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Jacobi wavelets method for numerical solution of fractional population growth model

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Abstract

This paper deals with the generalization of the fractional operational matrix of Jacobi wavelets. The fractional population growth model was solved by using this operational matrix and compared with other existing methods to illustrate the applicability of the method. Then, convergence and error analysis of this procedure were studied.

Keywords. Hybrid functions, Jacobi polynomials, Operational matrix, Population growth model, Wavelets.
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1. Introduction

Differential equations and fractional calculus have various applications in the field of knowledge and technology [11, 12]. Many physical problems are organized by fractional order differential equations (FDE) such as gas dynamics, wave and diffusion equations, cable equations, and so on. The analytical solutions of most fractional order differential equations are not easily obtainable. Hence, detecting the solutions of these equations have absorbed great consideration in late years [8, 15–17]. Lately, wavelet basis functions and their attributes have described their efficiency in various districts of knowledge and technology. The method of solving fractional differential equations employ wavelet operational matrices obtained from orthogonal functions (such as Legendre Wavelets [9, 27], Chebyshev Wavelets [2, 14, 28], CAS Wavelet [5], Jacobi Polynomials [23, 24, 30] and so on) have been suggested by various scholars and have been seen as a strong method to find an estimated solution for fractional integro-differential equations. Lately, Bhrawy et.al. in [3], have defined the operational matrix of fractional integration based on shifted Jacobi polynomials (SJP) on $[0, L]$.
Here, we give the definition of fractional population growth model (FPGM) [6, 10, 23, 25] as follows:

$$\begin{aligned} &\kappa D_*^{\alpha} P(X) - AP(X) + BP^2(X) + CP(X)\int_0^X P(\tau)d\tau = 0, \\ &P(0) = P_0, \quad 0 < \alpha \leq 1. \end{aligned} \tag{1.1}$$

where α is a fixed argument implicating the order of the time fractional derivative, $A > 0$ is the birth rate factor, $B > 0$ is the crowding factor, $C > 0$ is the toxicity factor, P_0 is the primary population, $\kappa = C/AB$ and $P(x)$ is the graduated population of the same individualist at a time X which express crowding and sensibility to the plenty of toxins constructed [13]. It is merit noting that in case $\alpha = 1$, the fractional equation changes to a classical logistic growth model. Here we use the scale time and population by recommending the non-dimensional variables $x = CX/B$

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and $u = BP/A$ to attain the following non-dimensional model:

$$\kappa^{C} D^{\alpha} u(x) - u(x) + u^2(x) + u(x) \int_0^x u(\tau) d\tau = 0, \tag{1.2}$$
$$u(0) = u_0, \quad 0 < \alpha \leq 1.$$

One may show that the only trivial solution of (1.2) is $u(t) = 0$ and the nontrivial analytical solution for $\alpha = 1$ is [10, 18]:

$$u(x) = u_0 \exp\left(\frac{1}{\kappa} \int_0^x \left(1 - u(\tau) - \int_0^\tau u(s)ds\right) d\tau\right). \tag{1.3}$$

Recently, various numerical procedures have been suggested to approximate the solution of the classical and fractional PGM, for example, the reader is guided to view [6, 10, 23, 29], and commission therein. As well, recently in [4, 9, 14, 15, 25], the novelist suggested a novel procedure based on operational matrices to solve nonlinear fractional integro-differential problems [1].

The purpose of this manuscript is to apply the operational matrix of fractional integration to convert the nonlinear fractional integro-differential equation to a system of nonlinear algebraic equations [19, 31]. Therefore, in this essay, the fractional operational matrix of integration for Jacobi wavelets (JWs) was constructed in the block structure. Thus, by applying this matrix a computational method for solving FPGM (1.2) was presented.

The remaining of this manuscript is as follows. In section 2, we stated some basic mathematical preliminaries that we need to establish our method which is followed by recalling the essential definitions from Jacobi polynomials, wavelets, and fractional calculus. In section 3, we applied the suggested method to solve FPGM. In section 4 the convergence and error analysis were presented. Finally, in section 5, the conclusion was derived.

2. Prelimineries

In this part, the definition of fractional calculus, Jacobi polynomials, wavelets and their attributes were explained. As well, the essential descriptions from the operational matrix of integral and fractional population growth model were defined to formulate our technique.

2.1. **Fractional calculus.** The Riemann-Liouville fractional integral I^α of order $0 \leq \alpha < 1$ is presented with[20]:

$$I^\alpha u(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u(s) ds & \alpha > 0, \\ u(x) & \alpha = 0. \end{cases} \tag{2.1}$$

One of the fundamental attributes of the operator I^α is:

$$I^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)} x^{\beta+\alpha}. \tag{2.2}$$

The Riemann-Liouville and Caputo fractional derivatives of order $\alpha > 0$ is respectively determined as[20]:

$$D^\alpha u(t) = \frac{d^n}{dt^n} I^{n-\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s) ds, \tag{2.3}$$

$${}^{C}D^\alpha u(t) = I^{n-\alpha} \frac{d^n}{dt^n} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds, \tag{2.4}$$

where n is an integer $(n-1 < \alpha \leq n)$ and $u^{(n)} \in L^1[0,b]$.

The main relationship among the Riemann- Liouville integral operator and Caputo differential is as follows:

$$I^\alpha \, {}^{C}D^\alpha u(t) = u(t) - \sum_{r=0}^{n-1} u^{(r)}(0^+) \frac{t^r}{r!}, \qquad (n-1 < \alpha \leq n), \tag{2.5}$$

where $u^{(r)} \in L^1[0,b]$.

2.2. **Shifted Jacobi polynomials (SJPs) and their attributes.** The analytical form of SJPs , on $[0,1]$ is:

$$P_j^{(\mu,\nu)}(x)=\sum_{k=0}^{j}\gamma_k^j x^k, \tag{2.6}$$

where

$$\gamma_k^j=\frac{(-1)^{j-k}\Gamma(j+\nu+1)\Gamma(j+k+\mu+\nu+1)}{\Gamma(k+\nu+1)\Gamma(j+\mu+\nu+1)(j-k)!k!},\quad j=0,1,... \tag{2.7}$$

The shifted Jacobi polynomials on $[0,1]$ are orthogonal [24]:

$$\int_0^1 P_j^{(\mu,\nu)}(x)P_k^{(\mu,\nu)}(x)w^{(\mu,\nu)}(x)dx=\theta_j\delta_{jk}, \tag{2.8}$$

where $w^{(\mu,\nu)}(x)=(1-x)^\mu x^\nu,\quad x\in[0,1]$ and δ_{jk} is the well-known Dirac function and

$$\theta_j=\frac{\Gamma(j+\mu+1)\Gamma(j+\nu+1)}{\Gamma(2j+\mu+\nu+1)j!\Gamma(j+\mu+\nu+1)}. \tag{2.9}$$

The SJPs on $[0,1]$ form an orthogonal base for $L^2[0,1]$, however are not normal. To simplify the next calculations, they were normalized as the following formula

$$\bar{P}_j^{(\mu,\nu)}(x)=\frac{1}{\sqrt{\theta_j}}P_j^{(\mu,\nu)}(x)=\frac{1}{\sqrt{\theta_j}}\sum_{k=0}^{j}\gamma_k^j x^k. \tag{2.10}$$

The polynomials $\bar{P}_j^{(\mu,\nu)}(x)$ on $[0,1]$ form an orthonormal base for $L^2[0,1]$. A function $u(x)$, square integrable on $[0,1]$, can be explained in phrases of normalized SJPs as:

$$u(x)=\sum_{j=0}^{\infty}C_j\bar{P}_j^{(\mu,\nu)}(x), \tag{2.11}$$

where,

$$C_j=\int_0^1 u(x)\bar{P}_j^{(\mu,\nu)}(x)w^{(\mu,\nu)}(x)dx,\qquad j=0,1,... \tag{2.12}$$

generally in perfect, only the first $N+1$ terms of normalized SJPs are used, thus:

$$u(x)\simeq u_N(x)=\sum_{j=0}^{N}C_j\bar{P}_j^{(\mu,\nu)}(x)=\Psi^T(x)\mathbf{C}=\mathbf{C}^T\Psi(x), \tag{2.13}$$

where the vectors $\mathbf{C}$ and $\Psi(x)$ are defined by:

$$\begin{aligned}&\mathbf{C}=[C_0,C_1,...,C_N]^T,\\&\Psi(x)=[\bar{P}_0^{(\mu,\nu)}(x),\bar{P}_1^{(\mu,\nu)}(x),...,\bar{P}_N^{(\mu,\nu)}(x)]^T.\end{aligned} \tag{2.14}$$

Some efficient lemmas and theorems, which were introdused by novelist in [24], are described as follows.

Lemma 2.1. *If $\bar{P}_i^{(\mu,\nu)}(x)$, $\bar{P}_j^{(\mu,\nu)}(x)$ and $\bar{P}_k^{(\mu,\nu)}(x)$ are severally i,j and k-th normalized SJPs, then*

$$\begin{aligned}q_{ijk}&=\int_0^1\bar{P}_i^{(\mu,\nu)}(x)\bar{P}_j^{(\mu,\nu)}(x)\bar{P}_k^{(\mu,\nu)}(x)w^{(\mu,\nu)}(x)dx\\&=\frac{1}{\sqrt{\theta_i\theta_j\theta_k}}\sum_{r=0}^{i}\sum_{s=0}^{j}\sum_{t=0}^{k}\gamma_r^i\gamma_s^j\gamma_t^k\frac{\Gamma(r+s+t+\nu+1)\Gamma(\mu+1)}{\Gamma(r+s+t+\mu+\nu+2)}.\end{aligned} \tag{2.15}$$

In practical calculation on the Jacobi bases, we mostly experienced the product of $\Psi(x)$ and $\Psi^T(x)$, called the operational matrix of product. In this part, several essential matrix relations will be obtained. First, a common law was expressed for finding the $(N+1)\times(N+1)$ operational matrix of product $\tilde{V}$ as:

$$\Psi(x)\Psi^T(x)V \simeq \tilde{V}\Psi(x). \tag{2.16}$$

The next theorem describes the elements of the matrix $\tilde{V}$.

Theorem 2.2. *The elements of matrix $\tilde{V}$ in equation(2.16) are calculated as*

$$\tilde{V}_{jk} = \sum_{i=0}^{N} V_i q_{ijk}, \qquad j,k = 0,1,...,N, \tag{2.17}$$

where q_{ijk} is obtained in equation (2.15) and V_i, $\quad i = 0,1,...,N$ are the elements of vector V in equation (2.16).

Proof. View [4]. □

Also, we know a general form of operational matrix of fractional integration for normalized SJPs is as follows:

$$I^\alpha\Psi(x) = \mathbf{P}^\alpha\Psi(x), \tag{2.18}$$

where $\mathbf{P}^\alpha$ is the operational matrix of fractional integral with dimension $(N+1)\times(N+1)$ on $[0,\,1]$. The next theorem presents the elements of matrix $\mathbf{P}^\alpha$.

Theorem 2.3. *The elements of matrix $\boldsymbol{P}^\alpha$ in equation (2.18) are calculated as follows*

$$\begin{aligned}
\boldsymbol{P}^\alpha_{ij} &= \left\langle I^\alpha \bar{P}_i(x), \bar{P}_j(x)\right\rangle \\
&= \frac{1}{\sqrt{\theta_i\theta_j}}\sum_{r=0}^{i}\sum_{s=0}^{j}\gamma_k^i\gamma_s^j\frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)}\int_0^1(1-x)^\mu x^\nu x^{k+\alpha}x^s dx \\
&= \frac{1}{\sqrt{\theta_i\theta_j}}\sum_{r=0}^{i}\sum_{s=0}^{j}\gamma_k^i\gamma_s^j\frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)}\frac{\Gamma(k+s+\alpha+\nu+1)\Gamma(\mu+1)}{\Gamma(k+s+\alpha+\mu+\nu+2)}.
\end{aligned} \tag{2.19}$$

2.3. **Jacobi wavelets (JWs).** In late years, applications of wavelets have appeared in various fields of knowledge and technology. Wavelets are a family of functions created from the dilation and translation of a single function called the mother wavelet [5, 7].
The Jacobi wavelets $\varphi_{nm}(t) = \varphi(k,n,m,t)$ [22, 30] have four parameters as $k \in N$, $n = 1,...,N$, m is the order of Jacobi polynomials, t is the time and $N = 2^{k-1}$. They are defined on $[0,L]$ by

$$\varphi_{nm}(t) = \begin{cases} \sqrt{\frac{N}{L}}\bar{P}_m^{(\mu,\nu)}(\frac{N}{L}t-n+1), & \frac{(n-1)L}{N} \le t < \frac{nL}{N}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.20}$$

where $m = 0,...,M-1$ and $n = 1,...,N$.

The JWs construct an orthonormal base for $L^2_{\omega_n}[0,L]$, i.e.[26].

$$\int_0^L w_n(t)\varphi_{nm}(t)\varphi_{n'm'}(t)dt = \begin{cases} 1, & (n,m)=(n',m'), \\ 0, & (n,m)\neq(n',m'), \end{cases} \tag{2.21}$$

where

$$w_n(t) = \begin{cases} (n-\frac{N}{L}t)^{\mu}\times(\frac{N}{L}t-n+1)^{\nu}, & \frac{(n-1)L}{N}\leq t<\frac{nL}{N}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.22}$$

2.4. **Function estimation.** Each function $y(t)\in L^2[0,L]$ that is infinitely differentiable can be expanded by means of JWs as:

$$y(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} a_{nm}\varphi_{nm}(t), \tag{2.23}$$

where $a_{nm}=\langle y(t),\varphi_{nm}(t)\rangle$ and $\langle .,.\rangle$ indicates the inner product in $L^2[0,L]$ with respect to the weight function $w_n^{(\alpha,\beta)}$. By shorten the infinite series in (2.23), we gain an estimate form of $y(t)$ as follows [26]:

$$y(t)\approx\sum_{n=1}^{N}\sum_{m=0}^{M-1} a_{nm}\varphi_{nm}(t). \tag{2.24}$$

For simplicity (2.24) can be rewritten as:

$$y(t)\approx\sum_{l=1}^{NM} c_l\varphi_l(t) = C^T\Phi(t), \tag{2.25}$$

where $c_l=a_{nm}$ and $\varphi_l(t)=\varphi_{nm}(t)$ and l is defined by $l=M(n-1)+m+1$. Here C and $\Phi(t)$ are column vectors with NM members as follows.

$$\begin{aligned} C^T &= [a_{10},...a_{1(M-1)},a_{20},...,a_{2(M-1)},...,a_{N0},...,a_{N(M-1)}], \\ \Phi(t) &= [\varphi_{10}(t),...,\varphi_{1(M-1)}(t),\varphi_{20}(t),...,\varphi_{2(M-1)}(t),...,\varphi_{N0}(t),...,\varphi_{N(M-1)}(t)]^T. \end{aligned} \tag{2.26}$$

A general structure of operational matrix of integral for JWs is as follows [22]:

$$I_t^{\alpha}\Phi(t)\approx \mathbf{P}_L^{\alpha}\Phi(t), \tag{2.27}$$

where $\mathbf{P}_L^{\alpha}$ is called the operational matrix of fractional integral of order α for JWs with dimension $MN\times MN$ on the interval $[0,1]$.
The next theorem generalizes the structure and entries of the operational matrix of integration $\mathbf{P}_L^{\alpha}$.

Theorem 2.4. *The matrix $\boldsymbol{P}_L^\alpha$ in (2.27) has the following structure.*

$$\boldsymbol{P}_L^\alpha = \left(\frac{L}{N}\right)^\alpha \begin{pmatrix} \boldsymbol{P}^\alpha & \boldsymbol{H}_1 & \boldsymbol{H}_2 & \boldsymbol{H}_3 & \boldsymbol{H}_4 & \cdots & \boldsymbol{H}_{N-1} \\ & \boldsymbol{P}^\alpha & \boldsymbol{H}_1 & \boldsymbol{H}_2 & \boldsymbol{H}_3 & \cdots & \boldsymbol{H}_{N-2} \\ & & \boldsymbol{P}^\alpha & \boldsymbol{H}_1 & \boldsymbol{H}_2 & \cdots & \boldsymbol{H}_{N-3} \\ & & & \boldsymbol{P}^\alpha & \boldsymbol{H}_1 & \ddots & \vdots \\ & & & & \ddots & \ddots & \boldsymbol{H}_2 \\ & & & & & \ddots & \boldsymbol{H}_1 \\ & & & & & & \boldsymbol{P}^\alpha \end{pmatrix}. \tag{2.28}$$

Whereas $\boldsymbol{P}^\alpha$ is the $M \times M$ operational matrix of fractional integration of order α for normalized SJPs on the interval $[0,1]$ which was defined in (2.18) and $\boldsymbol{H}_k$ $(k = 1, 2, ..., N-1)$ is the matrix with dimension $M \times M$ as follows:

$$(\boldsymbol{H}_k)_{mm'} = \int_0^1 I_m^\alpha(s+k)\bar{P}_{m'}(s)w(s)ds, \quad k = 1, ..., N-1, \tag{2.29}$$

and

$$I_m^\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^1 (t-r)^{\alpha-1}\bar{P}_m(r)dr. \tag{2.30}$$

Proof. For detailed proof, we refer to [21]. □

The next theorem determines the elements of $\tilde{A}$, the operational matrix of product for JWs defined as follows:

$$\Phi(x)\Phi^T(x)A \simeq \tilde{A}\Phi(x). \tag{2.31}$$

Theorem 2.5. *The matrix $\tilde{A}$ in (2.31) has the following structure.*

$$\tilde{A} = \sqrt{\frac{N}{L}} \begin{pmatrix} \tilde{V}_1 & & & \\ & \tilde{V}_2 & & \\ & & \ddots & \\ & & & \tilde{V}_N \end{pmatrix}. \tag{2.32}$$

where $\tilde{V}_k$, (k=1,2,...,N) is the $M \times M$ product operational matrix for normalized SJPs was defined in (2.17), whose entries are the combination of $(a_{k0}, a_{k1}, ..., a_{k(M-1)})$.

Proof. To construct the entries $\widetilde{\mathbf{A}}_{ij}$, let $i = (n-1)M + m + 1$ *and* $j = (n'-1)M + m' + 1$, then

$$\begin{aligned} \widetilde{\mathbf{A}}_{ij} &= \langle \varphi_{nm}(t) \sum_{r=0}^{M-1} a_{nr}\varphi_{nr}(t), \varphi_{ns}(t)\rangle_{w_n} \\ &= \sum_{r=0}^{M-1} a_{nr} \int_{\frac{(n-1)L}{N}}^{\frac{nL}{N}} \varphi_{nm}(t)\varphi_{nr}(t)\varphi_{ns}(t)w_n(t)dt \\ &= \sqrt{\tfrac{N}{L}} \sum_{r=0}^{M-1} a_{nr} \int_0^1 \overline{P}_m(X)\overline{P}_r(X)\overline{P}_s(X)w(X)dX \\ &= \sqrt{\tfrac{N}{L}} \sum_{r=0}^{M-1} a_{nr}q_{mrs}, \end{aligned}$$

where $X = \frac{N}{L}t - n + 1$ and q_{mrs} are defined in (2.15). Then, we concluded that the product operational matrix $\widetilde{\mathbf{A}}$ for JWs can be rewritten to the matrix format (2.32). □

3. Implementation of the Method and illustrations

Here, we use JWs to solve the FPGM defined in (1.2) where $u(t)$ is the unknown function to be found, $\alpha \in R^+$ positive real numeral and $n-1<\alpha\leq n$. To solve (1.2) with JWs, we commence with:

$$^{C}D^{\alpha}u(t)=\mathbf{C}^{T}\Phi(t), \tag{3.1}$$

where $\mathbf{C}$ is the vector of unknown coefficients and $\Phi(t)$ is the Jacobi wavelets vector. We apply Riemann-Liouville fractional integration I^{α} to the equation in (3.1):

$$u(t)-u_0=\mathbf{C}^{T}I^{\alpha}\Phi(t)=\mathbf{C}^{T}\mathbf{P}_L^{\alpha}\Phi(t), \tag{3.2}$$

then

$$u(t)=\mathbf{C}^{T}\mathbf{P}_L^{\alpha}\Phi(t)+V_0^{T}\Phi(t)=\mathbf{V}_1^{T}\Phi(t), \tag{3.3}$$

where $\mathbf{V}_1^T=\mathbf{C}^T\mathbf{P}_L^{\alpha}+V_0^T$ and $u_0=V_0^T\Phi(t)$. Also,

$$u^2(t)=u(t)u(t)=V_1^T\Phi(t)\ \Phi(t)^TV_1=\mathbf{V}_1^T\tilde{V}_1\Phi(t), \tag{3.4}$$

where $\tilde{V}_1$ is operational matrix of product defined in (2.17).
To calculate the integral part of (1.2) we have:

$$\int_0^t u(\tau)d\tau=\mathbf{V}_1^T\int_0^t\Phi(\tau)d\tau=\mathbf{V}_1^T\mathbf{P}_L^1\Phi(t). \tag{3.5}$$

Now, we substitute (3.1)-(3.5) in (1.2):

$$\kappa C^T\Phi(t)-V_1^T\Phi(t)+V_1^T\tilde{V}_1\Phi(t)+V_1^TP_L^1\Phi(t)\Phi^T(t)V_1=0, \tag{3.6}$$

and then

$$\kappa C^T\Phi(t)-V_1^T\Phi(t)+V_1^T\tilde{V}_1\Phi(t)+V_1^TP_L^1\tilde{V}_1\Phi(t)=0, \tag{3.7}$$

finally, we obtained a system of algebraic equations as follows:

$$\kappa\mathbf{C}^T-\mathbf{V}_1^T+\mathbf{V}_1^T\tilde{\mathbf{V}}_1+\mathbf{V}_1^T\mathbf{P}_L^1\tilde{\mathbf{V}}_1=0. \tag{3.8}$$

Equation (3.8) is a system of nonlinear equations which can be solved by appropriate softwares such as Matlab and Maple.
As a numeral instance, we investigate the nonlinear FPGM (1.2) with the primary condition $u(0)=0.1$. Here, our intention is to study the mathematical treatment of the solution of this FPGM as the order of the fractional derivative changes. To view the treatment of the solution for various values of α, we consider the following particular modes.

Mode 1. We study the equation (1.2) with $\alpha=0.9$ for several various small values κ. The treatment of the numerical solutions for $M=4, K=6$ and $N=32$ is shown in Figure 1. From Figure 1, it can be seen that as κ increases, the domain of $u(t)$ reduces.

Mode 2. In this case, we study the equation (1.2) for various values of α and κ. From Figures 2 and 3, it is easy to show that when the order of the fractional derivative decreases, the domain of $u(t)$ reduces.

4. Convergence analysis and error analysis

In this section, we provide some theorems for the convergence of the approximate solution. First, we prove the following theorem.

Theorem 4.1. If a continuous function $y(x)\in L^2[0,T]$ be bounded, then the JWs expansion (2.24) converges to $y(x)$ [26].

TABLE 1. Comparison of exact value of u_{max} with the proposed method (**JWs**), RHM, ADM and HPM [6].

	$\alpha = 1$				
κ	Exact	RHM	ADM	HPM	**JWs**
0.02	0.923427170	0.9234473	0.9234270	0.9229420	0.9234613
0.04	0.873719983	0.8737197	0.8612401	0.8737253	0.8737375
0.1	0.769741449	0.7697404	0.7651130	0.7651131	0.7697734
0.2	0.659050382	0.6590500	0.6579123	0.6590504	0.6590475
0.5	0.485190291	0.4851895	0.4852823	0.4851903	0.4851259
	$\alpha = 0.75$				
κ		RHM		**JWs**	
0.02		0.9186276		0.9170625	
0.04		0.8590308		0.8510732	
0.1		0.7517051		0.7566549	
0.2		0.6362737		0.6344228	
0.5		0.4475551		0.4472055	

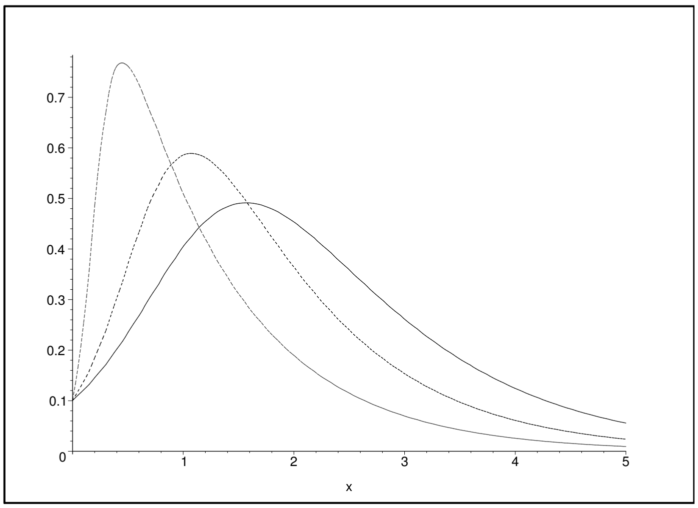

FIGURE 1. Numerical solutions of the fractional population growth model with $\alpha = 0.9$ for $\kappa = 0.1$ (Dash), 0.3 (Dashdot), 0.5 (Solid).

Proof. Let $y(x)$ be a bounded real valued function on $[0, L]$. The JWs coefficients of continuous functions $y(x)$ in (2.24) has the following upper bound.

$$\begin{aligned}
|a_{nm}| &= |\langle y(t), \varphi_{nm}(t)\rangle_{w_n}| \\
&= \left| \int_{\frac{(n-1)L}{N}}^{\frac{nL}{N}} y(t)\varphi_{nm}(t)w_n(t)dt \right| \\
&\leq \sqrt{\frac{N}{L}} \int_{\frac{(n-1)L}{N}}^{\frac{nL}{N}} |y(t)| \left| \overline{P}_m^{(\mu,\nu)}(\frac{N}{L}t - n + 1) \right| w(\frac{N}{L}t - n + 1)dt \\
&= \sqrt{\frac{L}{N}} \int_0^1 |\overline{P}_m^{(\mu,\nu)}(s)| \left| y(\frac{L}{N}(s + n - 1)) \right| ds,
\end{aligned} \tag{4.1}$$

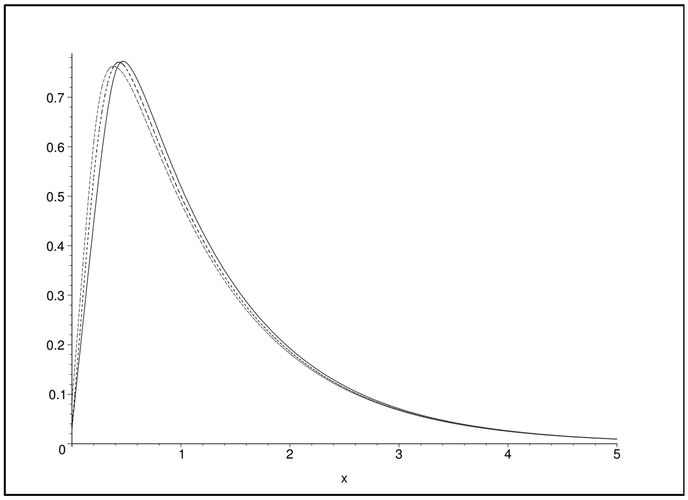

FIGURE 2. Numerical solutions of the fractional population growth model with $\kappa = 0.1$ for $\alpha = 0.6$ (Dash), 0.8 (Dashdot), 1 (Solid).

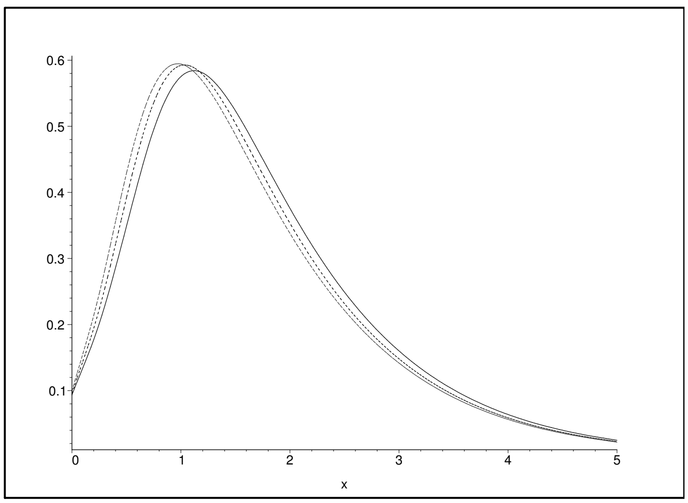

FIGURE 3. Numerical solutions of the fractional population growth model with $\kappa = 0.3$ for $\alpha = 0.6$ (Dash), 0.8 (Dashdot), 1 (Solid).

where $s = \frac{N}{L}t - n + 1$. Using generalized mean value theorem for integrals, we have

$$|a_{nm}| \leq \sqrt{\frac{L}{N}} \left| y(\frac{L}{N}(z+n-1)) \right| \int_0^1 |\bar{P}_m^{(\mu,\nu)}(s)| ds, \qquad z \in [0,1], \tag{4.2}$$

considering $\int_0^1 |\bar{P}_m^{(\mu,\nu)}(s) ds| \leq \gamma < \infty$, for $m = 0, 1, ..., M-1$, we have

$$|a_{nm}| \leq \sqrt{\frac{L}{N}} \left| y(\frac{L}{N}(z+n-1)) \right| \gamma \leq \sqrt{\frac{L}{N}} M_y \gamma = \frac{\sqrt{L} M_y \gamma}{2^{\frac{k-1}{2}}}, \tag{4.3}$$

where $N = 2^{k-1}$ and $M_y = \max_{0 \leq x \leq L} |y(x)|$.
Since y is bounded, therefore $\sum_{n,m=0}^{\infty} |a_{nm}|$ is absolutely convergent, then the series $\sum_{n,m=0}^{\infty} a_{nm}$ is convergent.

Hence the JWs expansion of $y(x)$ converges uniformly.

For error analysis, we consider equation (1.2). Let $e_m(t) = u_m(t) - u(t)$ be the error function, where $u_m(t)$ is the approximation of the true solution $u(t)$ which is as follows:

$$\begin{aligned} &\kappa\ {}^{C}D^{\alpha}u_m(t) - u_m(t) + u_m^2(t) + u_m(t)\int_0^t u_m(\tau)d\tau = r_m(t), \\ &u_m(0) = u_0, \quad 0 < \alpha \le 1, \end{aligned} \tag{4.4}$$

where $r_m(t)$ is the perturbation function that depends only on $u_m(t)$. With (1.2) and (4.4), we have:

$$\kappa\ {}^{C}D^{\alpha}(u_m(t) - u(t)) - (u_m(t) - u(t)) + \left(u_m^2(t) - u^2(t)\right) + u_m(t)\int_0^t u_m(\tau)d\tau - u(t)\int_0^t u(\tau)d\tau = r_m(t). \tag{4.5}$$

Then we have:

$$\kappa\ {}^{C}D^{\alpha}e_m(t) - e_m(t) + (u_m(t) - u(t))\,(u_m(t) + u(t)) \tag{4.6}$$

$$\begin{aligned} &+ \int_0^t \left(u_m(t)u_m(\tau) - u_m(t)u(\tau) + u_m(t)u(\tau) - u(t)u(\tau)\right) d\tau = r_m(t), \\ &e_m(0) = 0, \quad 0 < \alpha \le 1. \end{aligned} \tag{4.7}$$

From $u(t) = u_m(t) - e_m(t)$ and $u_m^2(t) - u^2(t) = (u_m(t) - u(t))\,(u_m(t) + u(t))$, we have:

$$\begin{aligned} &\kappa\ {}^{C}D^{\alpha}e_m(t) - e_m(t) + e_m(t)\,(2u_m(t) - e_m(t)) + u_m(t)\int_0^t e_m(\tau)d\tau + e_m(t)\int_0^t u(\tau)d\tau = r_m(t), \\ &e_m(0) = 0, \quad 0 < \alpha \le 1, \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} &\kappa\ {}^{C}D^{\alpha}e_m(t) - e_m(t) + 2u_m(t)e_m(t) - e_m^2(t) + u_m(t)\int_0^t e_m(\tau)d\tau + e_m(t)\int_0^t u_m(\tau)d\tau - e_m(t)\int_0^t e_m(\tau)d\tau = r_m(t), \\ &e_m(0) = 0, \quad 0 < \alpha \le 1, \end{aligned} \tag{4.9}$$

we have:

$$\kappa\ {}^{C}D^{\alpha}e_m(t) + \left(2u_m(t) - 1 - \int_0^t u_m(\tau)d\tau\right) e_m(t) - e_m^2(t) + (u_m(t) - e_m(t))\int_0^t e_m(\tau)d\tau \quad = r_m(t). \tag{4.10}$$

Equation (4.10) is a non homogeneous population type equation, where $r_m(t)$ and $u_m(t)$ are known functions and $e_m(t)$ is the unknown function, which can be solved by the proposed method to find $e_m(t)$ as error estimation.

5. Conclusion

In this article, we introduced a numerical plan for solving FPGM. The method is based on the operational matrix of fractional integration of JWs. The suggested method provides a careful estimate of the solution. In addition, the solution attained using the presented method is in marvelous agreement with the formerly existing ones. The important feature of the new technique is to convert the model in the current study to a system of nonlinear algebraic equations by introducing the operational matrix of fractional integration for these basic functions. Using plotted Figures, analysis of the treatment of the model shows that as κ increases, the maximum of $u(t)$ reduces and also when the order of the fractional derivative α decreases the domain of solution reduces.

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