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Stability for neutral-type integro-differential neural networks with random switches in noise and delay

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Abstract

This paper focuses on existence, uniqueness, and stability analysis of solutions for a new kind of delayed integro-differential neural networks with Markovian switches in delays and noises. The studied system combines many types of integro-differential neural network treatises in the literature. After having presented the studied system, the existence and uniqueness of solutions are shown under Lipschitz condition. By using the Lyapunov-Krasovskii functional, some stochastic analysis techniques and the M-matrix approach, stochastic stability, and general decay stability are established. Finally, a numerical example is given to validate the main established theoretical results.

Keywords. Neural networks, Markovian jumps systems, Lévy noise, Gaussian noise, Neutral-type systems, Time-varying delays, General decay stability.

2010 Mathematics Subject Classification. 37H05, 37B25, 37C75.

1. Introduction

After its appearance in the 19^{th} century, the study of dynamical systems has caught the attention of a group of researchers, due to its application in different fields [6, 13, 22, 25, 30].

Several important subclasses of hybrid dynamical systems have been widely studied, due to their importance in the real world and their usefulness in the development of several other areas. In addition, the study of the hybrid dynamical systems is important, because practical experiences show that even if the subsystems are stable, the switching system can be unstable, which leaves a large group of researchers interested in studying this kind of systems [1, 20, 23, 24]. There are two categories of hybrid dynamical systems, which are hybrid dynamical systems with deterministic commutations (the commutations are generally controlled), and the hybrid dynamical systems with random switching. But the study of a hybrid system with random switching is more representative than the study of a hybrid system with controlled switching, seeing that almost all the models that exist in reality are subject to randomness. Therefore, after the emergence of dynamic systems with random commutations, and due to their application in many fields, a large part of researchers are interested in the study of these kinds of dynamical systems, such as [11, 17, 18, 21, 29, 41].

Neural networks with random switching are a class of hybrid dynamical systems with random switching which have recently experienced wide applications in several areas such as distributed networks, digital communications, securing communication systems, signal processing, population dynamics systems, chemical process control, image processing, and among others, see [9, 10, 27] for a brief account.

As time-delay assignment in neural networks can cause oscillation and instability behavior, many researchers have been interested in the study of delayed neural networks systems with Markovian switching, but, in most work, the delay is considered as a constant, as constants in the case of a multi-delay system, or the functions depend on t [8, 33, 40]. In this work the delay is taken depends on the Markov process. Moreover, the system in which the delay

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is not only depends on the current state but also depends on the past state, namely neutral-type system, in particular neutral-type neural networks that model several realistic phenomena such as population ecology, propagation, and diffusion models, the study of the motion for a particle in a fluid [5, 15, 28]. So, neutral-type neural networks are taking into account in the study of this paper with the delays obeying a Markov process.

In some areas such as confocal microscopy and image processing, the importance of Lévy noise is based on the detection of photons. Thus, the addition of Lévy noise is more representative of the model than the addition of a Gaussian noise [12], and the modeling of the number of phone calls occurring over a certain period time is done using a model with Lévy noise [31]. Moreover, Gaussian noise and Lévy noise are parameters that can confuse the stability of the equilibrium point of a hybrid dynamic system. So, both noises are also taken into account in the study of different types of stability and stabilization of neural networks using different techniques as Lyapunov functional theory, generalized Itô's formula, sliding mode control and M-matrix [2–4, 16, 32]. Taking into account both noises in such systems is more reliable to apply in reality. But, we remark that the most of works that exist in the literature take these parameters in a classical way, that is, these parameters are permanently fixed to the system, while in some real systems, noises are not present in the system all the time, such as wind, heavy rain, heavy snow and other natural phenomena which are considered as noises for some systems. So, can a dynamic system keep stability when it is randomly impacted by different types of noise? Finally, some interesting results have been obtained on the various qualitative properties of nonlinear integro-differential equations and a stochastic differential equation of second order in [26, 34–38]

Considering the following system simulated under Matlab which illustrates well that when the system is affected randomly once by Gaussian noise and once by Lévy noise, stability is not always guaranteed.

$$dx(t) = Ax(t)dt + (2 - r(t))Bx(t)dw(t) + (r(t) - 1)\int_{-1}^{2} h(x(t), y)N(dt, dy), \tag{1.1}$$

with $(x_1(0), x_2(0)) = (0.2, -0.1)$, where $A = \begin{pmatrix} -0.9 & 0 \\ 1.3 & -1.3 \end{pmatrix}$, $B = \begin{pmatrix} 0.4 & 0.2 \\ 0.1 & 0.3 \end{pmatrix}$, $h(x(t), y) = x - y$, and $r(t)$ is the random process taking values in $\{1, 2\}$.

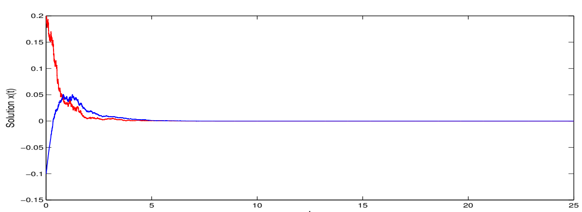

FIGURE 1. Stochastic approximation solution of System (1.1) with Gaussian noise.

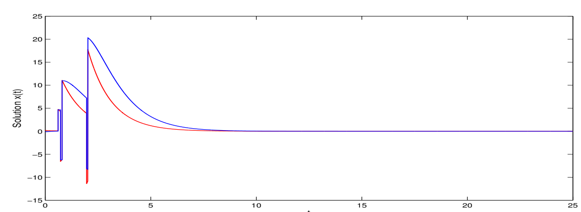

FIGURE 2. Stochastic approximation solution of System (1.1) with Lévy noise.

Compared to other related topics on neutral stochastic neural networks, the main contributions in this paper are highlighted from the following five aspects.

- To make the model more general, neutral-type, Gaussian noise and Lévy noise, Markovian switching and time delays are taken into account. Furthermore, the homogeneous Markov process is considered in a form which allows the model to change its nature randomly which has been almost never considered. In fact, it is a model which takes four states; delayed neutral-type neural networks without noise, delayed neutral-type neural networks with Gaussian noise, delayed neutral-type neural networks with Lévy noise and delayed neutral-type neural networks with Gaussian and Lévy noises.
- Time varying delay is considered continuous and supposed to be dependent on the Markovian process and is not necessarily derivable.
- Existence and uniqueness of solutions for the model are shown.

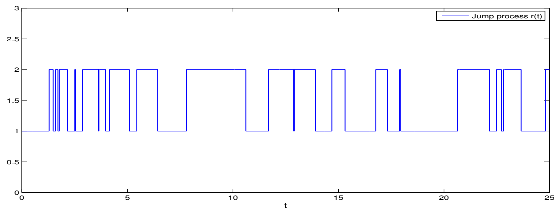

FIGURE 3. Trajectory of jump process $r(t)$.

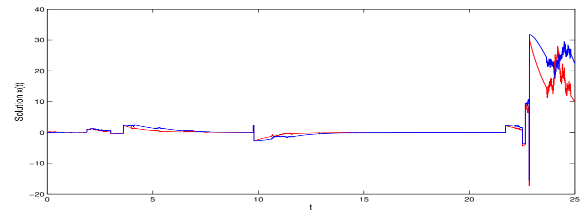

FIGURE 4. Stochastic approximation solution of System (1.1) with randomly switched Gaussian and Lévy noises.

- Basing on the Lyapunov-Krasovskii functional, M-matrix technique and some stochastic analysis techniques, stochastic stability is proved.
- By using M-matrix technique, Stability with a general decay in order μ and in order $\frac{\mu}{p}$ are derived.

The remainder of this paper is organized as follows. Some notations and tools which are used in the formulation of the studied system are presented in the next section. A description of the model and some useful lemmas are too introduced. In section 3, the main results are presented in three subsections as follows. In subsection 3.1, existence and uniqueness of solutions are established. Stochastic stability is proved in subsection 3.2. Stability with a general decay is studied in subsection 3.3. Section 4 provides a numerical example. In section 5, the conclusion is given.

2. Notations and preliminaries

The following notations are used throughout this paper. $\mathbb{R}^{n\times m}$ is the set of $n\times m$ real matrices and $|.|$ is the norm of the n-dimensional real Euclidean space $\mathbb{R}^n$. We use $a\wedge b$ $(a\vee b)$ to denote the minimum (maximum) for $a,b\in\mathbb{R}$. The trace norm of the matrix A is defined by $|A| = \big(Trace(A^TA)\big)^{\frac{1}{2}}$, where A^T is its transpose, $A>0$ $(A<0)$ means that A is a symmetric positive defined (negative defined) matrix. By $A \gg 0$ we mean that all element of A are positive.

Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Denote by $W(t)$ a m-dimensional $\mathcal{F}_t$-adapted Brownain motion defined on the probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},P)$, let $N(dt,dz)$ be a $\mathcal{F}_t$-adapted Poisson random measure on $[0,+\infty)\times\mathbb{R}$ with compensator $\tilde{N}$ which satisfies $\tilde{N}(dt,dz) = N(dt,dz)-\pi(dz)dt$, $t\in\mathbb{R}_+, y\in\mathbb{R}$ and $\pi(dz)dt = \mu\varphi(dz)dt$ is a Lévy measure associated to N, where μ is the intensity of the Poisson process and φ is the probability distribution of the random variable z which satisfies $\int_Z \pi(dz) < \infty$, where $Z\subset\mathbb{R}$. $\mathbb{E}(.)$ is the mathematical expectation with respect to the given probability measure P.
We define two irreducible Markov chains on the probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},P)$, the first one is the right-continuous Markov process $\alpha(t)$ which takes values in $\mathcal{S}_1 = \{1,2,3,4\}$ with generator $\Theta = (\theta_{\varsigma\ell})_{4\times 4}$ defined as

$$P(\alpha(t+dt)=\ell/\alpha(t)=\varsigma) = \begin{cases}\theta_{\varsigma\ell}dt + o(dt), & \varsigma\neq\ell\\ 1+\theta_{\varsigma\varsigma}dt+o(dt) & \text{otherwise,}\end{cases}$$

where $dt>0$ and $\theta_{\varsigma\ell}>0$ is the transition rate from ς to ℓ. If $\ell=\varsigma$, it follows that $\theta_{\varsigma\varsigma} = -\sum\limits_{\ell=1,\varsigma\neq\ell}^{4}\theta_{\varsigma\ell}$.
Let $\hbar_2(t) = J_1(t)J_2(t)$ and $\hbar_3(t) = J_1(t)J_3(t)$, with $J_1(t)$, $J_2(t)$ and $J_3(t)$ are defined as

$$\begin{cases} J_1(t) = & \dfrac{(4-\alpha(t))^{\alpha(t)-1}}{(\alpha(t)-1)^{3-\alpha(t)}+(3-\alpha(t))^{\alpha(t)-1}},\\[2ex] J_2(t) = & (\alpha(t)-1)^{3-\alpha(t)},\\[2ex] J_3(t) = & (2-\alpha(t))^2.\end{cases}$$

The second one is the right-continuous Markov process $r(t)$ which takes values in a finite state space $\mathcal{S}_2 = \{1, 2, ..., \mathcal{S}\}$ with generator $\Gamma = (\gamma_{\iota k})_{\mathcal{S}\times\mathcal{S}}$ as follows:

$$P(r(t+dt) = k/r(t) = \iota) = \begin{cases} \gamma_{\iota k} dt + o(dt), & \iota \neq k \\ 1 + \gamma_{\iota\iota} dt + o(dt) & \text{otherwise}, \end{cases}$$

where $dt > 0$ and $\gamma_{\iota k} > 0$ is the transition rate from ι to k. If $\iota = k$, it follows $\gamma_{\iota\iota} = -\sum\limits_{k=1, k\neq\iota}^{\mathcal{S}} \gamma_{\iota k}$.

$\mathcal{C} := \mathcal{C}([-\tilde{\sigma}, 0]; \mathbb{R}^n)$ denotes the family of bounded continuous functions ϕ defined on $[-\tilde{\sigma}, 0]$ endowed with the norm $|\phi|_c = \sup\limits_{-\tilde{\sigma}\leq\theta\leq 0} |\phi(\theta)|$. $\sigma_{t,\alpha(t)}$ is the transmission delay that depends on t and $\alpha(t)$ with $0 \leq \sigma_{t,\alpha(t)} \leq \tilde{\sigma}$ and $\tilde{\sigma}$ is a constant.

$x_t : [-\tilde{\sigma}, 0] \longrightarrow \mathbb{R}^n; \theta \longmapsto x_t(\theta) = x(t+\theta); -\tilde{\sigma} \leq \theta \leq 0$ is regarded as a $\mathcal{C}$-valued stochastic process, where $x_t(\theta) = (x_t^1(\theta), x_t^2(\theta), ..., x_t^n(\theta))^T$. We define the initial data of the stochastic process as a bounded continuous function by $x_0(\theta) = \xi(\theta)$ for $-\tilde{\sigma} \leq \theta \leq 0$.

For any $R > 0$, define the stopping time τ_x^R as follows:

$$\tau_x^R = \inf\{t \geq 0 : |x(t)| \leq R, |\xi|_c > R \text{ almost surely}\}.$$

We consider the following neutral-type Markovian switched neural networks system with random time-varying delay:

$$\begin{aligned} d[x(t) - \tilde{A}_{\alpha(t)} x(t - \sigma_{t,\alpha(t)})] = \big[&- \tilde{B}_{r(t)} x(t) + \tilde{C}_{r(t)} f(x(t)) + \tilde{D}_{r(t)} f(x(t - \sigma_{t,\alpha(t)})) \\ &+ \int_{t-\sigma_{t,\alpha(t)}}^{t} \tilde{E}_{r(t)} K(t-u) f(x(u)) du \big] dt \\ &+ \hbar_2(t) \big(\tilde{M}_{r(t)} g(x(t)) + \tilde{Q}_{r(t)} g(x(t - \sigma_{t,\alpha(t)})) \big) dW(t) \\ &+ \hbar_3(t) \int_Z \big(h(x(t), x(t-\sigma_{t,\alpha(t)}), z, r(t)) \big) N(dt, dz), \end{aligned} \tag{2.1}$$

where $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{R}^n$ is the state vector associated with the neurons, for all $i \in \mathcal{S}_2$, $\tilde{B}_i = diag(\tilde{b}_1^i, \tilde{b}_2^i, ..., \tilde{b}_n^i) \gg 0$ is the firing rate of the neurons, $\tilde{C}_i = (\tilde{c}_{jk}^i)_{n\times n}$ and $\tilde{M}_i = (\tilde{m}_{jk}^i)_{n\times n}$ are the connection weight matrices, $\tilde{D}_i = (\tilde{d}_{jk}^i)_{n\times n}, \tilde{E}_i = (\tilde{e}_{jk}^i)_{n\times n}$ and $\tilde{Q}_i = (\tilde{q}_{jk}^i)_{n\times n}$ are the delay connection weight matrices, and for all $i \in \mathcal{S}_1$ the matrix $\tilde{A}_i = (\tilde{a}_{jk}^i)_{n\times n}$ is the neutral-type parameter. $f(x(t)) = \big(f_1(x_1(t)), f_2(x_2(t)), ..., f_n(x_n(t))\big)^T$ is the neuron activation vector function, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is the continuous noise intensity function satisfying $g(0) = 0$ and $h : \mathbb{R}^n \times \mathbb{R}^n \times Z \times \mathcal{S}_2 \to \mathbb{R}^n$ is the Lévy noise intensity function satisfying $h(0, 0, z, i) = 0$ for all $(z, i) \in Z \times \mathcal{S}_2$. $K_{ij} : [0, \tilde{\sigma}) \longrightarrow [0, +\infty)$ $(i, j = 1, 2, ..., n)$ are piecewise continuous functions on $[0, \tilde{\sigma})$, subject to

$$\int_0^{\sigma_{t,\ell}} K_{ij}(s) ds \leq 1 \quad \text{for} \quad i, j = 1, 2, ..., n \text{ and } \ell \in \mathcal{S}_1, \tag{2.2}$$

and $\alpha(t)$ is a Markov process taking values in $\{1, 2, 3, 4\}$. Then, we have

$$\alpha(t) = 1 \Rightarrow \begin{cases} J_1(t) = & 1 \\ J_2(t) = & 0 \\ J_3(t) = & 1 \end{cases} \text{and} \begin{cases} \hbar_2(t) = & 0, \\ \hbar_3(t) = & 1, \end{cases}$$

$$\alpha(t) = 2 \Rightarrow \begin{cases} J_1(t) = & 1 \\ J_2(t) = & 1 \\ J_3(t) = & 0 \end{cases} \text{and} \begin{cases} \hbar_2(t) = & 1, \\ \hbar_3(t) = & 0, \end{cases}$$

$$\alpha(t) = 3 \Rightarrow \begin{cases} J_1(t) = & 1 \\ J_2(t) = & 1 \\ J_3(t) = & 1 \end{cases} \text{and} \begin{cases} \hbar_2(t) = & 1, \\ \hbar_3(t) = & 1, \end{cases}$$

and if $\alpha(t) = 4$, we get $J_1(t) = 0$, so $\hbar_2(t) = 0$ and $\hbar_3(t) = 0$. Then, we have the following result.

Remark 2.1. We notice that if $\alpha(t) = 1$, $\alpha(t) = 2$, $\alpha(t) = 3$ or $\alpha(t) = 4$, then system (2.1) is reduced, respectively, to a delayed Markovian switched neutral-type neural networks system with Lévy noise, with Gaussian noise, with both Gaussian and Lévy noises or without any noise.

In this paper, it is assumed that the processes $\alpha(t)$, $r(t)$, $W(t)$ and $N(dt, dz)$ are independent.

Definition 2.2. [7] The trivial solution of system (2.1) with initial data ξ is said to be stochastically stable if for every real pair $\epsilon \in (0, 1)$ and R, there exists a $\mathcal{H} = \mathcal{H}(\epsilon, R) > 0$, such that

$$P\Big\{|x(t,\xi)| < R, t \geq 0\Big\} \geq 1 - \epsilon,$$

where $|\xi|_c < \mathcal{H}$.

Now, we display some interesting lemmas which are used to demonstrate the main results.

Lemma 2.3. *[39] Let $a_i \in \mathbb{R}$, $s, p \in \mathbb{Z}$ with $p \geq 1$, then*

$$\Big|\sum_{i=1}^{s} a_i\Big|^p \leq s^{p-1} \sum_{i=1}^{s} |a_i|^p.$$

Lemma 2.4. *[39] For any $p \geq 2$, $q \geq 1$ with $p > q$ we have*

$$|x|^{p-q}|y|^q \leq \frac{p-q}{p}|x|^p + \frac{q}{p}|y|^p.$$

For the purpose of stability study, we impose the following assumptions.

Assumption 2.1. *For each $x, \bar{x}, y, \bar{y} \in \mathbb{R}^n$, $p \geq 2$, $j \in \{1, 2, ..., n\}$, $\ell \in \mathcal{S}_1$, and $k \in \mathcal{S}_2$*

$$|f_j(x_j) - f_j(y_j)|^p \vee |g_j(x_j) - g_j(y_j)|^p \leq L_{1j}|x_j - y_j|^p,$$

that is,

$$|f(x) - f(y)|^p \vee |g(x) - g(y)|^p \leq L_1|x - y|^p, \tag{2.3}$$

where $L_1 = \max\limits_{1 \leq j \leq n}\{L_{1j}\}$.

$$\int_Z |h(x, y, k) - h(\bar{x}, \bar{y}, k)|^p \pi(dz) \leq L_2(|x - \bar{x}|^p + |y - \bar{y}|^p). \tag{2.4}$$

Moreover, for all $x_t, y_t \in \mathcal{C}$

$$|\tilde{A}_\ell x_t - \tilde{A}_\ell y_t|^p \leq L_\ell |x_t - y_t|^p_{\mathcal{C}}, \tag{2.5}$$

with $\tilde{A}_\ell$ satisfies $\rho(\tilde{A}_\ell) \leq L_\ell < 1$, where $\rho(\tilde{A}_\ell)$ is the spectral radius of $\tilde{A}_\ell$.

Assumption 2.2. *For any $\ell \in \mathcal{S}_1$, there is a $\lambda = \big(\lambda_1, \lambda_1, \cdots, \lambda_S\big) \in \mathbb{R}^S_+$ such that*

$$\Delta^\ell = \Lambda^\ell \lambda \geq 0,$$

where

$$\Lambda^{\ell} := -\, diag\big(\chi_1^{\ell}, \chi_2^{\ell}, \cdots, \chi_S^{\ell}\big) - \zeta^{\ell},$$

$$\chi_k^{\ell} := -\, p\tilde{b}_{min}(k) + (2^{p-1}-1)\pi(Z) + 2^p L_2\big(\hbar_3(t)\big)^p + n^2.2^{p-1}\Big[\bar{K} + \bar{K}L_{\ell} + 2\big|\theta_{\ell\ell}\big| + \sum_{\varsigma=1}^{4}\big|\theta_{\ell\varsigma}\big|L_{\varsigma}\Big]$$

$$+ pL_1\Big[|\tilde{C}_k| + |\tilde{D}_k| + n^2|\tilde{E}_k| + \frac{1}{4}\big(\hbar_2(t)\big)^2\big(n(p-2)+1\big)\big(\big|\tilde{M}_k\big|^2 + \big|\tilde{Q}_k\big|^2\big)\Big],$$

$$\zeta^{\ell} := \big(\zeta_{k\iota}^{\ell}\big)_{\mathcal{S}\times\mathcal{S}},$$

such that $\zeta_{k\iota}^{\ell} := \gamma_{k\iota} + 2^{p-1}n^2\big(1+L_{\ell}\big)\big|\gamma_{k\iota}\big|$ *and* $k \in \mathcal{S}_2$.

3. Main results

In this section, we establish existence and uniqueness of solutions for system (2.1), then, we deal with stochastic stability.

3.1. Existence and uniqueness of solutions.

Theorem 3.1. *Assume that Assumption 2.1 holds. Then, system (2.1) has a unique global solution on* $[-\tilde{\sigma}, +\infty)$ *with initial conditions* $\xi \in \mathcal{C}, \alpha(0) = \alpha_0$ *and* $r(0) = r_0$.

Proof. $\alpha(t)$ and $r(t)$ are right continuous Markov jumps. Then, there are two sequences $\{\varrho_k\}_{k\geq 0}$ and $\{\tau_{k^*}\}_{k^*\geq 0}$ of stopping times such that the processes $\alpha(t)$ and $r(t)$ are random constants, respectively, on every interval $[\varrho_k, \varrho_{k+1})$, and $[\tau_{k^*}, \tau_{k^*+1})$. So, $\alpha(t) = \alpha(\varrho_k)$ on $\varrho_k \leq t < \varrho_{k+1}$ and $r(t) = r(\tau_{k^*})$ on $\tau_{k^*} \leq t < \tau_{k^*+1}$ for any $k, k^* \geq 0$.
We consider $[a_0^*, a_1^*) = [\varrho_0, \varrho_1) \cap [\tau_0, \tau_1)$. For the sake of simplicity, we denote $\alpha_0 = \alpha(\varrho_0), r_0 = r(\tau_0), x = x(t)$ and $x_t = x(t - \sigma_{t,\alpha(t)})$.
For any $t \in [a_0^*, a_1^*]$, we have from (2.1) that

$$d[x - \tilde{A}_{\alpha_0}x_t] = \tilde{F}(x, x_t, \alpha_0, r_0)dt + \tilde{G}(x, x_t, \alpha_0, r_0, \hbar_2(t))dW(t) + \hbar_3(t)\int_Z h(x, x_t, z, r_0)N(dt, dz), \tag{3.1}$$

with initial conditions $(\xi, \alpha_0, r_0) \in \mathcal{C} \times \mathcal{S}_1 \times \mathcal{S}_2$, where

$$\tilde{F}(x, x_t, \alpha_0, r_0) = -\tilde{B}_{r_0}x + \tilde{C}_{r_0}f(x) + \tilde{D}_{r_0}f(x_t) + \int_{t-\sigma_{\alpha_0}}^{t} \tilde{E}_{r_0}K(t-s)f(x(s))ds,$$

and

$$\tilde{G}(x, x_t, \alpha_0, r_0, \hbar_2(t)) = \hbar_2(t)\big(\tilde{M}_{r_0}g(x) + \tilde{Q}_{r_0}g(x_t))\big).$$

Now, we show the conditions that guarantee existence and uniqueness of solutions for system (3.1) on $[a_0^*, a_1^*]$. For any $\psi, \varphi \in \mathcal{C}$, using Lemma 2.3 and condition (2.3), we prove that

$$|\tilde{F}(\psi(0), \psi, \alpha_0, r_0) - \tilde{F}(\varphi(0), \varphi, \alpha_0, r_0)|^2 \vee |\tilde{G}(\psi(0), \psi, \alpha_0, r_0) - \tilde{G}(\varphi(0), \varphi, \alpha_0, r_0)|^2 \leq \bar{\delta}|\psi - \varphi|_c^2, \tag{3.2}$$

where $\bar{\delta} = \max\{\delta_1, \delta_2\}$, with $\delta_1 = 4\big(|\tilde{B}_{r(\iota_0)}|^2 + L_1(|\tilde{C}_{r(\iota_0)}|^2 + |\tilde{D}_{r(\iota_0)}|^2 + n^4|\tilde{E}_{r(\iota_0)}|^2)\big)$ and $\delta_2 = 2L_1(|\tilde{M}_{r_0}|^2 + |\tilde{Q}_{r_0}|^2)$.
Employing (2.4), we have

$$\begin{aligned}\int_Z |h(\psi(0), \psi, z, r_0) - h(\varphi(0), \varphi, z, r_0)|^p \pi(dz) &\leq L_2|\psi(0) - \varphi(0)|^p + L_2|\psi - \varphi|_c^p \\ &\leq 2L_2|\psi - \varphi|_c^p,\end{aligned} \tag{3.3}$$

and from (2.5), one has

$$|\tilde{A}_{\ell}\psi - \tilde{A}_{\ell}\varphi|^p \leq L_{\ell}|\psi - \varphi|_c^p. \tag{3.4}$$

Therefore, it is known that conditions (3.2), (3.3) and (3.4) assure that system (3.1) has a unique solution on $[a_0^*, a_1^*]$. For more information see [19].
The second step, we take system on $[a_1^*, a_2^*]$ with initial condition $x_{a_1^*}$ and by the same method, we show that it has a

unique solution on $[a_1^*, a_2^*]$.
Repeating the argument on the intervals $[a_i^*, a_{i+1}^*]$ with $(i \geq 2)$, we establish that system (2.1) has a unique solution on $[-\tilde{\sigma}, +\infty)$. $\square$

3.2. Stochastic stability.

Theorem 3.2. *Assume that Assumption 2.1 and Assumption 2.2 hold, then system (2.1) is stochastically stable.*

Proof. Let

$$V\big(t, x, \alpha(t), r(t)\big) = \frac{\lambda_{r(t)}}{p}\big|x + \tilde{A}_{\alpha(t)} x_{t,\alpha(t)}\big|^p + \frac{\lambda_{r(t)}}{p}\int_t^{t+\sigma_{t,\alpha(t)}} \sum_{i=1}^n \sum_{j=1}^n K_{ij}(s-t)\big|x_j(s) - \sum_{v=1}^n \tilde{a}_{jv} x_v(2t-s)\big|^p ds,$$

so

$$V\big(t, x - \tilde{A}_{\alpha(t)} x_t, \alpha(t), r(t)\big) = \tfrac{\lambda_{r(t)}}{p}\big|x\big|^p + \tfrac{\lambda_{r(t)}}{p}\int_t^{t+\sigma_{t,\alpha(t)}} \sum_{i=1}^n \sum_{j=1}^n K_{ij}(s-t)\big|x_j(s) - \sum_{v=1}^n \tilde{a}_{jv} x_v(2t-s)\big|^p ds.$$

For any $\epsilon \in (0.1)$, and $R \geq 0$, we take

$$2^{p-1} n^2 \bar{K}(1 + L_\ell)|\xi|_c^p < \epsilon R^p.$$

For system (1), the generalized Itô formula of the function $V\big(t, x, \alpha(t), r(t)\big)$ which is an element of $C^{1,2}\big(\mathbb{R}^+ \times \mathbb{R}^n \times S_1 \times S_2 : \mathbb{R}^+\big)$ is given by:

$$\begin{aligned} V\big(t, x - \tilde{A}_{\alpha(t)} x_t, \alpha(t), r(t)\big) =& V\big(0, x(0) - \tilde{A}_{\alpha(0)} x(-\sigma_{\alpha(0)}), \alpha(0), r(0)\big) \\ &+ \int_0^t \mathcal{L}V\big(s, x(s), x(s - \sigma_{s,\alpha(s)}), \alpha(s), r(s)\big) ds + G_t, \end{aligned} \tag{3.5}$$

where G_t is defined by

$$\begin{aligned} G_t =& \int_0^t V_x(s, \tilde{x}(s), \alpha(s), r(s)).\hbar_2(s)\big(\tilde{M}_{r(s)} g(x(s)) + \tilde{Q}_{r(s)} g(x(s - \sigma_{s,\alpha(s)}))\big) dw(s) \\ &+ \int_0^t \int_Z \Big[V\big(s, \tilde{x}(s) + \hbar_3(s) h\big(x(s), x(s - \sigma_{s,\alpha(s)}), z, r(s)\big), \alpha(s), r(s)\big) \\ &- V\big(s, \tilde{x}(s), \alpha(s), r(s)\big)\Big] \tilde{N}(ds, dz) \\ &+ \int_0^t \int_{\mathbb{R}} \big[V\big(s, \tilde{x}(s), \alpha(s), r_0 + \hbar_1(r(s), u)\big) - V\big(s, \tilde{x}(s), \alpha(s), r(t)\big)\big] \mu_1(ds, du) \\ &+ \int_0^t \int_{\mathbb{R}} \big[V\big(s, \tilde{x}(s), \alpha_0 + \hbar_4(\alpha(s), u), r(s)\big) - V\big(s, \tilde{x}(s), \alpha(s), r(t)\big)\big] \mu_2(ds, du), \end{aligned}$$

where $\mu_1(ds, du), \mu_2(ds, du)$ are two martingale measures, and the details of the functions $\mu_1, \mu_2, \hbar_1$ and $\hbar_4$ can be found in [14]. We can easily prove that G_t is a local martingale with $G_0 = 0$, because if a stochastic process is a martingale, then it is a local martingale.
Let $\tilde{x}(t) = x(t) - \tilde{A}_{\alpha(t)} x(t - \sigma_{t,\alpha(t)})$. The infinitesimal operator $\mathcal{L}V\big(t, x(t), x(t - \sigma_{t,\alpha(t)}), \alpha(t), r(t)\big)$ is defined for $\ell \in S_1$

and $k \in S_2$ by

$$\begin{aligned}\mathcal{L}V(t,x(t),x(t-\sigma_{t,\ell}),\ell,k) =& V_t(t,\tilde{x}(t),\ell,k) + V_x(t,\tilde{x}(t),\ell,k)\Big[-\tilde{B}_k x(t) + \tilde{C}_k f\big(x(t)\big) + \tilde{D}_k f\big(x_t(-\sigma_{t,\ell})\big) \\ &+ \int_{t-\sigma_{t,\ell}}^{t} \tilde{E}_k K(t-s) f\big(x(s)\big)ds\Big] \\ &+ \frac{1}{2} trace\Big[\hbar_2(t)\Big(\tilde{M}_k g(x(t)) + \tilde{Q}_k g(x_t(-\sigma_{t,\ell}))\Big)^T V_{xx}(t,\tilde{x}(t),\ell,k) \\ &\qquad \hbar_2(t)\Big(\tilde{M}_k g(x(t)) + \tilde{Q}_k g(x_t(-\sigma_{t,\ell}))\Big)\Big] \\ &+ \int_Z \Big[V\big(t,\tilde{x}(t)+\hbar_3(t)h\big(x(t),x_t(-\sigma_{t,\ell}),z,k\big),\ell,k\big) - V(t,\tilde{x}(t),\ell,k)\Big]\pi(dz) \\ &+ \sum_{\varsigma=1}^{4} \theta_{\ell\varsigma} V(t,\tilde{x}(t),\varsigma,k) + \sum_{\iota=1}^{S} \gamma_{k\iota} V(t,\tilde{x}(t),\ell,\iota),\end{aligned}$$

where $V_t(t,x,\ell,k) = \frac{\partial V(t,x,\ell,k)}{\partial t}$,
$V_x(t,x,\ell,k) = col\Big(\frac{\partial V(t,x,\ell,k)}{\partial x_1}, \frac{\partial V(t,x,\ell,k)}{\partial x_2}, ..., \frac{\partial V(t,x,\ell,k)}{\partial x_n}\Big)$, $V_{xx}(t,x,\ell,k) = \Big(\frac{\partial^2 V(t,x,\ell,k)}{\partial x_i \partial x_j}\Big)_{n\times n}$.
For the sake of simplicity, consider the notations $\mathcal{L}V(t,x(t),x_t(-\sigma_{t,\ell}),\ell,k) = \mathcal{L}V$, $x(t) = x = (x_1,x_2,\cdots,x_n)$ and $x_t(-\sigma_{t,\ell}) = x_{t,\ell} = (x^1_{t,\ell}, x^2_{t,\ell}, \cdots, x^n_{t,\ell})$. One has that

$$\begin{aligned}\mathcal{L}V =& \frac{\lambda_k}{p}\sum_{i=1}^{n}\sum_{j=1}^{n} K_{ij}(\sigma_{t,\ell})\Big|x_j(t+\sigma_{t,\ell}) - \sum_{v=1}^{n}\tilde{a}_{jv}x^v_{t,\ell}\Big|^p - \frac{\lambda_k}{p}\sum_{i=1}^{n}\sum_{j=1}^{n} K_{ij}(0)\Big|x_j - \sum_{v=1}^{n}\tilde{a}_{jv}x_v\Big|^p \\ &+ \lambda_k|x|^{p-2}x^T\Big[-\tilde{B}_k x + \tilde{C}_k f(x) + \tilde{D}_k f\big(x_{t,\ell}\big) + \int_{t-\sigma_{t,\ell}}^{t}\tilde{E}_k K(t-s) f\big(x(s)\big)ds\Big] \\ &+ \frac{\lambda_k}{2} trace\Big[\big(\hbar_2(t)\big)^2\Big(\tilde{M}_k g(x) + \tilde{Q}_k g(x_{t,\ell})\Big)^T\Big((p-2)\big|x\big|^{p-4}x.x^T + \big|x\big|^{p-2}\Big) \qquad (3.6) \\ &\times\Big(\tilde{M}_k g(x) + \tilde{Q}_k g(x_{t,\ell})\Big)\Big] + \frac{\lambda_k}{p}\int_Z\Big[\big|x + \hbar_3(t)h\big(x,x_{t,\ell},z,k\big)\big|^p - |x|^p\Big]\pi(dz) \\ &+ \frac{\lambda_k}{p}\sum_{\varsigma=1}^{4}\theta_{\ell\varsigma}\int_t^{t+\sigma_{t,\varsigma}}\sum_{i=1}^{n}\sum_{j=1}^{n}K_{ij}(s-t)\Big|x_j - \sum_{v=1}^{n}\tilde{a}_{jv}x_v(2t-s)\Big|^p ds \\ &+ \sum_{\iota=1}^{S}\gamma_{k\iota}\frac{\lambda_\iota}{p}\Big[|x|^p + \int_t^{t+\sigma_{t,\ell}}\sum_{i=1}^{n}\sum_{j=1}^{n}K_{ij}(s-t)\Big|x_j - \sum_{v=1}^{n}\tilde{a}_{jv}x_v(2t-s)\Big|^p ds\Big]. \qquad (3.7)\end{aligned}$$

Using Lemma 2.3 and Assumption 2.1, an upper bound of the first term is

$$\begin{aligned}&\frac{\lambda_k}{p}\sum_{i=1}^{n}\sum_{j=1}^{n} K_{ij}(\sigma_{t,\ell})\Big|x_j(t+\sigma_{t,\ell}) - \sum_{v=1}^{n}\tilde{a}_{jv}x^v_{t,\ell}\Big|^p - \frac{\lambda_k}{p}\sum_{i=1}^{n}\sum_{j=1}^{n} K_{ij}(0)\Big|x_j - \sum_{v=1}^{n}\tilde{a}_{jv}x_v\Big|^p \\ &\qquad\leq \frac{\lambda_k}{p}\sum_{i=1}^{n}\sum_{j=1}^{n} K_{ij}(\sigma_{t,\ell})\Big|x_j(t+\sigma_{t,\ell}) - \sum_{v=1}^{n}\tilde{a}_{jv}x^v_{t,\ell}\Big|^p \\ &\qquad\leq \frac{\lambda_k}{p}2^{p-1}\big(n^2\bar{K}|x|^p + n^2\bar{K}L_\ell|x_{t,\ell}|^p\big) \\ &\qquad\leq \frac{\lambda_k}{p}2^{p-1}n^2\bar{K}\big(1+L_\ell\big)|x_t|^p_c. \qquad (3.8)\end{aligned}$$

Applying Lemma 2.4 and Assumption 2.1, the second term has an upper bound as follows:

$$\begin{aligned}
&\lambda_k|x|^{p-2}x^T\Big[-\tilde{B}_kx+\tilde{C}_kf(x)+\tilde{D}_kf\big(x_{t,\ell}\big)+\int_{t-\sigma_{t,\ell}}^{t}\tilde{E}_kK(t-s)f\big(x(s)\big)ds\Big]\\
&\qquad=\lambda_k|x|^{p-2}\Big[-\sum_{i=1}^{n}\tilde{b}_i(k)x_i^2+\sum_{i=1}^{n}\sum_{j=1}^{n}x_i\tilde{c}_{ij}(k)f_j(x_j)+\sum_{i=1}^{n}\sum_{j=1}^{n}x_i\tilde{d}_{ij}(k)f_j(x_{t,\ell}^j)\\
&\qquad\qquad+\int_{t-\sigma_{t,\ell}}^{t}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{v=1}^{n}\tilde{e}_{iv}(k)K_{vj}(t-s)x_if_j(x_j(s))ds\Big]\\
&\qquad\le-\lambda_k\tilde{b}_{min}(k)|x|^p+\lambda_kL_1|\tilde{C}_k||x|^p+\lambda_k\frac{L_1}{2}|\tilde{D}_k||x|^p+\lambda_k\frac{L_1}{2}\frac{(p-2)}{p}|\tilde{D}_k||x|^p\\
&\qquad+\lambda_k\frac{L_1}{p}|\tilde{D}_k||x_{t,\ell}|^p+\lambda_kn^2L_1|\tilde{E}_k|\Big(\frac{p-2}{p}|x|^p+\frac{2}{p}|x_{t,\ell}|^p\Big)\\
&\qquad\le\lambda_k\Big[-\tilde{b}_{min}(k)+L_1|\tilde{C}_k|+L_1|\tilde{D}_k|+n^2L_1|\tilde{E}_k|\Big]|x_t|_c^p.
\end{aligned}\tag{3.9}$$

By Lemma 2.4, Assumption 2.1 and the Cauchy-Schwarz inequality, the third term in (3.6) has the following upper bound

$$\begin{aligned}
&\frac{\lambda_k}{2}trace\Big[\big(\hbar_2(t)\big)^2\Big(\tilde{M}_kg(x)+\tilde{Q}_kg(x_{t,\ell})\Big)^T\Big((p-2)\big|x\big|^{p-4}x.x^T+\big|x\big|^{p-2}\Big)\Big(\tilde{M}_kg(x)+\tilde{Q}_kg(x_{t,\ell})\Big)\Big]\\
&\quad=\frac{p-2}{2}\lambda_k\big|x\big|^{p-4}\big(\hbar_2(t)\big)^2\Big[\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\tilde{m}_{ij}(k)g_j(x_j)+\tilde{q}_{ij}(k)g_j(x_{t,\ell}^j)\Big)x_i\Big]^2\\
&\quad+\frac{\lambda_k}{2}\big|x\big|^{p-2}\big(\hbar_2(t)\big)^2\sum_{i=1}^{n}\Big[\sum_{j=1}^{n}\Big(\tilde{m}_{ij}(k)g_j(x_j)+\tilde{q}_{ij}(k)g_j(x_{t,\ell}^j)\Big)\Big]^2\\
&\quad\le\big(\hbar_2(t)\big)^2.\frac{n(p-2)}{4}\lambda_k\big|x\big|^{p-4}\Big[\sum_{i=1}^{n}x_i^2\Big(\sum_{j=1}^{n}\tilde{m}_{ij}(k)g_j(x_j)\Big)^2+\sum_{i=1}^{n}x_i^2\Big(\sum_{j=1}^{n}\tilde{q}_{ij}(k)g_j(x_{t,\ell}^j)\Big)^2\Big]\\
&\quad+\big(\hbar_2(t)\big)^2.\frac{\lambda_k}{4}\big|x\big|^{p-2}\Big[\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}\tilde{m}_{ij}(k)g_j(x_j)\Big)^2+\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}\tilde{q}_{ij}(k)g_j(x_{t,\ell}^j)\Big)^2\Big]\\
&\quad\le\big(\hbar_2(t)\big)^2\lambda_k\Big[\frac{n(p-2)}{4}L_1\big|\tilde{M}_k\big|^2|x|^p+\frac{n(p-2)^2}{4p}L_1\big|\tilde{Q}_k\big|^2|x|^p+\frac{n(p-2)}{2p}L_1\big|\tilde{Q}_k\big|^2|x_{t,\ell}|^p\\
&\qquad\qquad+\frac{L_1}{4}\big|\tilde{M}_k\big|^2|x|^p+\frac{p-2}{4p}L_1\big|\tilde{Q}_k\big|^2|x|^p+\frac{L_1}{2p}\big|\tilde{Q}_k\big|^2|x_{t,\ell}|^p\Big]\\
&\quad\le\big(\hbar_2(t)\big)^2.\frac{\lambda_k}{4}L_1\Big(n(p-2)+1\Big)\Big(\big|\tilde{M}_k\big|^2+\big|\tilde{Q}_k\big|^2\Big)\big|x_t\big|_c^p.
\end{aligned}\tag{3.10}$$

Similarly the fourth term in (3.6) has the following upper bound

$$\begin{aligned}
&\frac{\lambda_k}{p}\int_Z\Big[\big|x+\hbar_3(t)h(x,x_{t,\ell},z,k)\big|^p-|x|^p\Big]\pi(dz)\\
&\quad\le\frac{\lambda_k}{p}\Big[(2^{p-1}-1)\pi(Z)|x|^p+2^{p-1}\big(\hbar_3(t)\big)^p\int_Z\big|h\big(x,x_{t,\ell},z,k\big)\big|^p\pi(dz)\Big]\\
&\quad\le\frac{\lambda_k}{p}\Big[(2^{p-1}-1)\pi(Z)|x|^p+2^{p-1}L_2\big(\hbar_3(t)\big)^p\Big(|x|^p+|x_{t,\ell}|^p\Big)\Big]\\
&\quad\le\frac{\lambda_k}{p}\Big[(2^{p-1}-1)\pi(Z)+2^pL_2\big(\hbar_3(t)\big)^p\Big]|x_t|_c^p.
\end{aligned}\tag{3.11}$$

For the fifth term, Lemma 2.3 and Assumption 2.1 allows us to write

$$\begin{aligned}
&\frac{\lambda_k}{p}\sum_{\varsigma=1}^{4}\theta_{\ell\varsigma}\int_t^{t+\sigma_{t,\varsigma}}\sum_{i=1}^{n}\sum_{j=1}^{n}K_{ij}(s-t)\Big|x_j-\sum_{v=1}^{n}\tilde{a}_{jv}x_v(2t-s)\Big|^p ds\\
&\qquad+\sum_{\iota=1}^{S}\gamma_{k\iota}\frac{\lambda_\iota}{p}\Big[|x|^p+\int_t^{t+\sigma_{t,\ell}}\sum_{i=1}^{n}\sum_{j=1}^{n}K_{ij}(s-t)\Big|x_j-\sum_{v=1}^{n}\tilde{a}_{jv}x_v(2t-s)\Big|^p ds\\
&\qquad\le\frac{\lambda_k}{p}2^{p-1}\Big(2n^2\big|\theta_{\ell\ell}\big||x|^p+n^2\sum_{\varsigma=1}^{4}\big|\theta_{\ell\varsigma}\big|L_\varsigma|x_t|_c^p\Big)+\sum_{\iota=1}^{S}\gamma_{k\iota}\frac{\lambda_\iota}{p}|x|^p\\
&\qquad+\sum_{\iota=1}^{S}\big|\gamma_{k\iota}\big|\frac{\lambda_\iota}{p}2^{p-1}\Big(n^2|x|^p+n^2L_\ell|x_t|_c^p\Big)\\
&\qquad\le\Big[\frac{\lambda_k}{p}2^{p-1}n^2\Big(2\big|\theta_{\ell\ell}\big|+\sum_{\varsigma=1}^{4}\big|\theta_{\ell\varsigma}\big|L_\varsigma\Big)+\sum_{\iota=1}^{S}\Big(\gamma_{k\iota}+2^{p-1}n^2\big(1+L_\ell\big)\big|\gamma_{k\iota}\big|\Big)\frac{\lambda_\iota}{p}\Big]|x_t|_c^p. \qquad (3.12)
\end{aligned}$$

Substituting (3.8) – (3.12) into (3.6) and from Assumption 2.2, we have

$$\begin{aligned}
\mathcal{L}V\le&\frac{1}{p}\Big\{\lambda_k\Big(-p\tilde{b}_{min}(k)+(2^{p-1}-1)\pi(Z)+2^pL_2\big(\hbar_3(t)\big)^p+n^2.2^{p-1}\Big[\bar{K}+\bar{K}L_\ell+2\big|\theta_{\ell\ell}\big|\\
&+\sum_{\varsigma=1}^{4}\big|\theta_{\ell\varsigma}\big|L_\varsigma\Big]+pL_1\Big[|\tilde{C}_k|+|\tilde{D}_k|+n^2|\tilde{E}_k|+\frac{1}{4}\big(\hbar_2(t)\big)^2\big(n(p-2)+1\big)\big(\big|\tilde{M}_k\big|^2+\big|\tilde{Q}_k\big|^2\big)\Big]\Big)\\
&+\sum_{\iota=1}^{S}\Big(\gamma_{k\iota}+2^{p-1}n^2\big(1+L_\ell\big)\big|\gamma_{k\iota}\big|\Big)\lambda_\iota\Big\}|x_t|_c^p\\
&=\frac{1}{p}\Big\{\chi_k^\ell\lambda_k+\sum_{\iota=1}^{S}\zeta_{k\iota}^\ell\lambda_\iota\Big\}|x_t|_c^p \qquad (3.13)\\
&=-\frac{1}{p}\Lambda_k^\ell|x_t|_c^p. \qquad (3.14)
\end{aligned}$$

By Dynkin formula, for any $t\ge 0$, it follows that

$$\begin{aligned}
\mathbb{E}V(t\wedge\tau_x^R,\tilde{x}(t\wedge\tau_x^R),\ell,k)&=\mathbb{E}V(0,\tilde{x}(0),\alpha(0),r(0))+\mathbb{E}\int_0^{t\wedge\tau_x^R}\mathcal{L}V(s,x,x_s,\alpha(s),r(s))ds\\
&\le\mathbb{E}V(0,\tilde{x}(0),\alpha(0),r(0))\\
&\le\frac{\lambda_k}{p}2^{p-1}n^2\bar{K}(1+L_\ell)|\xi|_c^p. \qquad (3.15)
\end{aligned}$$

From the definition of $V(t,\tilde{x}(t),\alpha(t),r(t))$, we have

$$\begin{aligned}
V(t\wedge\tau_x^R,\tilde{x}(t\wedge\tau_x^R),\ell,k)&\ge\frac{\lambda_k}{p}|x(t\wedge\tau_x^R)|^p\\
\mathbb{E}V(t\wedge\tau_x^R,\tilde{x}(t\wedge\tau_x^R),\ell,k)&\ge\frac{\lambda_k}{p}\mathbb{E}\big[\mathbb{1}_{\{\tau_x^R<t\}}|x(\tau_x^R)|^p\big]\\
&\ge\frac{\lambda_k}{p}R^pP\{\tau_x^R<t\}.
\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{\lambda_k R^p}{p} P\{\tau_x^R < t\} &\leq \frac{\lambda_k}{p} 2^{p-1} n^2 \bar{K}(1+L_\ell)|\xi|_c^p \\ \Leftrightarrow P\{\tau_x^R < t\} &\leq \frac{2^{p-1} n^2 \bar{K}(1+L_\ell)|\xi|_c^p}{R^p} \\ &< \epsilon.\end{aligned}$$

Letting $t \longrightarrow \infty$, we get $P\{\tau_x^R < \infty\} < \epsilon$, which is equivalent to

$$P\{|x(t,\xi)| \leq R, t \geq 0\} \geq 1 - \epsilon. \tag{3.16}$$

This completes the proof. □

3.3. **Stability with a general decay.**

Definition 3.3. The function $\psi : \mathbb{R} \to (0, \infty)$ is said to be ψ-type function if the function satisfies the following three conditions

(*i*) It is continuous and nondecreasing on $\mathbb{R}$ and differentiable on $\mathbb{R}^+$.
(*ii*) $\psi(0) = 1, \psi(\infty) = \infty$ and $\tilde{\beta} = \sup \frac{\psi'(t)}{\psi(t)} < \infty$.
(*iii*) For any $s, t \geq 0$, $\psi(t) \leq \psi(s)\psi(t-s)$.

Definition 3.4. Let the function $\psi \in C(\mathbb{R}^+, \mathbb{R}^+)$ be the ψ-type function. Then, for any initial data φ, the system is said to be p^{th} $(p \geq 2)$ moment stable with a decay $\psi(t)$ of order μ if

$$\limsup_{t\to\infty} \frac{\log \mathbb{E}|x(t)|^p}{\log \psi(t)} \leq -\mu.$$

Moreover, for any initial data φ, the system is said to be almost surely stable with decay $\psi(t)$ of order $\frac{\mu}{p}$ if

$$\limsup_{t\to\infty} \frac{\log |x(t)|}{\log \psi(t)} \leq \frac{-\mu}{p} \quad \text{almost surely.}$$

Remark 3.5. If we replace $\psi(t)$ by e^t or $1 + t$, then it leads to the usual exponential stability or polynomial stability, respectively. Our results will therefore be more general, because we have a large choice for ψ-type functions.

Assumption 3.1. *Assume that for any $\ell \in S_1$, there exists $\mu > 0$ and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_S) \in \mathbb{R}_+^S$ such that*

$$\Lambda^\ell = \mathcal{A}^\ell \lambda \geq 0,$$

where
$\mathcal{A}^\ell := -diag\big(\Upsilon_1^\ell, \Upsilon_2^\ell, \cdots, \Upsilon_S^\ell\big) - \zeta^\ell$ *and* $\Upsilon_k^\ell := \chi_k^\ell + \mu\tilde{\beta}\big[1 + 2^{p-1}n^2\big(1 + L_\ell\big)\big]$.

Theorem 3.6. *Let Assumption 2.1 and Assumption 3.1 be satisfied. Then for any initial data $\xi \in C$, the solution of system (2.1) is p^{th} $(p \geq 2)$ moment stable with the decay $\psi(t)$ of order μ.*

Proof. Consider the above Lyapunov function V and set the function

$$U(t, x(t), \alpha(t), r(t)) = \psi^\mu(t) V(t, x(t), \alpha(t), r(t)). \tag{3.17}$$

Therefore, from the definition of the ψ-function and (3.13), we can get for any $t \geq 0$ and $(\ell, k) \in S_1 \times S_2$ that

$$\begin{aligned}
\mathcal{L}U(t,x,x_{t,\ell},\ell,k) =& \psi^{\mu}(t)\Big(\mu\frac{\psi^{'}(t)}{\psi(t)}V(t,\tilde{x},\ell,k) + \mathcal{L}V(t,x,x_{t,\ell},\ell,k)\Big) \\
&\leq \psi^{\mu}(t)\Big(\mu\tilde{\beta}V(t,\tilde{x},\ell,k) + \frac{1}{p}\Big\{\chi_k^{\ell}\lambda_k + \sum_{q=1}^{S}\zeta_{kq}^{\ell}\lambda_q\Big\}|x_t|_c^p\Big) \\
&\leq \psi^{\mu}(t)\Big(\frac{1}{p}\Big\{\mu\tilde{\beta}\big[1+2^{p-1}n^2(1+L_{\ell})\big]\lambda_k\Big\}|x_t|_c^p + \frac{1}{p}\Big\{\chi_k^{\ell}\lambda_k + \sum_{q=1}^{S}\zeta_{kq}^{\ell}\lambda_q\Big\}|x_t|_c^p\Big) \\
&= \frac{1}{p}\psi^{\mu}(t)\Big\{\Big(\chi_k^{\ell} + \mu\tilde{\beta}\big[1+2^{p-1}n^2(1+L_{\ell})\big]\Big)\lambda_k + \sum_{q=1}^{S}\zeta_{kq}^{\ell}\lambda_q\Big\}|x_t|_c^p \\
&= \frac{-1}{p}\psi^{\mu}(t)\Lambda_k^{\ell}|x_t|_c^p. \qquad (3.18)
\end{aligned}$$

Then, by using (3.18), the generalised Itô formula of the function $U\big(t,x,\alpha(t),r(t)\big) \in C^{1,2}\big(\mathbb{R}^+ \times \mathbb{R}^n \times S_1 \times S_2, \mathbb{R}^+\big)$ allows us to write

$$\begin{aligned}
\psi^{\mu}(t)\frac{\lambda_k}{p}\mathbb{E}\big|x(t)\big|^p \leq& \mathbb{E}U(t,\tilde{x}(t),\ell,k) \\
&= \mathbb{E}U\big(0,\tilde{x}(0),\alpha(0),r(0)\big) + \mathbb{E}\int_0^t \mathcal{L}U\big(s,x(s),x\big(s-\sigma_{s,\alpha(s)}\big),\alpha(s),r(s)\big)ds \\
&\leq \mathbb{E}V\big(0,\tilde{x}(0),\alpha(0),r(0)\big) \\
&\leq \frac{\lambda_{r(0)}}{p}\Big(1+2^{p-1}n^2\big(1+L_{\alpha(0)}\big)\Big)\mathbb{E}\big|\xi\big|_c^p,
\end{aligned}$$

which gives

$$\mathbb{E}\big|x(t)\big|^p \leq \frac{\lambda_{r(0)}}{\lambda_{max}}\Big(1+2^{p-1}n^2\big(1+L_{\alpha(0)}\big)\Big)\mathbb{E}\big|\xi\big|_c^p\psi^{-\mu}(t),$$

which in its turn implies that the global solution is p^{th}-moment stable with a decay $\psi(t)$ of order μ. ☐

Theorem 3.7. *Let Assumption 2.1 and Assumption 3.1 be verified. Then for any initial data $\xi \in C$, the solution of System (2.1) is almost surely stable with a decay $\psi(t)$ of order $\frac{\mu}{p}$.*

Proof. Let us consider the Lyapunov function U used in the proof of Theorem 3.6. By using the Itô formula, we get from (3.18) that

$$\begin{aligned}
U(t,\tilde{x}(t),\alpha(t),r(t)) &= U\big(0,\tilde{x}(0),\alpha(0),r(0)\big) + \int_0^t \mathcal{L}U\big(s,x(s),x\big(s-\sigma_{s,\alpha(s)}\big),\alpha(s),r(s)\big)ds + M(t) \\
&\leq V\big(0,\tilde{x}(0),\alpha(0),r(0)\big) + M(t) \\
&\leq \frac{\lambda_{r(0)}}{p}\Big(1+2^{p-1}n^2\big(1+L_{\alpha(0)}\big)\Big)\big|\xi\big|_c^p + M(t), \qquad (3.19)
\end{aligned}$$

where $M(t) := \int_0^t \psi^{\mu}(s)dG_s$ is a local martingale with $M(0) = 0$.
Applying the nonnegative semi-martingale convergence theorem [14], it gives from (3.19) that
$\limsup_{t\to\infty} U(t,\tilde{x}(t),\alpha(t),r(t)) < \infty$ almost surely.
Therefore, there exists a finite positive random variable $\hbar$ such that for any $t \geq 0$

$$U(t,\tilde{x}(t),\alpha(t),r(t)) \leq \hbar \quad \text{almost surely.} \qquad (3.20)$$

We deduce from (3.17) and (3.20) that for any $t \geq 0$

$$\big|x(t)\big|^p \le \frac{p\hbar}{\lambda_{max}}\psi^{-\mu}(t) \qquad \text{almost surely,}$$

which implies that

$$\limsup_{t\to\infty}\frac{\log|x(t)|}{\log\psi(t)} \le \frac{-\mu}{p} \qquad \text{almost surely.}$$

□

4. Numerical simulation

Consider the following system

$$\begin{aligned} d[x(t)-\tilde{A}_{\alpha(t)}x(t-\sigma_{t,\alpha(t)})] &= \big[-\tilde{B}_{r(t)}x(t)+\tilde{C}_{r(t)}f(x(t))+\tilde{D}_{r(t)}f(x(t-\sigma_{t,\alpha(t)})) \\ &\quad + \int_{t-\sigma_{t,\alpha(t)}}^{t}\tilde{E}_{r(t)}K(t-u)f(x(u))du\big]dt \\ &\quad + \hbar_2(t)\big(\tilde{M}_{r(t)}g(x(t))+\tilde{Q}_{r(t)}g(x(t-\sigma_{t,\alpha(t)}))\big)dW(t) \\ &\quad + \hbar_3(t)\int_{-1}^{7}\big(h(x(t),x_t(-\sigma_{t,\alpha(t)}),z,r(t))\big)N(dt,dz), \end{aligned} \tag{4.1}$$

with $\alpha(t)$ and $r(t)$ take values in $\mathcal{S}_1=\{1,2,3,4\}$ and $\mathcal{S}_2=\{1,2\}$ respectively. We choose a system in dimension two. For $\alpha(t)\in\mathcal{S}_1$, we take

$$\tilde{A}_1=\begin{bmatrix}0.2 & 0\\ 0 & 0.1\end{bmatrix},\quad \tilde{A}_2=\begin{bmatrix}0.2 & 0\\ 0 & 0.3\end{bmatrix},\quad \tilde{A}_3=\begin{bmatrix}0.3 & 0\\ 0 & 0.1\end{bmatrix},\quad \tilde{A}_4=\begin{bmatrix}0.3 & 0\\ 0 & 0.2\end{bmatrix},$$

and $\sigma_{t,\alpha(t)}=\frac{3}{\alpha(t)+t}$.
For $r(t)\in\mathcal{S}_2$, we take the following specifications:

$$\tilde{B}_1=\begin{bmatrix}30 & 0\\ 0 & 34\end{bmatrix},\quad \tilde{C}_1=\begin{bmatrix}0.5 & 0.6\\ 0.7 & 0.6\end{bmatrix},\quad \tilde{D}_1=\begin{bmatrix}0.3 & 0.4\\ 0.2 & 0.5\end{bmatrix},\quad \tilde{E}_1=\begin{bmatrix}0.4 & 0.2\\ 0.5 & 0.3\end{bmatrix},$$

$$\tilde{M}_1=\begin{bmatrix}0.4 & 0.2\\ 0.3 & 0.4\end{bmatrix},\quad \tilde{Q}_1=\begin{bmatrix}0.7 & 0.2\\ 0.5 & 0.2\end{bmatrix},\quad h(x,x_t,z,1)=2x+x_t+z,$$

$$\tilde{B}_2=\begin{bmatrix}28 & 0\\ 0 & 39\end{bmatrix},\quad \tilde{C}_2=\begin{bmatrix}0.3 & 0.4\\ 0.5 & 0.6\end{bmatrix},\quad \tilde{D}_2=\begin{bmatrix}0.2 & 1\\ 0.3 & 0.2\end{bmatrix},\quad \tilde{E}_2=\begin{bmatrix}0.2 & 0.2\\ 0.5 & 0.5\end{bmatrix},$$

$$\tilde{M}_2=\begin{bmatrix}0.3 & 0.2\\ 0 & 0.2\end{bmatrix},\quad \tilde{Q}_2=\begin{bmatrix}0.2 & 0.6\\ 0.3 & 0.8\end{bmatrix},\quad h(x,x_t,z,2)=x+\frac{1}{2}x_t-z.$$

The infinitesimal generators $\Theta=(\theta_{\varsigma\ell})_{4\times4}$ and $\Gamma=(\gamma_{\iota k})_{2\times2}$ of the Markov processes $\alpha(t)$ and $r(t)$, respectively, are given by:

$$\Theta=\begin{bmatrix}-1 & 0.2 & 0.4 & 0.4\\ 0.5 & -1.5 & 0.5 & 0.5\\ 0.7 & 0.5 & -2 & 0.8\\ 0.2 & 0.3 & 0.2 & -1\end{bmatrix},\qquad \Gamma=\begin{bmatrix}-0.7 & 0.7\\ 1 & -1\end{bmatrix}.$$

To have Assumption 2.1, let $f(x)=\sin(x)$, $g(x)=x$, $L_1=L_2=1$. Furthermore, we let $\lambda_1=0.8,\lambda_2=0.1$ and $K(t-s)=I_2e^{s-t}$ for any $s\le t$. Then, for any $\ell\in\mathcal{S}_1$, Λ^ℓ are the M-matrices as follows:

$$\Lambda^1=\begin{bmatrix}36.32 & -11.90\\ -17.00 & 41.82\end{bmatrix},\Lambda^2=\begin{bmatrix}15.82 & -17.50\\ -25.00 & 18.92\end{bmatrix},\Lambda^3=\begin{bmatrix}03.98 & -23.10\\ -33.00 & 04.68\end{bmatrix},\Lambda^4=\begin{bmatrix}15.50 & -28.70\\ -41.00 & 13.80\end{bmatrix}.$$

As a result, Theorem 3.2 guarantees that system (4.1) is stochastically stable. A sample of system (4.1) is generated, and the corresponding data is depicted in Fig. 8. It can be seen that the trajectory tends to zero as long as the time increases, thus corroborating the result of Theorem 3.2.

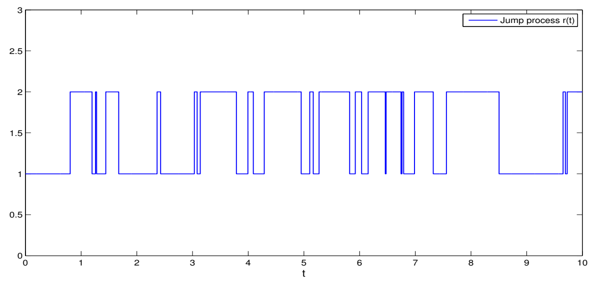

FIGURE 5. Trajectory of jump process $r(t)$ with $r(0) = 1$.

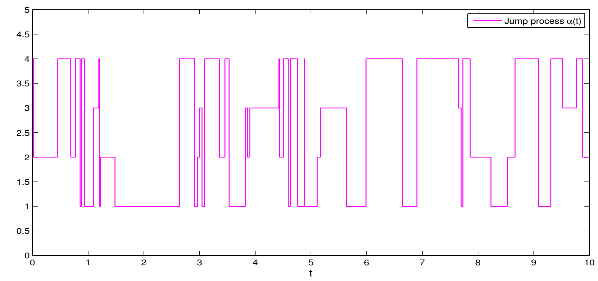

FIGURE 6. Trajectory of jump process $\alpha(t)$ with $\alpha(0) = 4$.

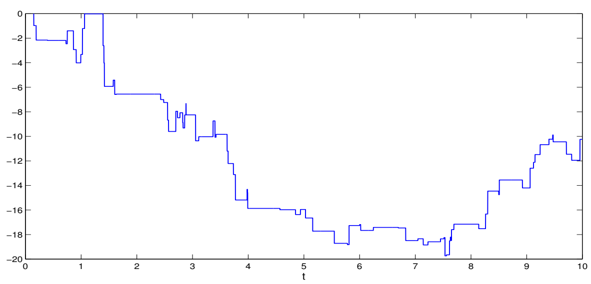

FIGURE 7. Poisson point process with normally distributed jump

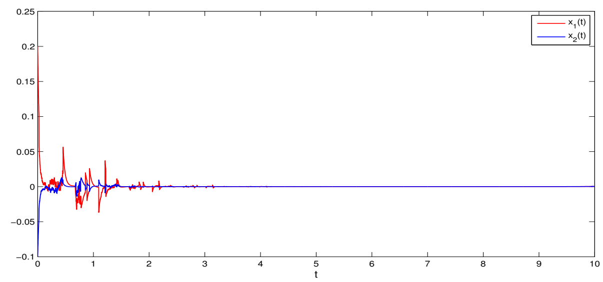

FIGURE 8. Stochastic approximate solution of System (4.1) with $(x_1(0), x_2(0)) = (0.2, -0.1)$

5. CONCLUSION

In this paper, we have shown the existence and uniqueness of solutions for a system that combines many classes of delayed neutral-type neural networks in one system in a random manner. The stochastic and general decay stabilities for the system are studied taking into account a measurable bounded time varying delay function. We have based our results on the M-matrix theory, a Lyapunov-Krasovskii functional, and some stochastic analysis techniques.
Our model takes four states; delayed neutral-type system without noise, delayed neutral-type system with Gaussian noise, delayed neutral-type system with Lévy noise, and delayed neutral-type system with the two noises. In addition, the time varying delay is also supposed to be dependent on the Markovian process. Those considerations represent the novelty of this paper.

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