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Parameter-uniform robust scheme for singularly perturbed parabolic convection-diffusion problems with large time-lag

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Abstract

This paper deals with a parameter uniform numerical method for singularly perturbed time delayed parabolic convection-diffusion problems. The method consists of a backward-Euler to discretize in temporal dimension and exponentially fitted B-spline collocation scheme for the spatial dimension on a uniform mesh. Parameter-uniform error estimates are obtained, and the method is proved uniformly convergent. The developed scheme is tested on various problems and observed to support the theoretical results. Finally, the numerical solutions are compared with the existing literature methods, and the present method is more accurate.

Keywords. Singularly perturbed parabolic convection-diffusion problem, Time delay, B-spline collocation, Exponentially fitted method, Parameter uniform convergence.

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1. Introduction

Singularly perturbed parabolic differential equations arise in the modeling of phenomena like convective heat transport problems [27], and physiological processes or diseases [17]. There are several approaches exist to deal with time dependent differential equations in which the highest-order derivative does not multiply by a small parameter $0 < \varepsilon \ll 1$ [24]. However, the numerical treatment of singularly perturbed parabolic partial differential equations with the negative shift started appearing in the late 2008s by Ramesh *et al.*[25]. Since then studied by Kumar and Kumar [15], Swaminathan *et al.* [29], Bansal and Sharma [3], Rao and Chakravarthy [26], and Daba and Duressa [6]. Singularly perturbed time-delayed parabolic partial differential equations exhibit much more complicated dynamics, as small delays can have large effects. Singularly perturbed time-delayed parabolic partial differential equations model arises in, for example, chemostat models [34], the respiratory system [31], neural networks [4], and control theory [11]. Many researchers tried to resolve the steadiness of equilibrium states with varied values of perturbations. An automatically controlled furnace in the form of a singularly perturbed time-delay parabolic partial differential equation was studied by Wu in [33]. The author stated that a simplified mathematical description of the overall control system might be given by:

$$\frac{\partial u(x,t)}{\partial t} = \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + v\left[g\left(u(x,t-\tau)\right)\right]\left(\frac{\partial u(x,t)}{\partial x}\right) + c\left[f\left(u(x,t-\tau)\right) - u(x,t)\right],$$

defined on a one dimensional spatial domain $0 < x < 1$, where v is the instantaneous material strip velocity depending on a prescribed spatial average of the time-delayed temperature distribution $u(x,t-\tau)$, and f represents a distributed temperature source function depending on $u(x,t-\tau)$. In parabolic partial differential equations, the existence of singular perturbation and time delay parameters causes uncontrolled oscillations in the calculated solution. Therefore, usually in thin transition layers, the solution alters rapidly or jumps steeply when away from the layers; solutions

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perform frequently and vary slowly. Since the numerical analysis of such problems presents many difficulties, the classical numerical methods fail to approximate their solution. A numerical analyst developed two classical approaches that capture the layers, which are the fitted mesh method [28] or graded meshes [9] and the fitted operator method [18]. In Ansari *et al.* [1], a singularly perturbed linear parabolic reaction-diffusion problem with time-delay is formulated. Das and Natesan [7], Gowrisankar and Natesan [12], Das and Natesan [8], Gelu and Duressa [10], Kumar and Kumari [16], Podila and Kumar [22], Babu and Bansal [2], Negero and Duressa [19–21], and Woldaregay *et al.* [32] extended the work to the singularly perturbed time-delayed of parabolic convection-diffusion type. The fact that most authors limit their research to adaptive grid methods motivates the current authors' research to solve the problem of singularly perturbed time-delay parabolic convection-diffusion. This article uses Euler's implicit finite difference scheme in the time direction and the B-spline collocation method of the exponential fitted in the space direction to test the robust numerical solution.

2. Problem Formulation

Let $\Omega_x = (0,1), D = \Omega_x \times (0,T], \bar{\Omega}_x = [0,1], \bar{D} = \bar{\Omega}_x \times [0,T], \Gamma = \Gamma_l \cup \Gamma_b \cup \Gamma_r$, where Γ_l and Γ_r are the left and the right side of the rectangular domain D corresponding to $x = 0$ and $x = 1$, respectively and $\Gamma_b = [0,1] \times [-\tau, 0]$. Consider the following singularly perturbed parabolic convection-diffusion problem with time delay:

$$\left(\frac{\partial}{\partial t} + L_\varepsilon\right) u(x,t) = -c(x,t)u(x,t-\tau) + f(x,t), (x,t) \in D, \tag{2.1}$$

with

$$\begin{cases} u(0,t) = \phi_l(t),\ \Gamma_l = \{(0,t) : 0 \le t \le T\}, \\ u(1,t) = \phi_r(t),\ \Gamma_r = \{(1,t) : 0 \le t \le T\}, \\ u(x,t) = \phi_b(x,t),\ (x,t) \in \Gamma_b, \end{cases} \tag{2.2}$$

where the differential operator L_ε is given by

$$L_\varepsilon u(x,t) = -\varepsilon u_{xx}(x,t) + a(x)u_x(x,t) + b(x,t)u(x,t), \tag{2.3}$$

$0 < \varepsilon \ll 1$ is a singular perturbation parameter and $\tau > 0$ represents the delay parameter and the functions $a(x), b(x,t), f(x,t)$ on $\bar{D}$ and $\phi_b(x,t), \phi_l(t), \phi_r(t)$ on Γ are sufficiently smooth, bounded functions that satisfy

$$a(x) \ge \alpha > 0, b(x,t) \ge \beta > 0, (x,t) \in \bar{D}.$$

The solution of the model problems (2.1) and (2.2) posses boundary layer of width $O(\varepsilon)$ along $x = 1$. The existence and uniqueness of the solution for the model problems (2.1) and (2.2) can be guaranteed by the sufficient smoothness of $\phi_l(t), \phi_b(x,t)$ and $\phi_r(t)$ along with appropriate compatibility conditions at the corner points $(0,0), (1,0), (0,-\tau)$ and $(1,-\tau)$, and delay terms as stated below:

$$\begin{cases} \phi_b(0,0) = \phi_l(0), \\ \phi_b(1,0) = \phi_r(0), \end{cases} \tag{2.4}$$

and

$$\begin{cases} \dfrac{\partial \phi_l(0)}{\partial t} - \varepsilon \dfrac{\partial^2 \phi_b(0,0)}{\partial x^2} + a(0)\dfrac{\partial \phi_b(0,0)}{\partial x} + b(0,0)\phi_b(0,0) = -c(0,0)\phi_b(0,-\tau) + f(0,0), \\ \dfrac{\partial \phi_r(0)}{\partial t} - \varepsilon \dfrac{\partial^2 \phi_b(1,0)}{\partial x^2} + a(1)\dfrac{\partial \phi_b(1,0)}{\partial x} + b(1,0)\phi_b(1,0) = -c(1,0)\phi_b(0,-\tau) + f(1,0). \end{cases} \tag{2.5}$$

The compatibility conditions (2.4) and (2.5) are guarantee for the existence of a constant C independent of ε for all $(x,t) \in \bar{\Omega}$,

$$|u(x,t) - u(x,0)| = |u(x,t) - u_0(x)| \le Ct$$

and

$$|u(x,t) - u(0,t)| = |u(x,t) - \phi_l(0,t)(x)| \le C(1-x).$$

By setting $\varepsilon = 0$, the reduced problem corresponding to Eqs. (2.1) and (2.2) is

$$\begin{cases} \dfrac{\partial u_0}{\partial t} + a(x)u_0(x,t) = -b(x,t)u_0(x,t-\tau) + f(x,t), (x,t) \in D, \\ u_0(x,t) = \phi_b(x,t), (x,t) \in \Gamma_b. \end{cases} \tag{2.6}$$

2.1. **Properties of continuous solution.**

Lemma 2.1. (Continuous maximum principle) *Assume that the function $\eta(x,t) \in C^2(D) \cap C^0(\bar{D})$. Suppose that $\left(\frac{\partial}{\partial t} + L_\varepsilon\right)\eta(x,t) \geq 0$, $\forall(x,t) \in D$ and $\eta(x,t) \geq 0$, $\forall(x,t) \in \Gamma$. Then $\eta(x,t) \geq 0$, $\forall(x,t) \in \bar{D}$.*

Proof. Let (x^*,t^*) be such that $\eta(x^*,t^*) = \min_{(x,t)\in\bar{D}} \eta(x,t)$ and suppose $\eta(x,t) < 0$. It is clear that $(x^*,t^*) \notin \Gamma$ as $\eta(x,t) \geq 0$ on Γ. Then, we have $\eta_x(x^*,t^*) = \eta_t(x^*,t^*) = 0$ and $\eta_{xx}(x^*,t^*) \geq 0$ and thus $\left(\frac{\partial}{\partial t} + L_\varepsilon\right)\eta_t(x^*,t^*) < 0$ which contradicts the given hypothesis and hence $\eta(x,t) \geq 0$, $\forall(x,t) \in \bar{D}$. □

Lemma 2.2. *Assume that $u(x,t)$ is the solution of the continuous problems (2.1) and (2.2). Then we have the bound*

$$|u(x,t)| \leq C, (x,t) \in \bar{D}.$$

Proof. The result follows from Lemma 2.1 and compatibility conditions. See the detailed proof in Das and Natesan [7]. □

Lemma 2.3. *[5] Under the assumption of Lemma 2.1 and Lemma 2.2, the bound on the derivative of u with respect to t is given by*

$$|u_t(x,t)| \leq C, (x,t) \in \bar{D}.$$

In order to study the convergence at the spatial discretization stage, we need to know the asymptotic behavior of the solution and its spatial derivatives.

Lemma 2.4. *The bounds on the derivatives of the solution $u(x,t)$ satisfies*

$$\left|\frac{\partial^{i+j}u}{\partial x^i \partial t^j}\right| \leq C\left(1 + \varepsilon^{-i} exp\left(-\alpha\left(1-x\right)/\varepsilon\right)\right), \forall(x,t) \in \bar{D},$$

where i and j are non-negative integers such that, $0 \leq i+j \leq 4$.

Proof. The required bound can be obtained by using the argument as given in [14] and bounds given in Lemmas 2.2 and 2.3. For detail proof see Kumar and Kumari [16]. □

3. Formulation of the numerical scheme

3.1. **The temporal semidiscretization.** The time interval $[0,T]$ is discretized using a uniform mesh with time step Δt as

$$\bar{\Omega}_t^M = \{t_n = n\Delta t, n = 0, 1, ..., T/\Delta t\},$$

and the interval $[-\tau, 0]$ is divided into s equal parts as

$$\bar{\Omega}_\Gamma^s = \{t_n = n\Delta t, n = 0, 1, ..., s, t_s = \tau, \Delta t = \tau/s\},$$

where M is the number of mesh points in time direction in $[0,T]$ such that $M = T/\Delta t$. Now, $u(x,t-\tau)$ is a known function on $[0,1]\times[0,\tau]$ and Eqs. (2.1)-(2.2) becomes a classical singularly perturbed partial differential equations, and can be treated using the known existing method. Applying implicit Euler method in the time variable discretization on $\Omega \times \bar{\Omega}^M$ produces the following semi-discretize problem,

$$\begin{cases} L_\varepsilon U^n(x) \equiv -\varepsilon\left(U_{xx}\right)^n(x) + a(x)\left(U_x\right)^n(x) + \nu^n(x)U^n(x) = F^n(x), \\ U^n(0) = \phi_l(t_n), U^n(1) = \phi_r(t_n), \\ U^n(x) = \phi_b(x,t_n), x \in \Omega_x, -(s+1) \leq n \leq -1, \end{cases} \tag{3.1}$$

where $\nu^n(x) = b^n(x) + \frac{1}{\Delta t}$ and $F^n(x) = -c^n(x)U^{n-s}(x) + f^n(x) + \frac{1}{\Delta t}U^{n-1}(x)$.
$F^n(x)$ can be defined with delayed term $U^{n-s}(x)$ which is evaluated as

$$F^n(x) = \begin{cases} -c^n(x)\phi_b(x,t_n) + f^n(x) + \frac{1}{\Delta t}U^{n-1}(x), \text{ if } t_n < s, \\ -c^n(x)U^{n-s}(x) + f^n(x) + \frac{1}{\Delta t}U^{n-1}(x), \text{ if } t_n \geq s. \end{cases}$$

Lemma 3.1. (Semi-discrete maximum principle) *Assume that $\eta^{n+1}(x)$ be a sufficiently smooth function on the domain $(\bar{D})$ such that $\eta^{n+1}(0) \geq 0$ and $\eta^{n+1}(1) \geq 0$. Then $(\delta_t^- + L_\varepsilon^M)\eta^{n+1}(x) \geq 0$ $\forall x \in D$, implies that $\eta^{n+1}(x) \geq 0$ $\forall x \in \bar{D}$, where $\delta_t^- = \frac{u^{n+1}-u^n}{\Delta t}$.*

Proof. Assume $(z^*) \in D$ such that $\eta^{n+1}(z^*) = \min_{(x)\in D}\eta^{n+1}(x)$ and suppose $\eta^{n+1}(x) < 0$. Now, it is clear that $(z^*, t_{n+1}) \notin \{(0,t_{n+1}),(1,t_{n+1})\}$ as $\eta^{n+1}(x) \geq 0$. Therefore, we have $\frac{d}{dx}\left(\eta^{n+1}(z^*)\right) = 0$ and $\frac{d^2}{dx^2}(\eta(z^*)) \geq 0$ and thus

$$\begin{aligned}(\delta_t^- + L_\varepsilon^M)\eta^{n+1}(z^*) = -\varepsilon\frac{d^2}{dx^2}\left(\eta^{n+1}(z^*)\right) + a(z^*)\frac{d}{dx}\eta^{n+1}(z^*) \\ + b^{n+1}(z^*)\eta^{n+1}(z^*) \leq b^{n+1}(z^*)\eta^{n+1}(z^*) < 0,\end{aligned}$$

this contradicts assumption and $\eta^{n+1}(z^*) \geq 0$, which implies that $\eta^{n+1}(x) \geq 0$ $\forall (x) \in \bar{D}$. □

Let $U^n(x)$ is the semi-discrete approximation to the exact solution $u(x,t_n)$ of the problem in Eqs. (2.1)-(2.2) at $t_n = n\Delta t$. The error estimates for the temporal semi-discretization (3.1) $e_{n+1} = U^n(x) - u(x,t_n)$ satisfy the following Lemma.

Lemma 3.2. (Local error estimate) *Assume that $\frac{\partial^k u(x,t)}{\partial t} \leq C, (x,t) \in \bar{D} \times (0,T]$, $0 \leq k \leq 2$. Then the local error estimate associated to the semi-discretized problem (3.1) is given by*

$$\|e_{n+1}\|_\infty \leq C(\Delta t)^2.$$

Proof. Applying Taylor's series expansion to $u(x,t_n)$ and substituting the result into the continuous problems (2.1) and (2.2) gives

$$\begin{aligned}u_t(x,t_n) + O\left((\Delta t)^2\right) &= \frac{u(x,t_{n+1}) - u(x,t_n)}{\Delta t} \\ &= \varepsilon u_{xx}(x,t_{n+1}) + a(x)u_x(x,t_{n+1}) - b(x,t_{n+1})u(x,t_{n+1}) \\ &- c(x,t_{n+1})u(x,t_{-s+n}) + f(x,t_{n+1}) + O\left((\Delta t)^2\right).\end{aligned}$$

Clearly $e_{n+1}(x)$ satisfies the semi-discrete operator

$$(\delta_t^- + L_\varepsilon^M)e_{n+1}(x) = O\left((\Delta t)^2\right).$$

Thus using maximum principle given at Lemma 3.1 we have,

$$\|e_{n+1}\|_\infty \leq C(\Delta t)^2.$$

□

Lemma 3.3. (Global error estimate) *The global error estimate E_n in the temporal direction is given by*

$$\|E_n\|_\infty \leq C(\Delta t).$$

Proof. The global error estimate is given by

$$\begin{aligned}\|E_n\|_\infty &= \left\|\sum_{k=1}^n e_k\right\|_\infty, n \leq \frac{T}{\Delta t} \\ &\leq \|e_1\|_\infty + \|e_2\|_\infty + ... + \|e_n\|_\infty.\end{aligned}$$

Using local error estimates given in Lemma (3.2),

$$\begin{aligned}
&\le C_1\left((n)\Delta t\right)(\Delta t)\\
&\le C_1 T\left(\Delta t\right), \text{since } \ n\left(\Delta t\right)\le T\\
&\le C\left(\Delta t\right), C=C_1 T,
\end{aligned}$$

where C is constant independent of ε and Δt. □

Lemma 3.4. *The solution $U^n(x)$ of semi-discretized scheme (3.1) and its derivatives satisfies*

$$\left|\frac{d^i U^n(x)}{dx^i}\right| \le C\left(1+\varepsilon^{-i}e^{-\alpha(1-x)/\varepsilon}\right), \text{for } 0\le i\le 4.$$

Proof. For the detail proof see Podila and Kumar [22]. □

3.2. **The spatial discretization.** Here we use cubic B-splines collocation and introduce the fitting factor for singularly perturbed parabolic convection-diffusion problems (2.1)-(2.2). Equation (3.1) can now be given as

$$\begin{cases}
\hat{L}_{\varepsilon,x}\hat{U}(x) \equiv -\sigma\left(x,\varepsilon\right)\hat{U}_{xx}\left(x\right)+a\left(x\right)\hat{U}_x\left(x\right)+\nu^n\left(x\right)\hat{U}\left(x\right)=\hat{F}\left(x\right),\\
\hat{U}\left(0\right)=\phi_l\left(t_n\right), \hat{U}\left(1\right)=\phi_r\left(t_n\right),\\
\hat{U}\left(x\right)=\phi_b\left(x,t_n\right), x\in\Omega_x, -(s+1)\le n\le -1,
\end{cases} \tag{3.2}$$

where $\hat{U}(x)\approx U^n\left(x\right)\approx U\left(x,t_n\right), \hat{F}\left(x\right)\approx F^n\left(x\right)\approx F\left(x,t_n\right)$.

Now, we consider the cubic B-splines at the equally spaced nodes x_m and $h=x_m-x_{m-1}=1/N$ where h is the piecewise uniform spacing are defined to form a basis over the spatial domain $[0,1]$, such that: $x_{-2}<x_{-1}<0=x_0<x_1<,...,<x_N=1<x_{N+1}<x_{N+2}$. Let $B_3\left(\bar{\Omega}_x\right)$ be the set of all cubic spline functions over the partition $\bar{\Omega}_x$ and $\Lambda_N\left(\bar{\Omega}_x\right)$ is $(N+3)$ dimensional subspace of $B_3\left(\bar{\Omega}_x\right)$. Approximating $\Phi(x)$ to the exact solution $\hat{U}(x)$ in terms of B-splines can be expressed as

$$\Phi(x)=\sum_{m=-1}^{N+1}\lambda_m B_m\left(x\right), \tag{3.3}$$

where λ_m's are unknown real coefficients, referred to as degrees of freedom, to be determined by requiring that $U(x)$ satisfies Eq. (3.2) at $N+1$ collocation points and boundary conditions. The cubic B-splines are defined by the following relation [23]:

$$B_m\left(x\right)=\frac{1}{h^3}\begin{cases}
\left(x_{m+2}-x\right)^3, x\in\left[x_{m+1},x_{m+2}\right],\\
\left(x_{m+2}-x\right)^3-4\left(x_{m+1}-x\right)^3, x\in\left[x_m,x_{m+1}\right],\\
\left(x_{m+2}-x\right)^3-4\left(x_{m+1}-x\right)^3+6\left(x_m-x\right)^3, x\in\left[x_{m-1},x_m\right],\\
\left(x_{m+2}-x\right)^3-4\left(x_{m+1}-x\right)^3+6\left(x_m-x\right)^3\\
-4\left(x_{m-1}-x\right)^3, x\in\left[x_{m-2},x_{m-1}\right],\\
0,\ \text{otherwise}.
\end{cases} \tag{3.4}$$

It is required that Eq. (3.3) satisfies the Eq. (3.2) at $x=x_m$, where x_m is an interior point. That is

$$\begin{cases}
\hat{L}_{\varepsilon,x_m}\Phi(x_m)\equiv -\sigma\left(x_m,\varepsilon\right)\Phi_{xx}\left(x_m\right)+a\left(x_m\right)\Phi_x\left(x_m\right)+\nu^n\left(x_m\right)\Phi\left(x_m\right)=\hat{F}\left(x_m\right),\\
\Phi\left(0\right)=\phi_l\left(t_n\right), \Phi\left(1\right)=\phi_r\left(t_n\right),\\
\Phi\left(x_m\right)=\phi_b\left(x_m,t_n\right), x_m\in\Omega_{x_m}, -(s+1)\le n\le -1.
\end{cases} \tag{3.5}$$

Putting the approximation (3.3) into collocation (3.5) at the mesh points $\bar{\Omega}^N$ with some manipulation yields

$$r_m^-\lambda_{m-1}+r_m^c\lambda_m+r_m^+\lambda_{m+1}=h^2\hat{F}_m \tag{3.6}$$

where

$$\begin{cases} r_m^- = -6\sigma_m - 3ha_m + h^2\left(\dfrac{1}{\Delta t} + b_m^n\right), \\ r_m^c = 12\sigma_m + 4h^2\left(\dfrac{1}{\Delta t} + b_m^n\right), \\ r_m^+ = -6\sigma_m + 3ha_m + h^2\left(\dfrac{1}{\Delta t} + b_m^n\right), \end{cases}$$

with $\sigma(x_m, \varepsilon) = \sigma_m = \frac{\varepsilon\rho a_m}{2}\coth\left(\frac{\rho a_m}{2}\right), a(x_m) = a_m, b(x_m)^n = b_m^n$ and $\hat{F}(x_m) = \hat{F}_m$.
The given boundary conditions become

$$\begin{cases} \lambda_{-1} + 4\lambda_1 + \lambda_2 = \phi_l(t_n), \\ \lambda_{N-1} + 4\lambda_N + \lambda_{N+1} = \phi_r(t_n). \end{cases} \tag{3.7}$$

Now, Eqs. (3.6) and (3.7) lead to an $(N+3)\times(N+3)$ system with $(N+3)$ unknowns $\{\lambda_{-1}, \lambda_0, ..., \lambda_{N+1}\}$. Eliminating λ_{-1} from first equation of (3.7) and λ_{N+1} from last equation of (3.7), we get

$$\begin{cases} (36\varepsilon + 12ha_0)\lambda_0 + 6ha_0\lambda_1 = h^2\hat{F}_0 \\ -6ha_N\lambda_{N-1} + (36\varepsilon - 12ha_N)\lambda_N = h^2\hat{F}_N. \end{cases} \tag{3.8}$$

The elimination of λ_{-1} and λ_{N+1} lead to a system of $(N+1)$ linear equations in $(N+1)$ unknowns $\lambda_0, \lambda_1, ..., \lambda_N$ which can be given as a linear system of the form

$$RX = Q, \tag{3.9}$$

where

$$R = \begin{pmatrix} 36\varepsilon + 12ha_0 & 6ha_0 & 0 & 0 & \dots & 0 \\ r_1^- & r_1^c & r_1^+ & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & r_m^- & r_m^c & r_m^+ & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & r_{N-1}^- & r_{N-1}^c & r_{N-1}^+ \\ 0 & \dots & 0 & 0 & -6ha_N & 36\varepsilon - 12ha_N \end{pmatrix}.$$

The column vectors X and Q are given as $X = [\lambda_0, \lambda_1, \lambda_N]^T$, $Q = [h^2\hat{F}_0, h^2\hat{F}_1, h^2\hat{F}_N]^T$. The matrix R in Eq. (3.9) is diagonally dominant and hence it is invertible giving rise to a unique approximate solution $U(x)$.

4. Stability and convergence analysis

This section provides stability and uniform convergence estimate in the maximum norm. Assume that a small error $\delta R, \delta Q$, has been made in the calculation of R, Q respectively. Let $\hat{X}$ be the solution of the perturbed system

$$(R + \delta R)\hat{X} = Q + \delta Q.$$

The collocation method for solving (3.9) is said to be stable, if the perturbed system has a unique solution for $\|\delta R\| \le C_3$ and

$$\|X - \hat{X}\| \le (C_1\|\delta R\|\|X\| + C_2\|\delta Q\|). \tag{4.1}$$

where C_1, C_2, and C_3 positive constants. In Eq. 3.9, we see that R is strictly diagonally dominant. Therefore, by a result in [30], for a sufficiently small value of h, we have

$$\|\delta R^{-1}\| \le \frac{C}{h^2} = \omega_1.$$

Since $\|R^{-1}\delta R\| \leq \|R^{-1}\|\|\delta R\| < \dfrac{1}{2}$. Choose a positive constant $\omega_2 < \dfrac{1}{2\omega_1}$. Then whenever $\|\delta R\| \leq \omega_2$, Eq. (4.1) has a unique solution, for

$$\| (R+\delta R)^{-1} \| = \| \left(I + R^{-1}\delta R\right)^{-1} R^{-1}\| \leq 2\omega_1.$$

As $(R+\delta R)\left(X - \hat{X}\right) = \delta R X - \delta Q$, it follows that

$$\|X - \hat{X}\| \leq 2\omega_1 \left(\|\delta R\|\|X\| + \|\delta Q\|\right).$$

which ensures the stability of the collocation system (3.9). The following lemma gives the properties of the B-splines which provides the ε-uniform convergence.

Lemma 4.1. *The B-splines set* $\hat{\Omega} = \{B_{-1}, B_0, ..., B_{N+1}\}$ *defined in Eq. (3.4), satisfy the inequality*

$$\sum_{m=-1}^{N+1} |B_m(x)| \leq 10, 0 \leq x \leq 1. \tag{4.2}$$

Proof. Note that

$$\left|\sum_{m=-1}^{N+1} B_m(x)\right| \leq \sum_{m=-1}^{N+1} |B_m(x)|.$$

Now, we have

$$\sum_{m=-1}^{N+1} |B_m(x)| = |B_{m-1}(x)| + |B_m(x)| + |B_{m+1}(x)| = 6 < 10.$$

This is for any nodal value x_m. Also, we have $|B_m(x)| \leq 4$ and $|B_{m-1}(x)| \leq 4$, for $x \in [x_m, x_{m+1}]$. Similarly, $|B_{m-1}(x)| \leq 1$ and $|B_{m+2}(x)| \leq 1$, for $x \in [x_m, x_{m+1}]$. Thus for any point $x_m \leq x_{m+1}$, we have

$$\sum_{m=-1}^{N+1} |B_m(x)| = |B_{m-1}(x)| + |B_m(x)| + |B_{m+1}(x)| + |B_{m+2}(x)| \leq 10.$$

□

Lemma 4.2. (Error in the spatial direction) *Let* $\Phi(x)$ *be the collocation approximation from the space of cubic splines to the solution* $\hat{U}(x) \approx U^n(x)$ *of Eq. (3.1) after temporal discretization. If* $\hat{F}(x) \in C^2[0,1]$, *then the parameter-uniform error estimate is given by*

$$\|\hat{U}(x_m) - \Phi(x_m)\|_\infty \leq Ch^2,$$

where C *is a positive constant independent of* h *and* ε.

Proof. Assume that $Y_N(x)$ is the unique spline interpolate from $\hat{U}(x)$ to the solution of our semi-discrete problem (3.1) is given by

$$Y_N(x) = \sum_{m=-1}^{N+1} \hat{\lambda}_m B_m(x). \tag{4.3}$$

If $\hat{F}(x) \in C^2[0,1]$, then $\hat{U}(x) \in C^4[0,1]$, and following the approach given as in Hall [13], the error estimates yields

$$\|D^j\left(\hat{U}(x) - Y_N\right)\| \leq \chi_m h^{4-j}, j = 0,1,2, \tag{4.4}$$

where χ_m are the constants. It follows immediately from the estimates (4.4) that

$$\|L\Phi(x_m) - LY_N(x)\| \leq \omega h^2, j = 0,1,2, \tag{4.5}$$

where $\omega = \varepsilon\chi_2 + h\chi_1\|a(x)\| + h\chi_0\|\nu(x)\|$. □

Assume that $LY_N(x) = \hat{F}_N(x_m)$ and $\hat{F}_N = [\hat{F}_0, \hat{F}_1, \hat{F}_N]^T$. Now, combining Eqs. (3.9) and (4.4) satisfies the inequality

$$\|R(X - y_N)\| \leq \omega h^4,$$

where $y_N = [\chi_0, \chi_1, \chi_N]^T, Ry_N = h^2\hat{F}_N(x_m), \forall m = 0, 1, 2, ..., N$. The matrix R in Eq. (3.9) is strictly diagonally dominant. Hence it is nonsingular. Also by Varah [30], we have

$$\|R^{-1}\|_\infty \leq \frac{C}{h^2}.$$

Again, using Eq. (3.9), it follows, the mth component of $R(X - y_N)$ can be written as

$$\begin{cases} \pi_0 = r_0^c\beta_0 + r_1^+\beta_1, \\ \pi_m = \left(-6\sigma_m - 3ha_m + h^2\nu_m\right)\beta_{m-1} + \left(12\sigma_m + 4h^2\nu_m\right)\beta_m + \left(-6\sigma_m + 3ha_m + h^2\nu_m\right)\beta_{m+1}, \\ \pi_N = r_{N-1}^-\beta_{N-1} + r_N^c\beta_N, \end{cases} \tag{4.6}$$

where $\pi_m = h^2\left[F(x_m) - \hat{F}(x_m)\right], 0 \leq m \leq N$, $r_0^c = 36\varepsilon + 12ha_0, r_1^+ = 6ha_0, r_{N-1}^- = -6ha_N, r_N^c = 36\varepsilon - 12ha_N$, and $\beta_m = \lambda_m - \hat{\lambda}_m, -1 \leq m \leq N+1$.

It is evident from inequality (4.5) that $\|\beta_m\| \leq h^2\|F(x_m) - \hat{F}(x_m)\| \leq \omega h^4$. Let $\pi = \max_{1\leq m\leq N-1}|\pi_m|$ and consider $\beta = [\beta_{-1}, \beta_0, \beta_1, \beta_N, \beta_{N+1}]^T$. Defining $P_m = |\pi_m|$ and $\hat{P} = \max_{1\leq m\leq N-1}|P_m|$. From Eq. (4.6), we have

$$\begin{aligned} &\left(12\sigma_m + 4h^2\nu_m\right)\beta_m = \\ &\left(6\sigma_m - h^2\nu_m\right)(\beta_{m-1} + \beta_{m+1}) + (3ha_m)(\beta_{m-1} - \beta_{m+1}) + \pi_m, 1 \leq m \leq N-1. \end{aligned} \tag{4.7}$$

Now, it is easy to see from Eq. (4.7),

$$\hat{P} \leq \frac{\omega h^3}{2h\tilde{\nu} - 6\tilde{a}}, P_0 \leq \frac{2\omega h^5\tilde{\nu}}{(36|\sigma_m| + 12h\tilde{a})(2h\tilde{\nu} - 6\tilde{a})}, P_N \leq \frac{2\omega h^5\tilde{\nu}}{(36|\sigma_m| - 12h\tilde{a})(2h\tilde{\nu} - 6\tilde{a})}.$$

Also, from the boundary conditions we have

$$P_{-1} \leq \frac{\omega h^3\omega_1}{2h\tilde{\nu} - 6\tilde{a}}, P_{N+1} \leq \frac{\omega h^3\omega_2}{2h\tilde{\nu} - 6\tilde{a}},$$

where $\omega_1 = \frac{9\sigma_0 + 3h\tilde{a} + 2h^2\tilde{\nu}}{9\sigma_0 + 3h\tilde{a}}$ and $\omega_2 = \frac{9\sigma_0 - 3h\tilde{a} + 2h^2\tilde{\nu}}{9\sigma_0 - 3h\tilde{a}}$.

Thus, using the value $\omega = \varepsilon\chi_2 + h\chi_1\|a(x)\| + h\chi_0\|\nu(x)\|$ one can easily show that

$$P = \max_{-1\leq m\leq N+1}\{P_m\} \leq \Theta h^2, \tag{4.8}$$

where Θ is independent of h and ε. Subtracting Eq. (4.3) from Eq. (3.3), we get

$$\Phi(x) - Y_N(x) = \sum_{m=-1}^{N+1}\left(\lambda_m - \hat{\lambda}_m\right)B_m(x).$$

This implies

$$\|\Phi(x) - Y_N(x)\|_\infty \leq \max_{-1\leq m\leq N+1}\left|\lambda_m - \hat{\lambda}_m\right|\sum_{m=-1}^{N+1}|B_m(x)|. \tag{4.9}$$

Now, combining Eqs. (4.2), (4.8), and (4.9) yields,

$$\|Y_N(x) - \Phi(x)\|_\infty \leq 10\Theta h^2,$$

with

$$\|\hat{U}(x) - Y_N(x)\|_\infty \leq \chi_0 h^4.$$

Using triangle inequality

$$\|\hat{U}(x) - \Phi(x)\| \leq \|\hat{U}(x) - Y_N(x)\| + \|Y_N(x) - \Phi(x)\|, \tag{4.10}$$

the above equation is combined to give

$$\|\hat{U}(x) - \Phi(x)\|_\infty \leq Ch^2, \tag{4.11}$$

where $C = 10\Theta + \chi_0 h^2$.

Lemma 4.3. (Error in the fully discrete scheme) *Let $u(x,t)$ be the solution of singularly perturbed time delay parabolic convection-diffusion problems (2.1) and (2.2) and $U(x_m, t_n)$ be the approximation to the solution $u(x_m, t_n)$ of the fully discretized scheme obtained after the temporal discretization. Then, the uniform error estimate for the fully discrete scheme is given by*

$$\|U(x_m, t_n) - u(x_m, t_n)\|_\infty \leq C\left(\Delta t + h^2\right), 0 \leq m \leq N.$$

Proof. The proof easily follows the estimates given in Lemma (3.2) and Lemma (4.2). □

5. Numerical results

The maximum pointwise error $E_\varepsilon^{N,\Delta t}$ and the corresponding convergence order $p_\varepsilon^{N,\Delta t}$ are defined by

$$E_\varepsilon^{N,\Delta t} = \max_{1\leq m,n\leq N-1,M-1}\left|(U_m^n)^{N,\Delta t} - (U_m^n)^{2N,\frac{\Delta t}{2}}\right|, \; p_\varepsilon^{N,\Delta t} = \frac{log\left(E_\varepsilon^{N,\Delta t}/E_\varepsilon^{2N,\frac{\Delta t}{2}}\right)}{log2},$$

and from these values we obtain the ε-uniform error $E^{N,\Delta t}$ and the corresponding ε-uniform order of convergence $p^{N,\Delta t}$ by

$$E^{N,\Delta t} = \max_\varepsilon E_\varepsilon^{N,\Delta t} \text{and } p^{N,\Delta t} = \frac{log\left(E^{N,\Delta t}/E^{2N,\frac{\Delta t}{2}}\right)}{log2}.$$

An analytical solution is unknown for these problems, therefore the maximum point wise errors and the corresponding numerical orders of convergence are calculated by using the double mesh principle [18].

Example 5.1. Consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \varepsilon\dfrac{\partial^2 u}{\partial x^2} + (2-x^2)\dfrac{\partial u}{\partial x} + xu(x,t) = -u(x,t-\tau) + 10t^2 exp(-t)x(1-x) \in (0,1)\times(0,2], \\ u(x,t) = 0, (x,t) \in [0,1]\times[-1,0], \\ u(0,t) = 0, u(1,t) = 0, t \in [0,2]. \end{cases}$$

Example 5.2. Consider problem given in

$$\begin{cases} \dfrac{\partial u}{\partial t} - \varepsilon\dfrac{\partial^2 u}{\partial x^2} + \dfrac{(5-x^2)}{3}\dfrac{\partial u}{\partial x} + tu(x,t) = -u(x,t-\tau) + t^3 x(1-x)\, sin(\pi x), (x,t) \in (0,1)\times(0,2], \\ u(0,t) = 0, u(1,t) = 0, t \in (0,2], \\ u(x,t) = 0, (x,t) \in [0,1]\times[-\tau,0]. \end{cases}$$

The computed maximum point wise errors $E_\varepsilon^{N,\Delta t}$ and the corresponding numerical rates of convergence $p_\varepsilon^{N,\Delta t}$ for Examples 5.1 and 5.2 calculated by scheme (3.6) tabulated in Tables 1 and 3 respectively. The two Tables 1 and 3 clearly indicates that for several values of ε, M and N, the proposed numerical method is parameter-uniform error as well as parameter-uniform rate of convergence. In Tables, 2 and 4, we are comparing maximum point wise error ($E_\varepsilon^{N,\Delta t}$) corresponding to two numerical results. It can also be noted that to solve the system of equations numerically, we have used the matrix inverse method. It can be observed from Tables 2 and 4 that maximum point-wise error of scheme in (3.6) is more accurate results than results in [2, 16]. In Tables 1 and 3, it is shown that the convergence order of the proposed numerical scheme is 1. Clearly, these results are in good agreement with our theoretical findings (see Lemma 4.3). To visualize the appearance of the boundary layers in the solutions of Examples 5.1 and 5.2, we have plotted the surface plots for $N = M = 64$ in Figures 1 and 2 for $\varepsilon = 1$ and $\varepsilon = 2^{-16}$. As observed from the two

Table 1. $E_\varepsilon^{N,\Delta t}$ and the corresponding $p_\varepsilon^{N,\Delta t}$ for Example 5.1.

ε	Number of intervals N/time step size Δt				
$\downarrow$	$16/\frac{1}{10}$	$32/\frac{1}{20}$	$64/\frac{1}{40}$	$128/\frac{1}{80}$	$256/\frac{1}{160}$
2^{-0}	$5.1006e-04$	$2.3974e-04$	$1.2585e-04$	$6.4539e-05$	$3.2675e-05$
	1.0892	0.92977	0.96346	0.98198	–
2^{-4}	$4.7329e-03$	$1.6262e-03$	$8.9236e-04$	$4.6710e-04$	$2.3903e-04$
	1.5412	0.86581	0.93389	0.9665	–
2^{-8}	$8.8017e-03$	$5.1912e-03$	$2.5903e-03$	$1.0822e-03$	$3.5995e-04$
	0.76171	1.0029	1.2592	1.5881	–
2^{-12}	$8.8017e-03$	$5.2758e-03$	$3.1195e-03$	$1.6970e-03$	$8.8451e-04$
	0.73839	0.75808	0.87833	0.9400	–
2^{-16}	$8.8017e-03$	$5.2758e-03$	$3.1195e-03$	$1.6970e-03$	$8.8451e-04$
	0.73839	0.75808	0.87833	0.9400	–
2^{-20}	$8.8017e-03$	$5.2758e-03$	$3.1195e-03$	$1.6970e-03$	$8.8451e-04$
	0.73839	0.75808	0.87833	0.9400	–
2^{-24}	$8.8017e-03$	$5.2758e-03$	$3.1195e-03$	$1.6970e-03$	$8.8451e-04$
	0.73839	0.75808	0.87833	0.9400	–
2^{-28}	$8.8017e-03$	$5.2758e-03$	$3.1195e-03$	$1.6970e-03$	$8.8451e-04$
	0.73839	0.75808	0.87833	0.9400	–
$E^{N,\Delta t}$	**8.8017e-03**	**5.2758e-03**	**3.1195e-03**	**1.6970e-03**	**8.8451e-04**
$p^{N,\Delta t}$	**0.73839**	**0.75808**	**0.87833**	**0.9400**	**-**

Table 2. Comparison $E_\varepsilon^{N,\Delta t}$ for Example 5.1.

$\varepsilon\downarrow$	$N=M=16$	$N=32$	$N=64$	$N=128$	$N=256$
Present method					
2^{-0}	$5.4727e-04$	$2.9416e-04$	$1.5573e-04$	$8.0256e-05$	$4.0737e-05$
2^{-4}	$4.8357e-03$	$2.0024e-03$	$1.1050e-03$	$5.8057e-04$	$2.9796e-04$
2^{-8}	$8.5779e-03$	$5.3292e-03$	$2.6204e-03$	$1.1346e-03$	$4.2387e-04$
2^{-12}	$8.5779e-03$	$5.4144e-03$	$3.2062e-03$	$1.7466e-03$	$9.1109e-04$
2^{-16}	$8.5779e-03$	$5.4144e-03$	$3.2062e-03$	$1.7466e-03$	$9.1129e-04$
2^{-20}	$8.5779e-03$	$5.4144e-03$	$3.2062e-03$	$1.7466e-03$	$9.1129e-04$
Result in [16]					
2^{-0}	$2.29e-03$	$1.31e-03$	$6.99e-04$	$3.61e-04$	$1.83e-04$
2^{-4}	$1.18e-02$	$9.18e-03$	$6.04e-03$	$3.61e-03$	$2.05e-03$
2^{-8}	$2.78e-02$	$1.39e-02$	$6.30e-03$	$2.68e-03$	$1.35e-03$
2^{-12}	$3.36e-02$	$1.81e-02$	$9.13e-03$	$4.49e-03$	$2.17e-03$
2^{-16}	$3.40e-02$	$1.84e-02$	$9.36e-03$	$4.66e-03$	$2.30e-03$
2^{-20}	$3.41e-02$	$1.84e-02$	$9.38e-03$	$4.67e-03$	$2.31e-03$

Figures 1 and 2, the solution of the two Examples 5.1 and 5.2 has layers at the right side of the rectangular domain. It is shown that the effect of perturbation parameter ε and delay parameter τ on the boundary layer of the solution for Examples 5.1 and 5.2 are illustrated in Figures 3A and 3B. It is observed that as the perturbation parameter ε goes to zero strong boundary layer is formed on the right side of the x-domain. The maximum pointwise errors are plotted in log-log scale in Figures 4A and 4B for the two Examples 5.1 and 5.2 respectively. It is shown that Figure 4 reveal the numerical order of convergence. The results in tables and figures obtained from the given two examples confirm the proposed numerical method is more efficient. Note that the current paper computations related to the given examples were performed using the MATLAB 2013A software package.

TABLE 3. $E_\varepsilon^{N,\Delta t}$ and the corresponding $p_\varepsilon^{N,\Delta t}$ for Example 5.2.

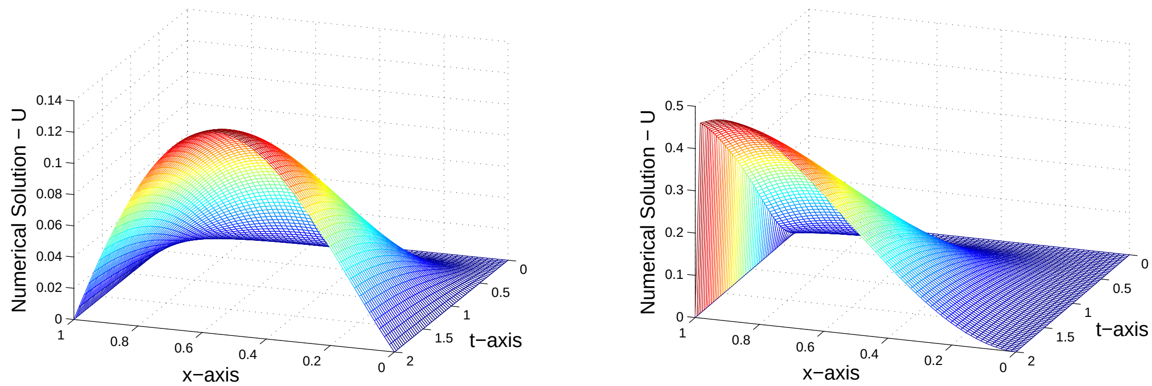
ε	Number of intervals N/time step size Δt				
$\downarrow$	$16/\frac{1}{10}$	$32/\frac{1}{20}$	$64/\frac{1}{40}$	$128/\frac{1}{80}$	$256/\frac{1}{160}$
2^{-0}	1.7528e-04	1.2993e-04	8.3665e-05	4.6776e-05	2.4655e-05
	0.43193	0.63504	0.83886	0.92389	
2^{-4}	2.6981e-03	9.2848e-04	5.6576e-04	3.2394e-04	1.7391e-04
	1.5390	0.71468	0.80446	0.89739	
2^{-8}	6.8523e-03	3.9013e-03	1.9354e-03	7.3687e-04	2.2450e-04
	0.81263	1.0113	1.3931	1.7147	
2^{-12}	6.8523e-03	3.9116e-03	2.0788e-03	1.0701e-03	5.4269e-04
	0.80883	0.91201	0.95801	0.97955	-
2^{-16}	6.8523e-03	3.9116e-03	2.0788e-03	1.0701e-03	5.4269e-04
	0.80883	0.91201	0.95801	0.97955	-
2^{-20}	6.8523e-03	3.9116e-03	2.0788e-03	1.0701e-03	5.4269e-04
	0.80883	0.91201	0.95801	0.97955	-
2^{-24}	6.8523e-03	3.9116e-03	2.0788e-03	1.0701e-03	5.4269e-04
	0.80883	0.91201	0.95801	0.97955	-
2^{-28}	6.8523e-03	3.9116e-03	2.0788e-03	1.0701e-03	5.4269e-04
	0.80883	0.91201	0.95801	0.97955	-
$E^{N,\Delta t}$	**6.8523e-03**	**3.9116e-03**	**2.0788e-03**	**1.0701e-03**	**5.4269e-04**
$p^{N,\Delta t}$	**0.80883**	**0.91201**	**0.95801**	**0.97955**	**-**

TABLE 4. Comparison $E_\varepsilon^{N,\Delta t}$ for Example 5.2.

$\varepsilon\downarrow$	$N=M=16$	$N=32$	$N=64$	$N=128$	$N=256$
Present method					
2^{-0}	2.3281e-04	1.7793e-04	1.0848e-04	5.9446e-05	3.1063e-05
2^{-4}	2.6790e-03	1.2062e-03	7.3732e-04	4.1419e-04	2.1984e-04
2^{-8}	6.6457e-03	3.8619e-03	1.9363e-03	7.4430e-04	2.3021e-04
2^{-12}	6.6458e-03	3.8721e-03	2.0782e-03	1.0751e-03	5.4655e-04
2^{-16}	6.6458e-03	3.8721e-03	2.0782e-03	1.0751e-03	5.4655e-04
2^{-20}	6.6458e-03	3.8721e-03	2.0782e-03	1.0751e-03	5.4655e-04
Result in [2]					
2^{-0}	2.3950e-02	1.7664e-02	1.1228e-02	6.4886e-03	3.5334e-03
2^{-4}	4.8048e-02	2.7869e-02	1.4847e-02	7.6292e-03	3.8619e-03
2^{-8}	4.9006e-02	2.8622e-02	1.5142e-02	7.7170e-03	3.8852e-03
2^{-12}	4.9006e-02	2.8622e-02	1.5141e-02	7.7173e-03	3.8858e-03
2^{-16}	4.9006e-02	2.8622e-02	1.5141e-02	7.7173e-03	3.8858e-03
2^{-20}	4.9006e-02	2.8622e-02	1.5141e-02	7.7173e-03	3.8858e-03

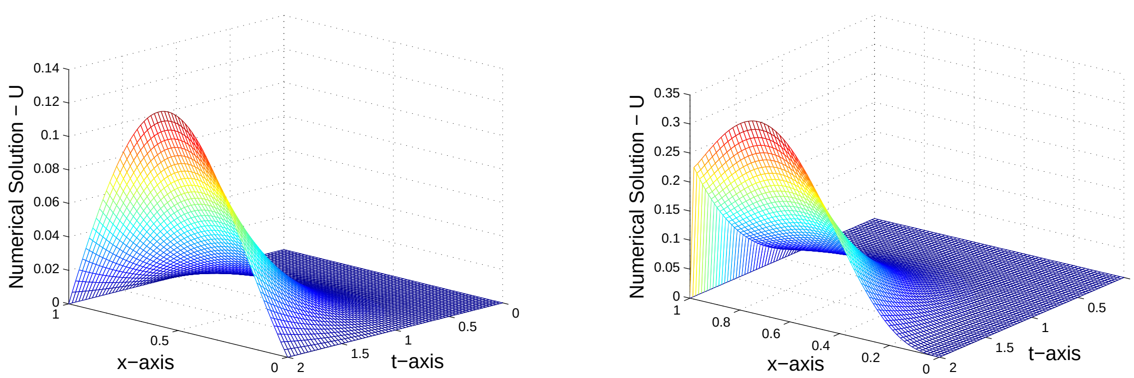
6. Conclusion

A robust numerical study is formulated to solve a one-dimensional singularly perturbed parabolic convection-diffusion with a large time delay. The scheme is based on the implicit Euler scheme in the time dimension and the exponentially fitted B-spline collocation scheme in the spatial dimension, both on a uniform grid. An error estimate for the numerical scheme is constructed and is accurate of order $O\left(\Delta t + h^2\right)$. Numerous numerical experiments were carried out to prove the robustness of the proposed method. The method turns out to be uniformly convergent and is also unconditionally stable. The advantage of the proposed method lies in its simplicity and accuracy. The



(A) (B)

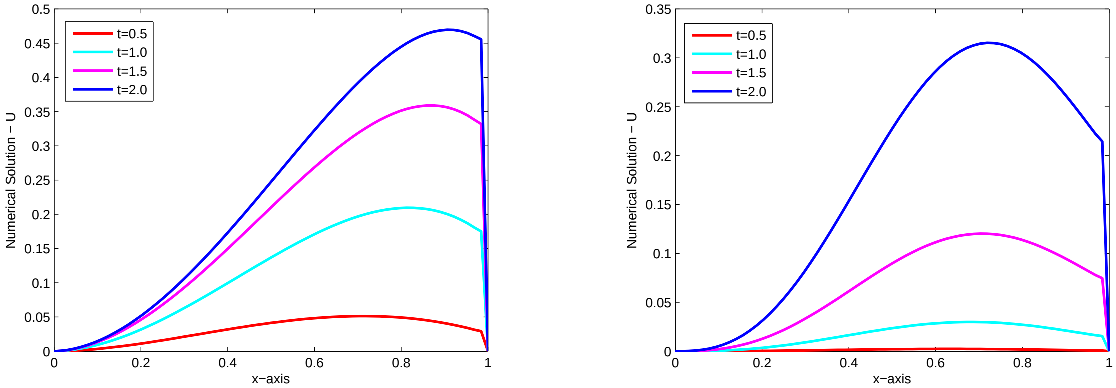
FIGURE 1. Solution based on scheme (3.6) for Example 5.1 for different values of ε and T **(A)** $\varepsilon = 1$ and **(B)** $\varepsilon = 2^{-16}$



(A) (B)

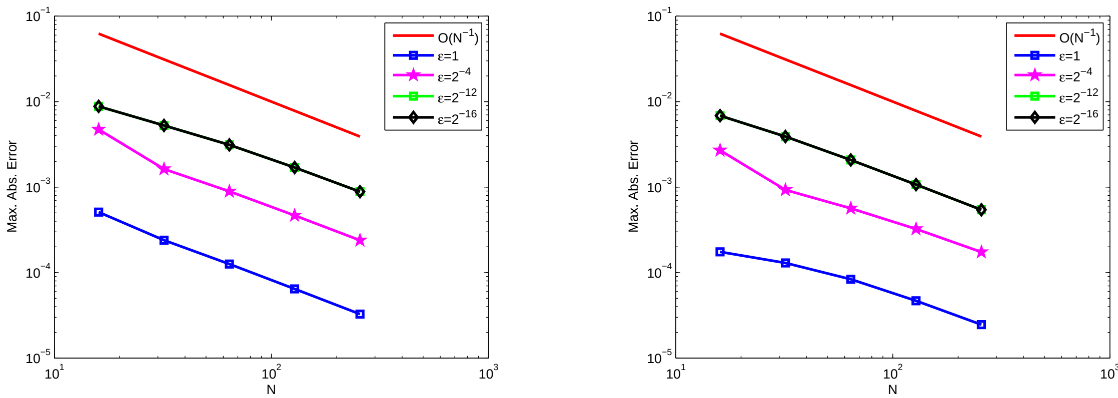
FIGURE 2. Solution based on scheme (3.6) for Example 5.2 for different values of ε and T **(A)** $\varepsilon = 1$ and **(B)** $\varepsilon = 2^{-16}$

numerical results show that the present method achieves higher accuracy than many other boundary-layer resolving finite difference methods.



(A) (B)

FIGURE 3. Scheme (3.6) solution of Example 5.1 on **(A)** and Example 5.2 on **(B)** for different values of T and $\varepsilon = 2^{-12}$.



(A) (B)

FIGURE 4. Example 5.1 on (A), Example 5.2 and on (B) Log-Log scale plot of the maximum absolute error for different values of ε.

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