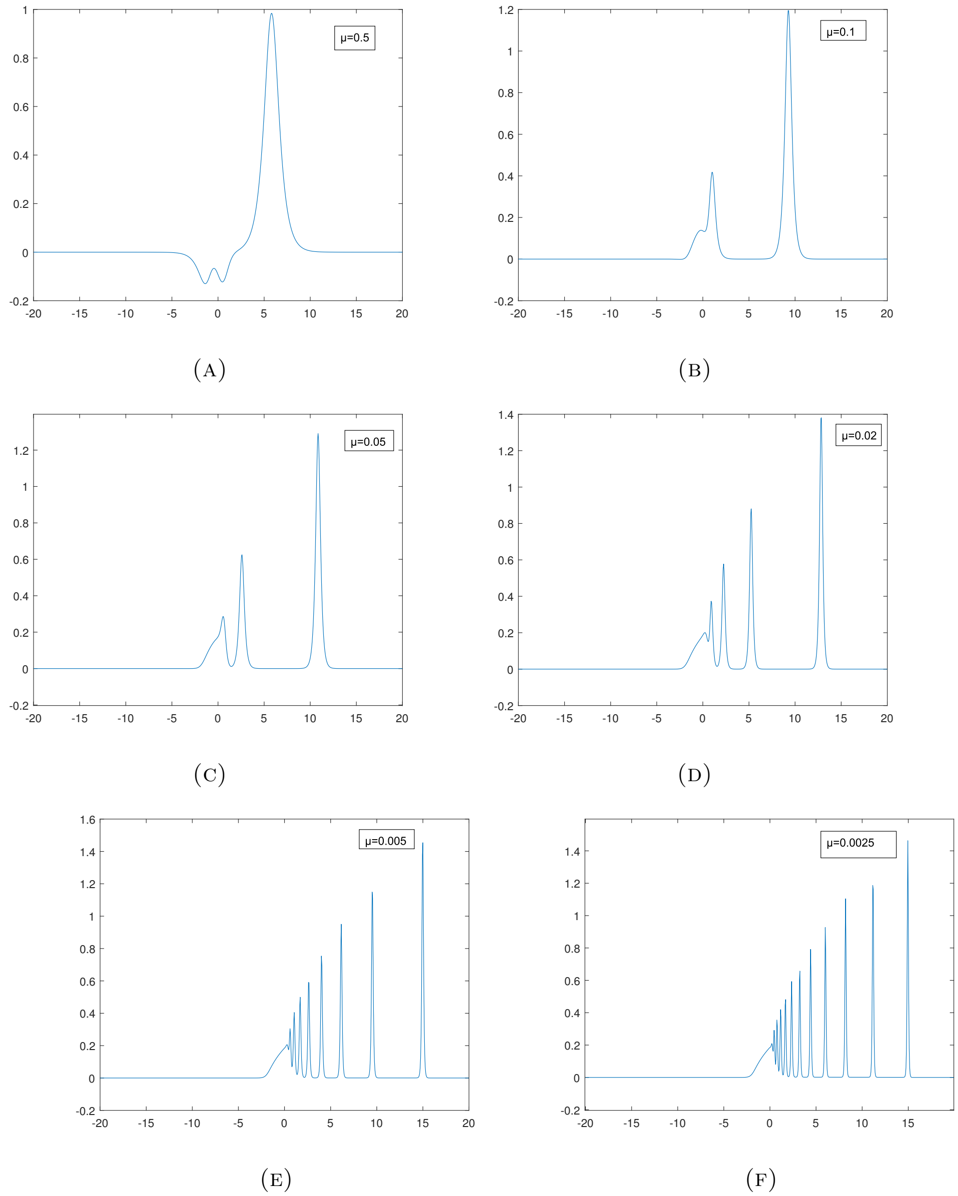


FIGURE 2. Maxwellian initial condition for different values of μ.

5. CONCLUSION

In this article, numerical computation of the MEW equation with Lie-Trotter splitting algorithm combined with finite element collocation method with quintic B-spline is investigated. The error norms L_2 and L_∞ and the conservation constants I_1, I_2, and I_3 are calculated to demonstrate the performance of this new algorithm. It can be clearly seen from the tables presented in the study that the newly obtained numerical solutions are good enough in comparison with some existing results in the literature. It can also be stated that this new technique can be easily applied to other partial differential equations used in other fields of science in terms of obtained results and computational cost.

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