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Optimal control of Volterra integro-differential equations based on interpolation polynomials and collocation method

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Abstract

In this paper, we introduce a new direct scheme based on Dickson polynomials and collocation points to solve a class of optimal control problems (OCPs) governed by Volterra integro-differential equations namely Volterra integro-OCPs (VI-OCPs). This topic requires to calculating the corresponding operational matrices for expanding the solution of this problem in terms of Dickson polynomials. Further, the highlighted method allows us to transform the VI-OCP into a system of algebraic equations for choosing the coefficients and control parameters optimally. The error estimation of this technique is also investigated which given the high efficiency of the Dickson polynomials to deal with these problems. Finally, some examples are brought to confirm the validity and applicability of this approach in comparison with those obtained from other methods.

Keywords. Dickson polynomials, Optimal control problem, Volterra integro-differential equation, Algebraic equations, Collocation points, Error estimation.

2010 Mathematics Subject Classification. 34K35, 49M25, 34H05.

1. Introduction

The optimal control problem refers to the minimization of a performance index subject to dynamic constraints on the state and the control variables. After the physical realization of OCPs in a diverse world [24, 29], over the past decades, many researchers have been trying to build an intelligent method for solving OCPs with computational power comparable to the hardware simplicity. These efforts are often divided into two main categories. The first category is for works that attempt to use indirect methods and the second category comprises studies focused on the direct methods [5, 28, 31, 35]. The importance of having practical and scalable hardware becomes clear when we look at the results of these two methods and the complexity of connections between them. Since it is difficult to achieve satisfactory performance in indirect methods, it is important to examine the accurate solutions of OCPs which are widely used in many physical and engineering phenomena in the real world. With an overview, you will find that direct methods have attracted more attention than indirect methods due to their greater convergence radius [23, 39]. Also, unlike indirect methods, direct methods are more strong to the primitive guess of parameters without general deformation of the total problem. The aforementioned properties of direct methods have encouraged some researchers to develop new computing architectures and techniques where the primary focus was on hardware simplicity.

Recently, many researchers have been fascinated by integral dynamics in the fields of applied sciences such as epidemiology, biology, economics, and memory effects [6, 15, 25]. This issue has been used through modeling many nonlinear physical applications evidently. More specifically, integral equations are used to illustrate the mechanics and dynamic systems [43–46]. A hybrid method based on the block-pulse functions proposed to solve two cases of integro-differential equations in [18]. Alpert wavelet system and Newton's iterative method are applied to solve a fractional nonlinear integro-differential equation in [30]. The authors in [16] have presented Chebyshev cardinal functions for the

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solution of integro-differential equations. For more details about this topic, we can refer interested readers to [40–42]. Optimal control of these equations has recently become a major topic of researches, for instance, one application of such equations was modeled by Kamien and Muller [11]. Nonetheless, it should be noted that due to the high complexity of integral terms in VI-OCPs, handling analytical solutions of these problems are completely tough and even impossible. To be prevailed through this challenge, researchers have resorted to approximate methods. Hence, they first generalized the methods used to solve OCPs such as homotopy analysis and parameterization method [1], reduction method [3], Legendre polynomials [36], triangular functions [20], Hybrid functions [22] and Muntz-Legendre polynomials [26]. Despite the existence of many applications for VI-OCPs in control systems but it is regrettable that extremely few publications for this problem were reported [21, 27]. Therefore, it is quite clear that the numerical studies of this problem are still in the early stages of growth. As a matter of fact, given the complex nature of these problems and compared to the direct methods, indirect ones have more complex dynamic characteristics. Therefore, merging the direct methods for solving VI-OCPs is highly anticipated. In this paper, we consider the following problem:

$$Min\ J(x(t),u(t)) = \int_a^b f(t,x(t),u(t))dt \tag{1.1}$$

subject to the nonlinear time-invariant system

$$\dot{x}(t) = g(t) + \int_a^t k(t,s,\, x(s),u(s))ds, \tag{1.2}$$

with the initial condition

$$x(a) = x_0, \qquad u(t) \in U, \tag{1.3}$$

where $x_0 \in \mathbb{R}$, a and b are two positive constants, g and k are assumed to be continuously differentiable functions in all arguments, the set $U \subset \mathbb{R}^m$ denotes the acceptable inputs and $x(t)$ is the state variable known as the optimal trajectory. The problem is to find $u(t)$ that will drive the system in (1.2) with the initial state (1.3) while minimizing the cost functional (1.1) where f is a continuously differentiable function. Motivated by the above discussions, at this time, we want to introduce a direct method based on Dickson polynomials approximation for solving VI-OCP (1.1)-(1.3) as follows:

$$x(t) \simeq x_M(t) = \sum_{i=0}^{M} D_i(t,\alpha)x_i \qquad u(t) \simeq u_M(t) = \sum_{i=0}^{M} D_i(t,\alpha)u_i, \tag{1.4}$$

where x_i and u_i, $i = 0,1,2,...,M$, are the unknown Dickson coefficients. Indeed, we have chosen the Dickson polynomials to estimate the offer state, control variable, and hence the objective function. Accordingly, to obtain an approximate solution via (1.4), we used the following collocation points:

$$t_j = a + (\frac{b-a}{M})j, \qquad j = 0,1,2,...,M, \tag{1.5}$$

where $a = t_0 < t_1 < t_2 < ... < t_M = b$. One of the main advantages of the Dickson collocation method is its efficiency and rapidly solving a wide range of problems. A numerical approach with error estimation was proposed to solve general integro-differential equations using Dickson polynomials in [12]. Authors in [13], studied a novel collocation method based on Dickson and Taylor polynomials to solve integro-differential equations. Some widely-used model problems consisting of linear, nonlinear differential, and integral equations with employing Dickson polynomials are investigated in [2, 8, 14]. In addition, these polynomials have simple forms and are computationally easy to use that vividly cause the solution procedure is either reduced and simplified. The proposed method allows us to transform the VI-OCPs into a system of algebraic equations with a matrix form of unknown coefficients for choosing the state and control parameters optimally. The error estimation of this technique is also investigated. The significant merits of this approach are swift calculations, efficiency, ease of implementation, and robustness. Indeed, it provides satisfactory results even a small number of the Dickson polynomials is used. Simple operations and ease of implementation are further characteristics of the mentioned polynomials. To attain these aims, the suitable choice of α, the parameter of Dickson polynomials, plays a crucial role to enhance the accuracy of the results evaluated by the current approach.

The overall layout of this manuscript is according to the following pattern. In section 2, the Dickson polynomials have been formulated and their properties, including the function approximation and the operational matrix of derivatives, are discussed. Also, we present a direct collocation scheme based on Dickson polynomials to solve the VI-OCP (1.1)-(1.3). The error estimation and the convergence analysis of this approach are carried out in section 3. The numerical results and comparison have been brought in section 4 to substantiate the efficiency of our results and then the conclusions are expressed in the last section.

2. Main matrix relation and method of solution

In this section, in order to construct the method of solution of VI-OCP (1.1)-(1.3), we first introduce the Dickson polynomials and then give their developed matrix relations.

Dickson polynomials $D_m(t,\alpha)$ are definable over a commutative ring R in which, if $R = \mathbb{C}$ be the set of complex numbers, $D_m(t,\alpha)$ is associated with the known Chebyshev polynomials of the first kind $T_m(t)$. Exactly, $D_m(2cos\theta, 1) = 2T_m(cos\theta)$ for any real number θ, and we have Lucas polynomials when $\alpha = -1$ [7]. For any integer $m \geq 1$ and any element α over finite fields, we define the first kind of Dickson polynomial of degree m as follow:

$$D_m(t,\alpha) = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m}{m-i} \binom{m-i}{i} (-\alpha)^i t^{(m-2i)}, \qquad -\infty < t < \infty, \tag{2.1}$$

where $\lfloor \frac{m}{2} \rfloor$ is the floor of $\frac{m}{2}$. Besides, $D_0(t,\alpha) = 2$, $D_1(t,\alpha) = t$ and for $m > 1$, we have the following recurrence relation [17]:

$$D_m(t,\alpha) = tD_{m-1}(t,\alpha) - \alpha D_{m-2}(t,\alpha), \qquad m \geq 2. \tag{2.2}$$

Further, the Dickson polynomials $D_m(t,\alpha)$ satisfy the following ordinary differential equations

$$(t^2 - 4\alpha)x'' + tx' - m^2 x = 0, \qquad m = 0, 1, 2, ... \tag{2.3}$$

The Dickson polynomials have the generating function

$$\sum_{m=0}^{\infty} D_m(t,\alpha) v^m = \frac{2 - t\,v}{1 - t\,v + \alpha\, v^2}. \tag{2.4}$$

The reader can refer to [4, 9, 10, 19, 33, 37, 38] for more information about the Dickson polynomials.

To perform the continuous functions $x(t)$ and $u(t)$ of VI-OCP (1.1)-(1.3) via truncated Dickson polynomials presented in (1.4), we have outline our approach in this section. Firstly, Eq. (1.4) can be rewritten in the following matrix form:

$$\begin{aligned} x(t) &\simeq x_M(t) = D(t,\alpha)X = Y(t)K(\alpha)X, \\ u(t) &\simeq u_M(t) = D(t,\alpha)U = Y(t)K(\alpha)U, \end{aligned} \tag{2.5}$$

where $X = [x_0, x_1, ..., x_M]^T$ and $U = [u_0, u_1, ..., u_M]^T$ are unknown coefficients, $Y(t) = [1, t, t^2, ..., t^M]$ and

$$D(t,\alpha) = [D_0(t,\alpha), D_1(t,\alpha), ..., D_M(t,\alpha)].$$

In addition, if M is even

$$K^T(\alpha) = \begin{bmatrix} 2 & 0 & 0 & 0 & ... & 0 \\ 0 & \frac{1}{1}\binom{1}{0}(-\alpha)^0 & 0 & 0 & ... & 0 \\ \frac{2}{1}\binom{1}{1}(-\alpha)^1 & 0 & \frac{2}{2}\binom{2}{0}(-\alpha)^0 & 0 & ... & 0 \\ 0 & \frac{3}{2}\binom{2}{1}(-\alpha)^1 & 0 & \frac{3}{3}\binom{3}{0}(-\alpha)^0 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{M}{M/2}\binom{M/2}{M/2}(-\alpha)^{M/2} & 0 & \frac{M}{(M/2)+1}\binom{(M/2)+1}{(M/2)-1}(-\alpha)^{(M/2)-1} & 0 & \cdots & \frac{M}{M}\binom{M}{0}(-\alpha)^0 \end{bmatrix},$$

and if M is odd

$$K^T(\alpha) = \begin{bmatrix} 2 & 0 & 0 & 0 & ... & 0 \\ 0 & \frac{1}{1}\binom{1}{0}(-\alpha)^0 & 0 & 0 & ... & 0 \\ \frac{2}{1}\binom{1}{1}(-\alpha)^1 & 0 & \frac{2}{2}\binom{2}{0}(-\alpha)^0 & 0 & ... & 0 \\ 0 & \frac{3}{2}\binom{2}{1}(-\alpha)^1 & 0 & \frac{3}{3}\binom{3}{0}(-\alpha)^0 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{M}{\lceil M/2 \rceil}\binom{\lceil M/2 \rceil}{\lfloor M/2 \rfloor}(-\alpha)^{\lfloor M/2 \rfloor} & 0 & \frac{M}{\lceil M/2 \rceil+1}\binom{\lceil M/2 \rceil+1}{\lfloor M/2 \rfloor-1}(-\alpha)^{\lfloor M/2 \rfloor-1} & \cdots & \frac{M}{M}\binom{M}{0}(-\alpha)^0 \end{bmatrix}.$$

Now, for the matrix form of first derivative we have:

$$\begin{aligned} \dot{x}(t) &\simeq \dot{x}_M(t) = \dot{D}(t,\alpha)X = \dot{Y}(t)K(\alpha)X = Y(t)CK(\alpha)X, \\ \dot{u}(t) &\simeq \dot{u}_M(t) = \dot{D}(t,\alpha)U = \dot{Y}(t)K(\alpha)U = Y(t)CK(\alpha)U, \end{aligned} \tag{2.6}$$

where

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 0 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & M \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Now, for solving the VI-OCP (1.1)-(1.3), we need to find the approximations presented in relation (1.4). For this purpose, based on these approximtions and also using relations (2.5)-(2.6), the performance index (1.1) can be rewrite as follow:

$$Min\ J(X,U) = \int_a^b f(t, Y(t)K(\alpha)X, Y(t)K(\alpha)U)dt \cong G(X,U). \tag{2.7}$$

In a similar way, taking into account the above approximations for the dynamical system (1.2), we have:

$$Y(t)CK(\alpha)X - g(t) - \int_a^t k(t,s,Y(s)K(\alpha)X, Y(s)K(\alpha)U)ds \cong \Lambda(t,X,U) \cong 0. \tag{2.8}$$

Furthermore, by taking the collocation points t_j, $j = 1, \cdots, M$, which is defined in relation (1.5) into Eq. (2.8), and employing the Simpson's integration rule, it leads to the following system of algebraic equations:

$$\Lambda_i \cong \Lambda(t_i, X, U) \cong 0, \qquad i = 1, ..., 2M. \tag{2.9}$$

Also, by doing the similar process for the initial conditions (1.3), we obtain

$$\Lambda_0 \cong Y(a)K(\alpha)X - x_0 = 0. \tag{2.10}$$

To get the approximate solutions of VI-OCP (1.1)-(1.3), we can adopt the Lagrange multipliers method for minimizing (2.7) subject to the conditions given in (2.9)-(2.10) as

$$J^*(X,U,\lambda) = G(X,U) + \lambda\Lambda, \tag{2.11}$$

where $\lambda = [\lambda_0, \lambda_1, ..., \lambda_{2M}]$ are the unknown Lagrange multipliers and $\Lambda = [\Lambda_0, \Lambda_1, ..., \Lambda_{2M}]$. The necessary conditions for the optimality of functional (2.11) are as follows:

$$\frac{\partial J^*}{\partial X} = 0, \qquad \frac{\partial J^*}{\partial U} = 0, \qquad \frac{\partial J^*}{\partial \lambda} = 0. \tag{2.12}$$

To solve Eq. (2.12), we can use today's mathematical packages such as Mathematica or Matlab.

3. Convergence analysis of the proposed scheme

Kürkcü et al, [14] performed a convergence study on the Dickson polynomial solution for integro equations by using the residual function in Banach space. Here, we investigate the convergence of these polynomial solutions for VI-OCPs. The residual function shows a distinctive reaction for the different values of M. Therefore, we can determine the behavior of our solutions.

Let P_M be the set of all Dickson polynomials of degree at most M. Without loss of generality suppose that $[a,b]=[0,1]$. If $f(t)$ be a function in $L^2[0,1]$, since P_M is a finite space, $f(t)$ has a best unique approximation out of P_M like as $\hat{f}(t)$ such that:

$$\forall g \in P_M: \qquad ||f-\hat{f}||_2 \leq ||f-g||_2.$$

Suppose that $f_m \in P_M$. Then, there exist coefficients $c_k, k=0,1,...,m$, such that

$$f_m(t) \approx \sum_{k=0}^{m} c_k D_k(t,\alpha), \tag{3.1}$$

in which c_k are real valued unknown coefficients and $D_k(t,\alpha)$ are the Dickson functions for $k=0,1,...,m$.

Theorem 3.1. Let $f \in L^2[0,1]$ has been approximated by f_M in terms of Dickson polynomials in which $f_M(t) = \sum_{k=0}^{M} c_k D_k(t,\alpha)$. If $e_M(t) = \|f(t)-f_M(t)\|$ then $\lim_{M\to\infty} e_M(t) = 0$.

Proof. We divide interval $[0,1]$ into subintervals $\left[\frac{n-1}{N}, \frac{n}{N}\right]$, $n=1,2,\cdots,N$, with the limitation that f_n approximates f over the subinterval $\left[\frac{n-1}{N}, \frac{n}{N}\right]$, $n=1,2,\cdots,N$ and $f(t) \simeq \sum_{n=1}^{N} f_n(t)$. Using the Taylor expansion, we define the following approximation of $f(t)$ out of P_N as follows:

$$f_n(t) = \sum_{k=0}^{M} \frac{(t-\frac{n-1}{N})^k}{\Gamma(k+1)} f^{(k)}(\frac{n-1}{N}).$$

Then we have:

$$|f(t)-f_n(t)| \leq \frac{(t-\frac{n-1}{N})^{M+1}}{\Gamma(M+2)} |f^{(M+1)}(\xi)|, \quad \xi \in \Big[\frac{n-1}{N}, \frac{n}{N}\Big].$$

Assume that $f_M(t) = \sum_{k=0}^{M} c_k D_k(t,\alpha)$ be the best approximation of f. Then we have:

$$\begin{aligned}
\|f(t)-f_M(t)\|_2^2 &= \int_0^1 |f(t)-f_M(t)|^2 dt = \sum_{n=1}^{N} \int_{\frac{n-1}{N}}^{\frac{n}{N}} |f(t)-f_M(t)|^2 dt \\
&\leq \sum_{n=1}^{N} \int_{\frac{n-1}{N}}^{\frac{n}{N}} |f(t)-f_n(t)|^2 dt \leq \sum_{n=1}^{N} \int_{\frac{n-1}{N}}^{\frac{n}{N}} |f^{(M+1)}(\xi)|^2 \Big(\frac{(t-\frac{n-1}{N})^{M+1}}{(M+1)!}\Big)^2 dt \\
&\leq \frac{L^2}{(2M+3)((M+1)!)^2 N^{2(M+1)}},
\end{aligned}$$

where $L = max|f^{(M+1)}(t)|$, $t \in [0,1]$. Now, by taking square roots we obtain

$$\|f(t)-f_M(t)\|_2 \leq \frac{L}{(M+1)! N^{(M+1)}} \sqrt{\frac{1}{(2M+3)}}. \tag{3.2}$$

This means that the error approximation with Dickson polynomials tends to zero when M and N are sufficiently increased. □

The above theorem confirms that $f_M(t)$ converges to $f(t)$. It is also easy to conclude that, by increasing the number of M, the series approximation for derivative of Dickson polynomials defined by the operational matrix in Eq. (2.6), converges to $\dot{f}(t)$. Two approaches can be made for verifying a numerical method, namely, the convergence and error bound provision. The error bounds for the Dickson polynomials approximation operator is presented. In the following, the convergence of the proposed approximate method are provided.

Consider a set of pairwise distinct points in $\Omega = [0, 1]$ such as $X = \{x_1, x_2, \cdots, x_M\}$ with the fill distance $h_{X,\Omega} = \sup_{x\in\Omega} \inf_{x_i\in X} \|x - x_i\|$. Indeed, we used the collocation points $x_i \in X$, in which $J(x_i) = 0$, $i = 1, 2, \cdots, M$. Let $\{\phi_1, \phi_2, \cdots, \phi_M\}$ be a collocation set of functions on Ω such that $\phi_i(x_j) = \delta_{i,j}$, in which $\delta_{i,j}$ is the cardinal function. Also, assume that $f \in C^m(\Omega)$. Then, for any $\epsilon > 0$ and $a \in \Omega$, there exists $c_i(a) \in \mathbb{R}$, $i = 1, 2, \cdots, M$, such that $f(x) - \sum_{i=1}^{M} c_i(a)\phi_i(x) = h_{f,m}(x, a)$, in which $|h_{f,m}(x, a)| \leq H_{f,m}(a, h_{X,\Omega})$ and $\lim_{h\to 0} H_{f,m}(a, h) = 0$. An immediate result from this discussion is that $|f(x) - \sum_{i=1}^{M} c_i\phi_i(x)| \leq H_{f,m}(h_{X,\Omega})$ for all $x \in \Omega$. So, we can define the previous continuous functions with this approximation as we denote by Dickson polynomials in (2.1). We continue the discussion by estimating the error bound of our numerical method.

Theorem 3.2. Assume that $X = \{x_1, x_2, \cdots, x_N\}$ be a set of pairwise distinct points in compact set Ω and $\{\phi_1, \phi_2, \cdots, \phi_M\}$ as before. If there exists $H_{f,m}(h_X, \Omega)$ such that $|f(x) - \sum_{i=1}^{M} f(x_i)\phi_i(x)| \leq H_{f,m}(h_{X,\Omega})$ for any $f \in C^m(\Omega)$, then, there exist w_i, $i = 1, 2, \cdots, M$, such that:

$$|\int_\Omega f(x)dx - \sum_{i=1}^{M} w_i f(x_i)| \leq H_{f,m}(h_{X,\Omega}). \tag{3.3}$$

Proof. It is directly provided from the above context. □

In the sequel, we provide a completely nonlinear equation and discuss its residual error. For this purpose, let $H_{u,m}(h_{X,\Omega}) = Ch^m_{X,\Omega}\|u\|_{H^m(\Omega)}$, $C > 0$, and consider the following general nonlinear equation:

$$Ly = f(y), \tag{3.4}$$

where L is a linear differential operator from $H^m(\Omega)$ to the Banach space of functions which is denoted by χ and f is a nonlinear operator. Using the proposed method in this paper, the approximation solution of Eq. (3.4) will be satisfied in the following system:

$$\sum_{i=1}^{M} c_i \int_\Omega L\phi_i(x)\phi_j(x)dx = \int_\Omega f\Big(\sum_{i=1}^{M} c_i\phi_i(x)\phi_j(x)\Big)dx. \tag{3.5}$$

By Theorem 3.2, we obtain

$$\sum_{i=1}^{M} c_i \sum_{k=1}^{M} w_k L\phi_i(x_k)\phi_j(x_k) = \sum_{k=1}^{M} w_k f\Big(\sum_{i=1}^{M} c_i\phi_i(x_k)\phi_j(x_k)\Big) \tag{3.6}$$

in whcih $x_k \in X$, $k = 1, 2, \cdots, M$. Furthermore, we know $\phi_j(x_i) = \delta_{i,j}$. So, if $w_j \neq 0$, we have:

$$\sum_{i=1}^{M} c_i L\phi_i(x_j) = f(c_j), \quad j = 1, 2, \cdots, M. \tag{3.7}$$

Now, we can determine the unknown coefficients from (3.7) and then approximate $u(x_i)$, $i = 1, 2, \cdots, M$. Generally, it can be concluded that the proposed method has consistency with the following error estimate:

$$\|Ly_M - f(y_M)\|_\chi \leq Ch^m_{X,\Omega}\|y\|_{H^m(\Omega)}, \tag{3.8}$$

where y_M is the approximate solution of (3.4) with Dickson polynomials.

Theorem 3.3. Suppose $\hat{x}_M(x)$ and $\hat{u}_M(x)$ are the Dickson polynomials approximate solutions for problem (1.1)-(1.3) which obtained from Eq. (2.12). While M tends to infinity, the state and control approximate variables converge to the exact values.

Proof. The proof is directly obtained from Theorem 3.1, Theorem 3.2 and the discussion given in [34]. □

As discussed in the above, one may conclude that by increasing the number of Dickson polynomials approximation, the approximate solution converges to the exact solution of the problem. Moreover, having no exact solution of a given OCP at hand, the method evaluates an accurate approximate solution of the problem. Let us now construct the residual error analysis for the Dickson polynomials. Given $e_1(t) = x(t) - x_M(t)$ and $e_2(t) = u(t) - u_M(t)$. So, the maximum absolute error can be evaluated as

$$e(t) = \max_{0 \leq t \leq 1} |e_1(t) + e_2(t)|. \tag{3.9}$$

Now, we can obtain the estimated error function as follow:

$$\mathbf{E}(t) = \sum_{k=0}^{M} e_k D_k(t, \alpha). \tag{3.10}$$

Consequently, the solution based on Dickson polynomials will be obtained as follows:

$$\hat{x}_M = x_M(t) + \mathbf{E}(t), \qquad \hat{u}_M = u_M(t) + \mathbf{E}(t).$$

Therefore, the corrected error function has been obtained by $\hat{e}_1(t) = x(t) - \hat{x}_M(t)$ and $\hat{e}_2(t) = u(t) - \hat{u}_M(t)$. The accuracy of this approximate solutions is also obtained by substituting the approximate solutions (x_M, u_M) into Eq. (1.2) as follow:

$$E_M = |\dot{x}_M(t) - g(t) - \int_a^t k(t, s,\, x_M(s), u_M(s))ds|. \tag{3.11}$$

It is expected that E_M to be zero at the collocation points. Indeed, the more accurate of the proposed method will be obtained for the approximate solutions when E_M sufficiently be close to zero.

4. Numerical results

We would test introducing method by several examples. We show the efficiency of this method by solving three non-trivial examples. In addition, we used the following uniform norms defining the absolute errors as:

$$E(x) = ||x - x^*||_2^2 = \int_a^b (x(t) - x^*(t))^2 dt,$$

and

$$E(u) = ||u - u^*||_2^2 = \int_a^b (u(t) - u^*(t))^2 dt, \tag{4.1}$$

where (x^*, u^*) and (x, u) denote the exact and approximate solutions, respectively. All numerical computations have been coded in Mathematica software. Also, we assume that the total error to be less than a given number ϵ. To evaluate the advantages of this method, we provide the following examples.

Example 4.1. For the first example, we consider

$$Min\ J(x, u) = \int_0^1 (tx(t) - u(t))^2 dt,$$

subject to

$$\dot{x}(t) = 1 - \frac{7}{12} t^4 + \int_0^t (s^2 t + s u(s)) \dot{x}(s) ds, \tag{4.2}$$

with boundary conditions

$$x(0) = 0. \tag{4.3}$$

Table 1. Numerical results of J_M with different values of α for Example 4.1.

Itr	$\alpha = -1$	$\alpha = 0$	$\alpha = 0.1$	$\alpha = 1$
$M = 2$	2.96059×10^{-16}	1.02968×10^{-29}	-1.38991×10^{-27}	-1.32764×10^{-28}

Table 2. Comparing the absolute errors with $\alpha = 0.1$ for Example 4.1.

Method	Itr	$E(x)$	$E(u)$
This study	$M = 2$	2.61098×10^{-32}	1.77735×10^{-29}
Method in [1]	$k = 2, M = 2$	4.04054×10^{-10}	3.2226×10^{-10}

Table 3. Accuracy errors with $\alpha = 0.1$ and $M = 2, 3$ at different values of t for Example 4.1.

t	0	0.2	0.4	0.6	0.8	1
$M = 2$	6.4746×10^{-15}	2.11771×10^{-16}	4.57494×10^{-16}	9.2826×10^{-16}	2.92655×10^{-15}	1.02668×10^{-15}
$M = 3$	6.66134×10^{-16}	1.54506×10^{-16}	1.3315×10^{-16}	3.84096×10^{-17}	1.41869×10^{-16}	3.05639×10^{-16}

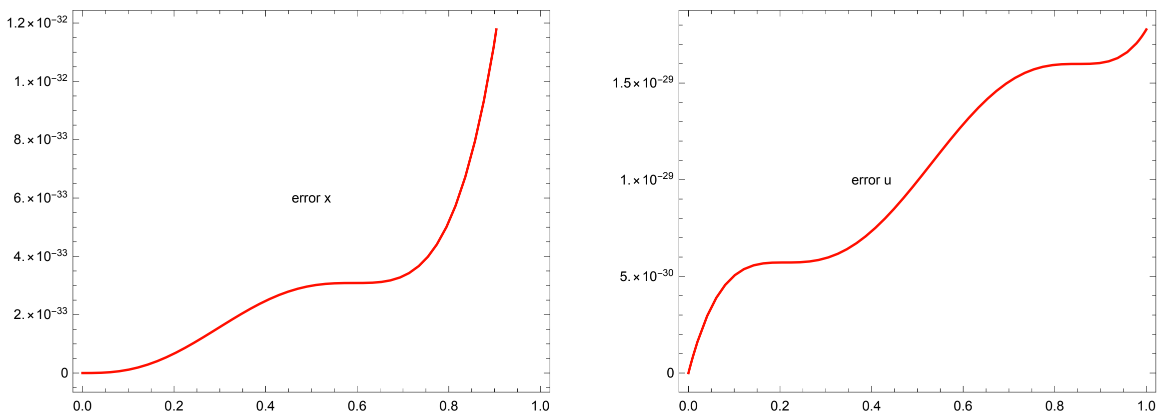

Figure 1. Evaluated error functions $x(t)$ and $u(t)$ with $\alpha = 0.1$ and $M = 2$ for Example 4.1.

The exact control functions and optimal trajectory are $u(t) = t^2$ and $x(t) = t$, respectively. The value J_M obtained based on our proposed method with $\epsilon = 10^{-12}$ and compared with the results reported in [1] respectively in Tables 1 and 2. Comparing the results reveal that the accuracy of the Dickson collocation method is higher than the method presented in [1]. The accuracy of these solutions for different values of t and considering $M = 2, 3$ are reported in Table 3. Also, the errors of control functions and trajectory for $M = 2$ are depicted in Figure 1. The effect of the parameter M on these approximations are also plotted in Figure 2. It can be seen from this figure that by increasing the value of M, the proposed method is convergent.

Example 4.2. As a second example let us consider:

$$Min\ J(x, u) = \int_0^1 (x(t) - sin(t))^2 + (u(t) - t^2)^2 dt,$$

subject to

$$\dot{x}(t) = g(t) + \int_0^t (tsx^3(s) + s^2u^2(s))ds. \tag{4.4}$$

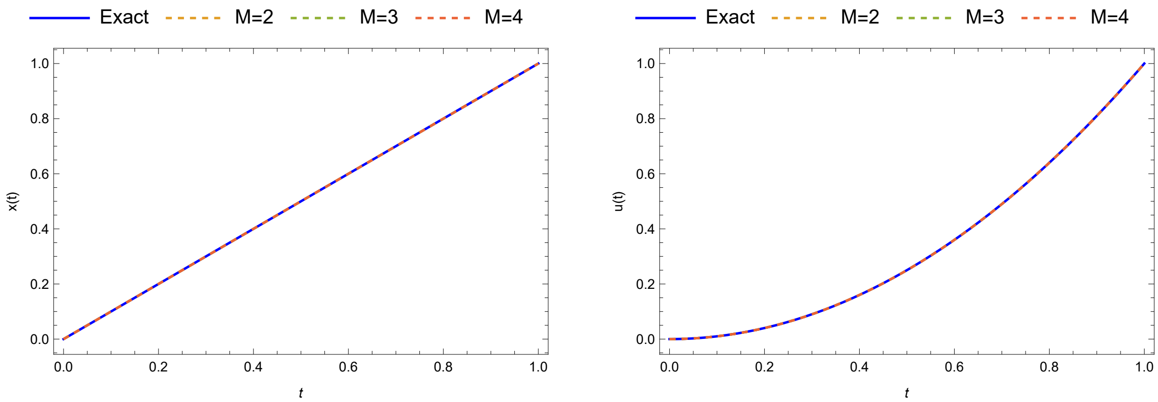

FIGURE 2. Approximate functions $x(t)$ and $u(t)$ with $\alpha = 0.1$ and different values of M for Example 4.1.

TABLE 4. Numerical results of J_M for Example 4.2.

Itr	$\alpha = -1$	$\alpha = 0$	$\alpha = 0.1$	$\alpha = 1$
$M = 4$	4.88411×10^{-6}	5.60165×10^{-4}	4.02883×10^{-7}	5.89004×10^{-7}
$M = 5$	9.46657×10^{-5}	1.20529×10^{-7}	2.81915×10^{-8}	8.02963×10^{-5}

TABLE 5. Comparing the absolute errors with $\alpha = 0.1$ for Example 4.2.

Method	Itr	$E(x)$	$E(u)$
Proposed method	$M = 4$	1.02426×10^{-8}	3.9264×10^{-7}
	$M = 5$	2.0124×10^{-11}	2.81713×10^{-8}
Method in [36]	$M = 4$	9.5×10^{-7}	1.2×10^{-7}

TABLE 6. Accuracy errors with $\alpha = 0.1$ for Example 4.2.

t	0	0.2	0.4	0.6	0.8	1
$M = 4$	0	1.27174×10^{-6}	1.44287×10^{-6}	3.49122×10^{-6}	4.46583×10^{-6}	3.81167×10^{-7}
$M = 5$	0	6.24637×10^{-7}	1.67263×10^{-6}	2.84911×10^{-6}	3.75231×10^{-6}	4.88179×10^{-7}

With initial conditions

$$x(0) = 0, \tag{4.5}$$

where $g(t) = sin(t) - \frac{1}{7}t^7 + \frac{1}{3}t^2 sin^2(t)cos(t) + \frac{2}{3}t^2 cos(t) - \frac{1}{9}tsin^3(t) - \frac{2}{3}tsin(t)$.

The exact control functions and optimal trajectory are $u(t) = t^2$ and $x(t) = sin(t)$, respectively. Applying the proposed method and considering $\epsilon = 10^{-7}$ for this problem leads to Table 4. A comparison is made between the absolute errors obtained by our method with the best results that achieved by Legendre polynomials [36] in Table 5. The accuracy of these solutions for different choices of M and considering $\alpha = 0.1$ are reported in Table 6. Figure 3 shows the convergence of approximate solutions $x(t)$ and $u(t)$ for $\alpha = 0.1$ and different values of M. Furthermore, Figure 4 shows the graphs of absolute errors for different choices of M. It is clear that by selecting small values of M, the error quickly tends to zero.

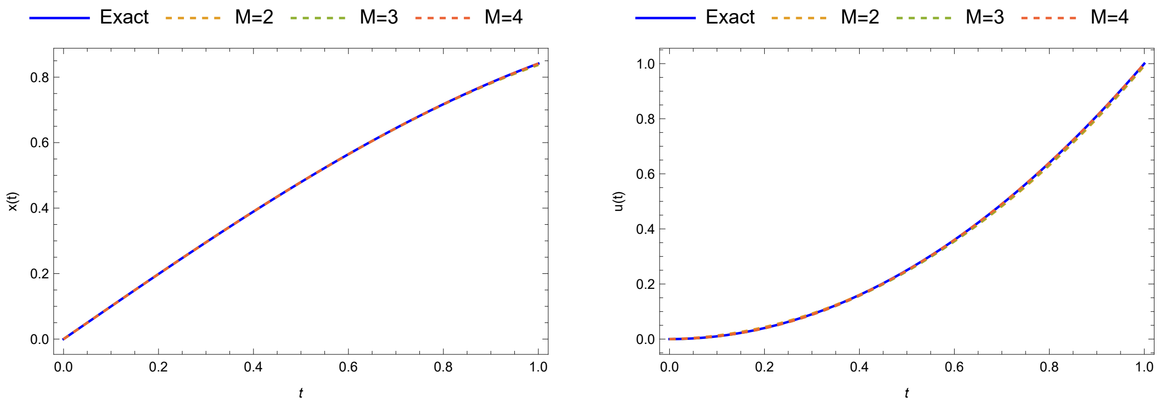

FIGURE 3. Approximate functions $x(t)$ and $u(t)$ with $\alpha = 0.1$ and different values of M for Example 4.2.

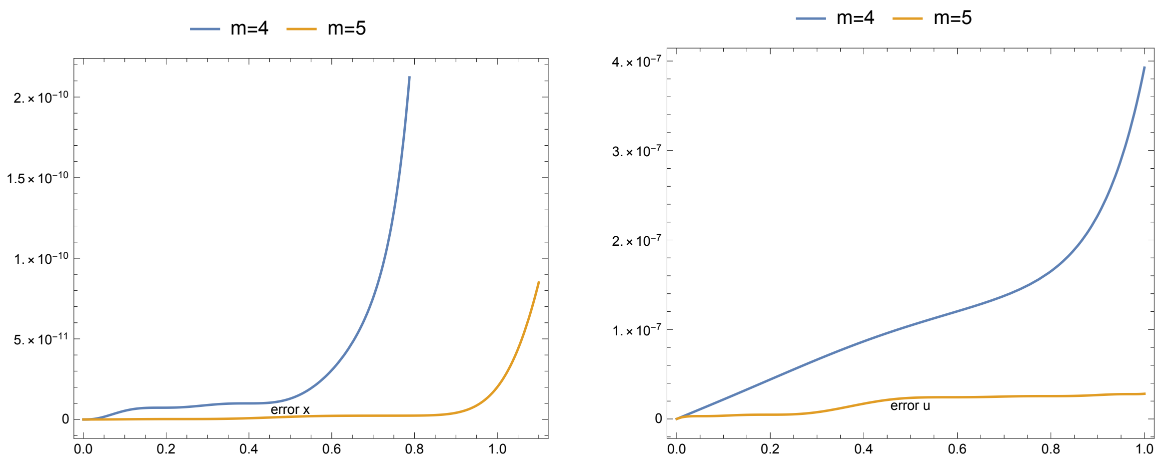

FIGURE 4. Evaluated error functions $x(t)$ and $u(t)$ with $\alpha = 0.1$ and $M = 4, 5$ for Example 4.2.

Example 4.3. In the last example we solved the following problem:

$$Min\ J(x,u) = \int_0^1 (x(t) - t - 1) + (u(t) - t^2 - t)^2 dt,$$

subject to

$$\dot{x}(t) = g(t) + \int_0^t (t^2 s x(s) u(s)) ds \tag{4.6}$$

where $g(t) = -\frac{1}{5}t^7 - \frac{1}{2}t^6 - \frac{1}{3}t^5 + t + 1$.

The exact control functions and optimal trajectory are $u(t) = t^2 + t$ and $x(t) = t + 1$, respectively. The computed results for J_M with different values of α and $\epsilon = 10^{-16}$ have been reported in Table 7. The absolute errors of these solutions for different choices of M and $\alpha = 0.1$ are reported in Table 8. Also, a comparison is made between the absolute errors obtained by our method with the results that achieved in [32] in this table. As can be seen, the proposed approach is more effective by selecting small values for M, notably improving previous results in the literature in terms of J. The accuracy of these solutions for $M = 2, 3$ and considering $\alpha = 0.1$ are reported in Table 9. In addition, the errors of $x(t)$ and $u(t)$ for $M = 4$ are depicted in Figure 5.

TABLE 7. Numerical results of J_M at various values of α and M for Example 4.3.

Itr	$\alpha = -1$	$\alpha = 0$	$\alpha = 0.1$	$\alpha = 1$
$M = 2$	0	-2.22045×10^{-16}	-1.38778×10^{-16}	-8.88178×10^{-16}
$M = 3$	-1.77636×10^{-15}	-1.94289×10^{-16}	-4.16334×10^{-17}	-1.77636×10^{-15}
$M = 4$	-8.88178×10^{-16}	-2.77557×10^{-16}	-1.11022×10^{-16}	-4.44178×10^{-16}

TABLE 8. Comparing the absolute errors with $\alpha = 0.1$ for Example 4.3.

Method	Itr	J_M	$E(x)$	$E(u)$
This study	$M = 2$	-1.38778×10^{-16}	1.51009×10^{-33}	4.12303×10^{-31}
	$M = 3$	-4.16334×10^{-17}	5.79045×10^{-29}	1.19787×10^{-27}
Method in [32]	$M = 3$	1.36165×10^{-6}	7.78602×10^{-7}	2.60418×10^{-3}
	$M = 5$	5.29848×10^{-9}	6.15457×10^{-10}	1.6276×10^{-4}

TABLE 9. Accuracy errors with $\alpha = 0.1$ for Example 4.3.

t	0	0.2	0.4	0.6	0.8	1
$M = 2$	3.1961×10^{-17}	4.74905×10^{-17}	5.59174×10^{-17}	9.4384×10^{-17}	1.74039×10^{-16}	9.81256×10^{-18}
$M = 3$	4.29468×10^{-17}	3.22159×10^{-17}	1.17212×10^{-17}	1.22228×10^{-17}	3.59305×10^{-17}	6.00312×10^{-17}

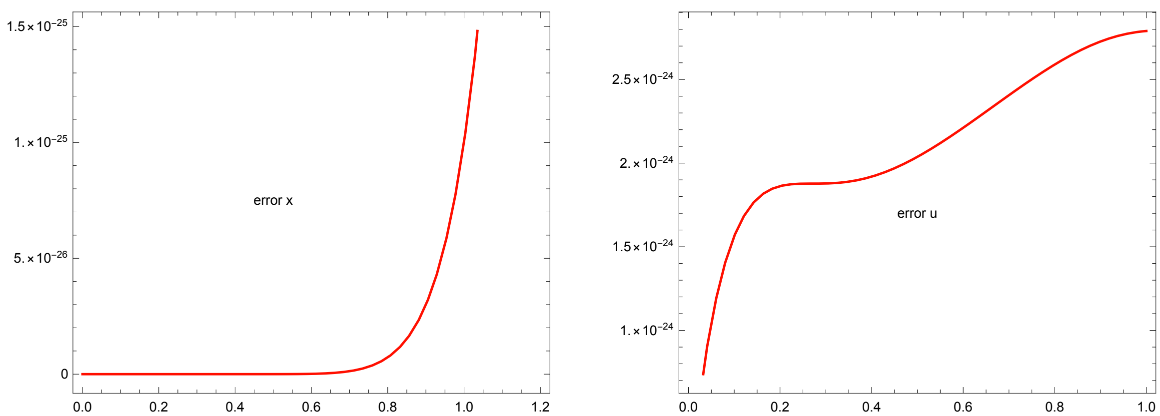

FIGURE 5. Evaluated error functions $x(t)$ and $u(t)$ with $\alpha = 0.1$ and $M = 4$ for Example 4.3.

5. CONCLUSION

We have presented Dickson polynomials with a collocation method to solve an OCPs governed by Volterra integro-differential equation. Our design uses the control variables and the state via a linear combination of Dickson polynomials as basic functions. The properties of these functions, allow us to reduce the VI-OCPs to a system of nonlinear algebraic equations for choosing the coefficients and control parameters optimally. Using Dickson polynomials via a collocation method bears some advantages such as simple evaluation of high order derivatives and integral terms of given differential equations and less expensive computational costs. Three examples are solved to illustrate the efficiency, applicability, and high performance of this approach. As can be seen in these examples, the parameter α plays an important role in the Dickson polynomials in a way that can change the behavior of the solution. The accuracy of the Dickson collocation method can be easily concluded from the improved results by our newly introduced method.

References

[1] M. Alipour, M. A. Vali, and A. H. Borzabadi, *A direct approach for approximate optimal control of integro-differential equations based on homotopy analysis and parameterization method*, IMA J. Math. Control Inf., *34*(2) (2017), 611–630.

[2] M. Alipour and S. Soradi-Zeid, *Optimal control of time delay Fredholm integro-differential equations*, J. Math. Model., *9*(2) (2021), 277–291.

[3] S. A. Belbas, *A reduction method for optimal control of Volterra integral equations*, Appl. Math. Comput., *197*(2) (2008), 880–890.

[4] A. Blokhuis, X. Cao, W. S. Chou, and X. D. Hou, *On the roots of certain Dickson polynomials*, J. Number Theory, *188* (2018), 229–246.

[5] O. Cots, J. Gergaud, and D. Goubinat, *Direct and indirect methods in optimal control with state constraints and the climbing trajectory of an aircraft*, Optim. Control. Appl. Methods, *39*(1) (2018), 281–301.

[6] R. Cont and E. Voltchkova, *Integro-differential equations for option prices in exponential Levy models*, Finance Stochastics, *9*(3) (2005), 299–325.

[7] W. S. Chou, *The factorization of Dickson polynomials over finite fields*, Finite Fields Appl., *3*(1) (1997), 84–96.

[8] S. B. Chen, S. Soradi-Zeid, M. Alipour, Y. M. Chu, J. F. Gomez-Aguilar, and H. Jahanshahi, *Optimal control of nonlinear time-delay fractional differential equations with Dickson polynomials*, Fractals, *29*(4) (2021), 2150079–270.

[9] L. E. Dickson, *The Analytic Representation of Substitutions on a Power of a Prime Number of Letters with a Discussion of the Linear Group*, Annal. Math., *11*(1/6) (1896), 65–120.

[10] A. Diene and M. A. Salim, *Fixed points of the Dickson polynomials of the second kind*, J. Appl. Math., 2013.

[11] M. I. Kamien and E. Muller, *Optimal control with integral state equations*, The Review of Economic Studies, *43*(3) (1976), 469–473.

[12] Ö. K. Kürkcü, E. Aslan, and M. Sezer, *A numerical approach with error estimation to solve general integro-differential-difference equations using Dickson polynomials*, App. Math. Comput., *276* (2016), 324-339.

[13] Ö. K. Kürkcü, E. Aslan and M. Sezer, *A novel collocation method based on residual error analysis for solving integro-differential equations using hybrid Dickson and Taylor polynomials*, Sains Malays, *46* (2017), 335–347.

[14] Ö. K. Kürkcü, E. Aslan, and M. Sezer, *A numerical method for solving some model problems arising in science and convergence analysis based on residual function*, App. Num. Math., *121* (2017), 134–148.

[15] P. Khodabakhshi and J. N. Reddy, *A unified integro-differential nonlocal model*, Int. J. Engineering Sci., *95* (2015), 60–75.

[16] M. Lakestani and M. Dehghan, *Numerical solution of fourth-order integro-differential equations using Chebyshev cardinal functions*, Int. J. Comput. Math., *87*(6) (2010), 1389–1394.

[17] R. Lidl, G. L. Mullen, and G. Turnwald, *Dickson polynomials*, *65*, Chapman and Hall/CRC, 1993.

[18] N. Mahmoodi Darani, *Hybrid collocation method for some classes of second-kind nonlinear weakly singular integral equation*, Comput. Methods Differ. Equ., 2022.

[19] J. Ma and G. Ge, *A note on permutation polynomials over finite fields*, Finite Fields Appl., *48* (2017), 261–270.

[20] K. Maleknejad and H. Almasieh, *Optimal control of Volterra integral equations via triangular functions*, Math.comput. model., *53*(9-10) (2011), 1902–1909.

[21] K. Maleknejad and A. Ebrahimzadeh, *The use of rationalized Haar wavelet collocation method for solving optimal control of Volterra integral equations*, J. Vib. Control, *21*(10) (2015), 1958–1967.

[22] S. Mashayekhi, Y. Ordokhani, and M. Razzaghi, *Hybrid functions approach for optimal control of systems described by integro-differential equations*, Appl. Math. Model. *37*(5) (2013), 3355–3368.

[23] H. Mirinejad and T. Inanc, *An RBF collocation method for solving optimal control problems*, Rob. Auton. Syst., *87* (2017), 219–225.

[24] A. S. Matveev and A. V. Savkin, *Application of optimal control theory to analysis of cancer chemotherapy regimens*, Syst. control lett., *46*(5) (2002), 311–321.

[25] J. Medlock and M. Kot, *Spreading disease: integro-differential equations old and new*, Math. biosciences, *184*(2) (2003), 201–222.

[26] N. Negarchi and K. Nouri, *A New Direct Method for Solving Optimal Control Problem of Nonlinear Volterra - Fredholm Integral Equation via the Muntz-Legendre Polynomials*, Bulletin of the Iranian Math. Society, *45*(3) (2019), 917–934.

[27] C. J. Oates, T. Papamarkou, and M. Girolami, *The controlled thermodynamic integral for Bayesian model evidence evaluation*, J. Am. Stat. Assoc., *111*(514) (2016), 634–645.

[28] B. Pan, Y. Ma, and Y. Ni, *A new fractional homotopy method for solving nonlinear optimal control problems*, Acta Astronaut., *161* (2019), 12–23.

[29] N. Patel and N. Padhiyar, *Modified genetic algorithm using Box Complex method: Application to optimal control problems*, J. Process Control, *26* (2015), 35–50.

[30] S. Paseban Hag, E. Osgooei, and E. Ashpazzadeh, *Alpert wavelet system for solving fractional nonlinear Fredholm integro-differential equations*, Comput. Methods Differ. Equ., *9*(3) (2021), 762–773.

[31] A. V. Rao, *A survey of numerical methods for optimal control*, Adv. Astronaut. Sci., *135*(1) (2009), 497–528.

[32] Z. Shabani and H.Tajadodi, *A numerical scheme for constrained optimal control problems*, Int. J. Ind. Electron. Control Optim., *2*(3) (2019), 233–238.

[33] T. Stoll, *Complete decomposition of Dickson-type recursive polynomials and a related Diophantine equation*, Formal Power Series and Algebraic Combinatorics, Tianjin, China, 2007.

[34] E. Safaie, M. H. Farahi, and M. F. Ardehaie, *An approximate method for numerically solving multi-dimensional delay fractional optimal control problems by Bernstein polynomials*, Comput. Appl. Math., *34*(3) (2015), 831–846.

[35] G. Tang and K. Hauser, *A data-driven indirect method for nonlinear optimal control*, Astrodynamics, *3*(4) (2019), 345–359.

[36] E. Tohidi and O. R. N. Samadi, *Optimal control of nonlinear Volterra integral equations via Legendre polynomials*, IMA J. Math. Control Inf., *30*(1) (2013), 67–83.

[37] Q. Wang and J. L. Yucas, *Dickson polynomials over finite fields*, Finite Fields Appl., *18*(4) (2012), 814–831.

[38] P. Wei, X. Liao, and K. W. Wong, *Key exchange based on Dickson polynomials over finite field with* 2^m, JCP, *6*(12) (2011), 2546–2551.

[39] E. M. Yong, L. Chen, and G. J. Tang, *A survey of numerical methods for trajectory optimization of spacecraft*, J. Astronaut., *29*(2) (2008), 397–406.

[40] S. Yüzbasi and N. Ismailov, *An operational matrix method for solving linear Fredholm-Volterra integro-differential equations*, Turkish J. Math., *42*(1) (2018), 243–256.

[41] S. Yüzbasi, *An exponential method to solve linear Fredholm-Volterra integro-differential equations and residual improvement*, Turkish J. Math., *42*(5) (2018), 2546–2562.

[42] S. Yüzbasi, *A shifted Legendre method for solving a population model and delay linear Volterra integro-differential equations*, Int. J. Biomath., *10*(07) (2017), 1750091.

[43] S. Yüzbasi and M. Sezer, *An exponential approach for the system of nonlinear delay integro-differential equations describing biological species living together*, Neural Comput. Appl., *27*(3) (2016), 769–779.

[44] S. Yüzbasi, *A collocation method based on Bernstein polynomials to solve nonlinear Fredholm-Volterra integro-differential equations*, Appl. Math. Comput., *273* (2016), 142–154.

[45] S. Yüzbasi, *Improved Bessel collocation method for linear Volterra integro-differential equations with piecewise intervals and application of a Volterra population model*, Appl. Math. Modell., *40* (9-10) (2016), 5349–5363.

[46] S. Yüzbasi, *Numerical solutions of system of linear Fredholm-Volterra integro-differential equations by the Bessel collocation method and error estimation*, Appl. Math. Comput., *250* (2015), 320–338.