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Approximate symmetry group analysis and similarity reductions of the perturbed mKdV-KS equation

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Abstract

In this paper, we apply the approximate symmetry transformation group to obtain the approximate symmetry group of the perturbed mKdV-KS equation which is a modified Korteweg-de Vries (mKdV) equation with a higher singularity perturbed term as the Kuramoto-Sivashinsky (KS) equation. Also, an optimal system of one-dimensional subalgebras of symmetry algebra is constructed and the corresponding differential invariants and some approximately invariant solutions of the equation are computed.

Keywords. Perturbed mKdV-KS equation, Approximate symmetry, Approximately invariant solution, Optimal system.

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1. Introduction

Some differential equations which appear in mathematics, physics, mechanics, and etc., have terms that involve a small parameter which is called perturbed term. An essential step in these studies is to compute the symmetries and invariant solutions of these equations. There are several methods to analyze these perturbed equations [1, 2, 15]. One of these methods is the approximate symmetry method, that is firstly introduced by Baikov, Gazizov, and Ibragimov in the 1980s [1, 2]. In fact, this method is based on the Lie symmetry method and the theory of perturbations. The Lie symmetry method is a very practical and important method that can be used to obtain and classify invariant group solutions for a differential equation. It is also widely used in calculating the conservation laws of a differential equation [7, 8]. To determine approximate symmetries, Fushehich and Shtelen [4] create a new method which was followed by Euler et al. [3]. We refer to [6, 12, 13, 16] to compare these methods.

In this study, by using the approximate symmetry method, we obtain the approximate symmetries of the perturbed mKdV-KS equation,

$$u_t + 6u^2 u_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0, \tag{1.1}$$

where ε is a small parameter. A general form of this equation has appeared some where like study of shallow water on tilted planes [14]. There have been many studies on the invariant group solutions and the conservation laws of different types of this equation [5, 9].

The outline of this paper is as follows. Section 2 is devoted to some definitions and basic concepts of the approximate symmetry method. In section 3, we analyze the approximate symmetry group of perturbed mKdV-KS equation by Baikov, Gazizov and Ibragimov method. An optimal system of one-dimensional subalgebras of the Lie symmetry algebra is obtained in section 4. Finally, in section 5 we construct the approximate differential invariants corresponding to the generators in the optimal system and obtain some similarity reductions of (1.1).

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2. Notations and Definitions

Some definitions and related results which we use through of this work are presented in this section. These concepts and results are brought from the reference [6].

Suppose $g(x,\varepsilon)$ are functions of n variables $x=(x^1,\dots,x^n)$ and a parameter ε which are locally considered in a neighborhood of $\varepsilon=0$. If the function $g(x,\varepsilon)$ satisfies the condition,

$$\lim_{\varepsilon\to 0}\frac{g(x,\varepsilon)}{\varepsilon^p}=0,$$

it is written $g(x,\varepsilon)=o(\varepsilon^p)$ and we say that g is of order less than ε^p. The functions g and h are called approximately equal and denoted by $g\approx h$, whenever

$$g(x,\varepsilon)=h(x,\varepsilon)+o(\varepsilon^p).$$

An equivalence relation is defined on such functions by this approximate equality. Suppose,

$$g_0(x)+\varepsilon g_1(x)+\cdots+\varepsilon^p g_p(x),$$

be the approximating polynomial of degree p in ε that results from the Taylor series expansion of a given function $g(x,\varepsilon)$ in powers of ε about $\varepsilon=0$. Then, any function $h\approx g$ has the form,

$$h(x,\varepsilon)=g_0(x)+\varepsilon g_1(x)+\cdots+\varepsilon^p g_p(x)+o(\varepsilon^p).$$

Consider the ordered sets of smooth vector functions,

$$g_0(x,a),g_1(x,a),\dots,g_p(x,a),$$

which depend on x's and a as the parameter of group, with coordinates:

$$g_0^i(x,a),g_1^i(x,a),\dots,g_p^i(x,a),\qquad i=1,\dots,n.$$

The one-parameter family of approximate transformations,

$$\tilde{x}^i\approx g_0^i(x,a)+\varepsilon g_1^i(x,a)+\cdots+\varepsilon^p g_p^i(x,a),\qquad i=1,\dots,n,$$

is the class of invertible transformations $\tilde{x}=g(x,a,\varepsilon)$ with vector function $g=(g^1,\dots,g^n)$ such that:

$$g^i(x,a,\varepsilon)\approx g_0^i(x,a)+\varepsilon g_1^i(x,a)+\cdots+\varepsilon^p g_p^i(x,a),\qquad i=1,\dots,n,$$

and verify the conditions $g(x,0,\varepsilon)\approx x$ and $g(g(x,a,\varepsilon),b,\varepsilon)\approx g(x,a+b,\varepsilon)$.

Definition 2.1. Let G be a one-parameter group of approximate transformation:

$$\tilde{z}^i\approx g(z,a,\varepsilon)\equiv g_0^i(z,a)+\varepsilon g_1^i(z,a),\qquad i=1,\dots,n.$$

The approximate equation,

$$H(z,\varepsilon)\equiv H_0(z)+\varepsilon H_1(z)\approx 0,\tag{2.1}$$

is called an approximately invariant with respect to G (or admits G) if

$$H(\tilde{z},\varepsilon)\approx H(g(z,a,\varepsilon),\varepsilon)=o(\varepsilon),$$

whenever $z=(z^1,\dots,z^n)$ satisfies (2.1).

If $z=(x,u,u_{(1)},\dots,u_{(k)})$, then according to the above definition, G is called an approximate symmetry group of the approximate k-order differential equation.

Theorem 2.2. *If* (2.1) *is an approximate invariant which admits G, then*

$$X=X_0+\varepsilon X_1\equiv \xi_0^i(z)\frac{\partial}{\partial z^i}+\varepsilon\xi_1^i(z)\frac{\partial}{\partial z^i},\tag{2.2}$$

is the generator of G if and only if,

$$\Big[X^{(k)}H(z,\varepsilon)\Big]_{H\approx 0}=o(\varepsilon),$$

or

$$\left[X_0^{(k)}H_0(z)+\varepsilon(X_1^{(k)}H_0(z)+X_0^{(k)}H_1(z))\right]_{(2.1)}=o(\varepsilon), \tag{2.3}$$

where $X^{(k)}$ is the k^{th} order prolongation of X [6].

If operator (2.2) satisfies the condition (2.3), we call it, an infinitesimal approximate symmetry admitted by (2.1).

Theorem 2.3. *([6]) If $X=X_0+\varepsilon X_1$ be a generator of the approximate transformation group for (2.1), where $X_0 \neq 0$, then the operator,*

$$X_0=\xi_0^i(z)\frac{\partial}{\partial z^i},$$

is an exact symmetry generator for the equation,

$$H_0(z)=0. \tag{2.4}$$

Equations (2.1) and (2.4) are called the perturbed equation and unperturbed equation, respectively. By assumptions of the Theorem 2.3, the operator X_0 is called a stable symmetry of Eq. (2.4). The approximate symmetry generator corresponding to X_0 is $X=X_0+\varepsilon X_1$, which is called X_0 deformation arising from perturbation $\varepsilon H_1(z)$. When all generators of the symmetry Lie algebra (2.4) are stable, the perturbed Eq. (2.1) is called to inherits the symmetries of the unperturbed equation.

3. Approximate Symmetry Analyzing of the Perturbed mKdV-KS Equation

In this section, we obtain the approximate symmetries of the Eq. (1.1). To this end, we first obtain the exact symmetries of the equation. The generator of the approximate transformation group admitted by (1.1) has the form,

$$X=X_0+\varepsilon X_1=(\tau_0+\varepsilon\tau_1)\frac{\partial}{\partial t}+(\xi_0+\varepsilon\xi_1)\frac{\partial}{\partial x}+(\phi_0+\varepsilon\phi_1)\frac{\partial}{\partial u}, \tag{3.1}$$

where τ_i, ξ_i and ϕ_i are unknown functions of t, x and u, for $i=0,1$. For obtaining the exact symmetry X_0 of the unperturbed equation, we must solve the determining equation,

$$X_0^{(3)}\left(u_t+6u^2u_x+u_{xxx}\right)\Big|_{u_t+6u^2u_x+u_{xxx}=0}=0. \tag{3.2}$$

Solving (3.2), by direct calculations we obtain,

$$\tau_0=-3c_1t+c_2, \qquad \xi_0=-c_1x+c_3, \qquad \phi_0=c_1u,$$

where c_1, c_2 and c_3 are arbitrary constants, so the infinitesimal symmetry generator X_0 is:

$$X_0=(-3c_1t+c_2)\partial_t+(-c_1x+c_3)\partial_x+c_1u\partial_u. \tag{3.3}$$

Therefore, (1.1) admits the three-dimensional Lie algebra which is generated by vector fields:

$$X_0^1=\partial_x, \qquad X_0^2=\partial_t, \qquad X_0^3=-3t\partial_t-x\partial_x+u\partial_u. \tag{3.4}$$

Now, we obtain the approximate symmetries of (1.1). At first, the auxiliary function I must be determined due to (2.3) by the equation,

$$I=\frac{1}{\varepsilon}\left[X_0^{(k)}(H_0(z)+\varepsilon H_1(z))\Big|_{H_0(z)+\varepsilon H_1(z)=0}\right].$$

By substituting the generator X_0 from (3.3) in the above equation, the auxiliary function is obtained as:

$$I=-c_1(u_{xx}-u_{xxxx}).$$

For calculating the operator X_1, we must solve the following inhomogenous determining equation:

$$X_1^{(k)}H_0(z)\Big|_{H_0(z)}+I=0.$$

Rewriting the above equation for (1.1) yields,

$$X_1^{(3)}\left(u_t + 6u^2u_x + u_{xxx}\right)\Big|_{u_t+6u^2u_x+u_{xxx}=0} - c_1\left(u_{xx} - u_{xxxx}\right) = 0,$$

and by solving this equation we have:

$$\tau_1 = -3A_1 t + A_2, \qquad \xi_1 = -A_1 x + A_3, \qquad \phi_1 = A_1 u,$$

where A_1, A_2, and A_3 are arbitrary constants. Therefore, the approximate symmetries of perturbed mKdV-KS equation are:

$$\begin{aligned} &V_1 = \partial_x, && V_4 = \varepsilon V_1,\\ &V_2 = \partial_t, && V_5 = \varepsilon V_2,\\ &V_3 = -3t\partial_t - x\partial_x + u\partial_u, && V_6 = \varepsilon V_3. \end{aligned} \tag{3.5}$$

Considering the first order of precision, the commutator Table 1 shows that the operators in (3.5) generate a 6-dimensional approximate symmetry Lie algebra. We denote this Lie algebra by $\mathfrak{g}$.

Table 1: The commutator table of approximate symmetry of (1.1).

	V_1	V_2	V_3	V_4	V_5	V_6
V_1	0	0	0	0	0	$-V_4$
V_2	0	0	$-3V_2$	0	0	$-3V_5$
V_3	0	$3V_2$	0	V_4	$3V_5$	0
V_4	0	0	$-V_4$	0	0	0
V_5	0	0	$-3V_5$	0	0	0
V_6	V_4	$3V_5$	0	0	0	0

It is clear that the perturbed equation inherits the symmetry of the unperturbed equation since all of the generators in (3.4) are stable. Also, $\mathfrak{g}$ is solvable and the finite sequence of ideals for $\mathfrak{g}$ is as follows:

$$0 \subset \langle V_4\rangle \subset \langle V_4, V_5\rangle \subset \langle V_4, V_5, V_6\rangle \subset \langle V_1, V_4, V_5, V_6\rangle \subset \langle V_1, V_2, V_4, V_5, V_6\rangle \subset \mathfrak{g}.$$

4. Optimal System of Perturbed mKdV-KS Equation

In this section, we construct the one-dimensional optimal system of Lie subalgebras for $\mathfrak{g}$.

Definition 4.1. Suppose G is a Lie group. An optimal system of s-parameter subgroups is a list of conjugacy inequivalent s-parameter subgroups with the property that each other subgroup is conjugated exactly to one subgroup of this list. Also, a list of s-dimensional subalgebras forms an optimal system if every s-dimensional subalgebra of $\mathfrak{g}$ is equivalent to a unique member of the list under some element of the adjoint representation: $\tilde{\mathfrak{h}} = \mathrm{Ad}g(\mathfrak{h})$ [11].

The reason for importance of the optimal system is that it gives a classification on the infinite group invariant solutions of the equation. This classification gives us the assurance that if two solutions are placed in two different classes, they will not be transformed to each other by any group action.In fact, this classification is the classification of orbits for the adjoint representation, which is done in a simple method [10, 11]. In this method, a general element of Lie algebra is considered and then we try to simplify it by applying the different adjoint transformations on it as far as possible. The optimal system of subalgebras is obtained by selecting one representative from every equivalence class. The adjoint representation is constructed by Lie series:

$$\mathrm{Ad}(\exp(\epsilon.V_i).V_j) = V_j - \epsilon.[V_i, V_j] + \frac{\epsilon^2}{2}.[V_i,[V_i,V_j]] - \cdots, \tag{4.1}$$

where $[V_i, V_j]$ is the commutator of $\mathfrak{g}$ (mentioned in Table 1) and ϵ is a parameter $(i, j = 1, ..., 6)$. Therefore, Table 2 is deduced, where its (i, j)-th entry imply $\mathrm{Ad}(\exp(\epsilon.V_i).V_j)$.

Table 2: Adjoint representation of $\mathfrak{g}$.

Ad	V_1	V_2	V_3	V_4	V_5	V_6
V_1	V_1	V_2	V_3	V_4	V_5	$V_6+\epsilon V_4$
V_2	V_1	V_2	$V_3+3\epsilon V_2$	V_4	V_5	$V_6+3\epsilon V_5$
V_3	V_1	$e^{-3\epsilon}V_2$	V_3	$e^{-\epsilon}V_4$	$e^{-3\epsilon}V_5$	V_6
V_4	V_1	V_2	$V_3+\epsilon V_4$	V_4	V_5	V_6
V_5	V_1	V_2	$V_3+3\epsilon V_5$	V_4	V_5	V_6
V_6	$V_1-\epsilon V_4$	$V_2-3\epsilon V_5$	V_3	V_4	V_5	V_6

Theorem 4.2. *An optimal system for one-dimensional subalgebras of the approximate Lie symmetry algebra of perturbed mKdV-KS equation is produced by:*

$$\begin{array}{llll} 1) & aV_1+V_4\pm V_5, & 5) & aV_1+V_2\pm V_5,\\ 2) & \pm V_2+V_4, & 6) & aV_1+V_2+bV_6,\\ 3) & V_4\pm V_5, & 7) & aV_1+V_5,\\ 4) & V_4, & 8) & aV_1+bV_3+cV_6, \end{array} \tag{4.2}$$

where a, b and c are arbitrary constants.

Proof. Suppose $F_i^\epsilon:\mathfrak{g}\to\mathfrak{g}$ be the adjoint transformation $V\mapsto \mathrm{Ad}(\exp(\epsilon.V_i).V)$. The matrices of F_i^ϵ with respect to the basis V_i, $i=1,...,6$ are as following:

$$M_1^{\epsilon_1}=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&\epsilon_1&0&1\end{pmatrix},\qquad M_4^{\epsilon_4}=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&\epsilon_4&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},$$

$$M_2^{\epsilon_2}=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&3\epsilon_2&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&3\epsilon_2&1\end{pmatrix},\qquad M_5^{\epsilon_5}=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&3\epsilon_5&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},$$

$$M_3^{\epsilon_3}=\begin{pmatrix}1&0&0&0&0&0\\0&e^{-3\epsilon_3}&0&0&0&0\\0&0&1&0&0&0\\0&0&0&e^{-\epsilon_3}&0&0\\0&0&0&0&e^{-3\epsilon_3}&0\\0&0&0&0&0&1\end{pmatrix},\qquad M_6^{\epsilon_6}=\begin{pmatrix}1&0&0&-\epsilon_6&0&0\\0&1&0&0&-3\epsilon_6&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}.$$

Let $V=\sum_{i=1}^6 a_iV_i$ be the general form of an element of $\mathfrak{g}$. We should simplify the coefficients of V as much as possible, by affecting the suitable F_i^ϵ on it.

- Suppose first that $a_4\neq 0$, $a_2\neq 0$ and $a_5\neq 0$, we can assume that $a_4=1$ by scaling V if necessary. Also, we can make the coefficients of V_6, V_3 and V_2 vanish using $F_1^{\epsilon_1}$, $F_2^{\epsilon_2}$ and $F_6^{\epsilon_6}$, by setting $\epsilon_1=-\frac{a_6}{a_4}$, $\epsilon_2=-\frac{a_3}{3a_2}$ and $\epsilon_6=\frac{a_2}{3a_5}$ respectively. Moreover, the coefficient of V_5 can be ± 1 using $F_3^{\epsilon_3}$, by setting $\epsilon_3=\frac{1}{3}\ln|a_5|$. So, V is reduced to the case (1).
- If $a_4\neq 0$, $a_2\neq 0$ and $a_5=0$, then we can make the coefficients of V_6, V_3 and V_1 vanish using $F_1^{\epsilon_1}$, $F_2^{\epsilon_2}$ and $F_6^{\epsilon_6}$, by setting $\epsilon_1=-\frac{a_6}{a_4}$, $\epsilon_2=-\frac{a_3}{3a_2}$ and $\epsilon_6=a_1$ respectively. Also, by setting $\epsilon_3=\frac{1}{3}\ln|a_2|$ in $F_3^{\epsilon_3}$, we can make the coefficient of V_2, ± 1. So by scaling if necessary, V is reduced to the case (2).
- If $a_4\neq 0$, $a_5\neq 0$ and $a_2=0$, then we can make the coefficients of V_6, V_3 and V_1 vanish using $F_1^{\epsilon_1}$, $F_5^{\epsilon_5}$ and $F_6^{\epsilon_6}$, by setting $\epsilon_1=-\frac{a_6}{a_4}$, $\epsilon_5=-\frac{a_3}{3a_5}$ and $\epsilon_6=a_1$ respectively. Also, by setting $\epsilon_3=\frac{1}{3}\ln|a_5|$ in $F_3^{\epsilon_3}$, we can make the coefficient of V_5, ± 1. So by scaling if necessary, V is reduced to the case (3).

- If $a_4 \neq 0$ and $a_2 = a_5 = 0$ then we can make the coefficients of V_6, V_3 and V_1 vanish by $F_1^{\epsilon_1}$, $F_4^{\epsilon_4}$ and $F_6^{\epsilon_6}$, by setting $\epsilon_1 = -\frac{a_6}{a_4}$, $\epsilon_4 = -a_3$ and $\epsilon_6 = a_1$ respectively. So by scaling if necessary, V is reduced to the case (4).
- If $a_2 \neq 0$, $a_5 \neq 0$ and $a_4 = 0$, one can vanish the coefficients of V_3 and V_6 using $F_2^{\epsilon_2}$, by setting $\epsilon_2 = -\frac{a_3}{3a_2}$ and $\epsilon_2 = -\frac{a_6}{3a_5}$ respectively. Also by setting $\epsilon_3 = \frac{1}{3}\ln|a_5|$ in $F_3^{\epsilon_3}$ we can make the coefficient of V_5, ± 1. So by scaling if necessary, V reduces to the case (5).
- If $a_2 \neq 0$ and $a_5 = a_4 = 0$, then we can make the coefficient of V_3 vanish by setting $\epsilon_2 = -\frac{a_3}{3a_2}$ in $F_2^{\epsilon_2}$. So by scaling if necessary, V is reduced to the case (6).
- If $a_5 \neq 0$ and $a_2 = a_4 = 0$, one can vanish the coefficients of V_6 and V_3 by setting $\epsilon_2 = -\frac{a_6}{3a_5}$ and $\epsilon_5 = -\frac{a_3}{3}$ in $F_2^{\epsilon_2}$ and $F_5^{\epsilon_5}$ respectively. So by scaling if necessary, V reduces to the case (7).
- If $a_2 = a_4 = a_5 = 0$, then V is reduces to the case (8).

More cases do not exist for study, so the proof is complete. $\square$

5. Approximate Invariant Solutions for the Perturbed mKdV-KS Equation

In this section, the approximately differential invariants and some approximate invariant solutions of (1.1) are computed. At first, consider the operator $X = aV_1 + V_4 + V_5$. The approximate invariants of X are differential functions such as $J = J_0 + \varepsilon J_1$ that satisfy the equation $X(J) = o(\varepsilon)$. So we have,

$$X(J) = (a\partial_x + \varepsilon\partial_x + \varepsilon\partial_t)(J_0 + \varepsilon J_1) = o(x).$$

Equivalently

$$a\partial_x J_0 = 0, \qquad \varepsilon\partial_x J_0 + \varepsilon\partial_t J_0 + a\varepsilon\partial_x J_1 = 0.$$

We can obtain two functionally independent solutions, $J_0 = t$ and $J_0 = u$ from the first equation. Substituting $J_0 = t$ in the second equation we obtain

$$1 + a\partial_x J_1 = 0 \quad \Rightarrow \quad \partial_x J_1 = -\frac{1}{a} \quad \Rightarrow \quad J_1 = -\frac{x}{a}.$$

So the first invariant is $t - \frac{\varepsilon}{a}x$. In a similar way, by substituting $J_0 = u$ in the second equation we have $0 + a\varepsilon\partial_x J_1 = 0$. So the simplest solution is $J_1 = 0$. Therefore, the second invariant is $u + \varepsilon(0)$. Then $\{t - \frac{\varepsilon}{a}x, u\}$ is a set of independent invariants for X. Consider the new coordinates with the variable $z = t - \frac{\varepsilon}{a}x$ and $f(z) = u$. Substituting the new variables in the perturbed mKdV-KS equation and using the chain rule and considering the first order of precision, the reduced equation is obtained as,

$$f' = \frac{-6\varepsilon}{az^4}(f^2 f' z^2 + f').$$

After simplifying this equation, a trivial approximate invariant solution is obtained for (1.1) as follows,

$$f = \pm\sqrt{\frac{-az^4 - 6\varepsilon}{6\varepsilon z^2}}.$$

In a similar manner, we construct approximate differential invariants and reduced equations with respect to the operators in optimal system and list them in Table 3.

Table 3: Approximate differential invariants and similarity reduction of (1.1).

Operator	Approximate invariants	Similarity reduction
$aV_1 + V_4 - V_5$	$\{t + \varepsilon \frac{x}{a}, u\}$	$f' = \frac{6\varepsilon}{az^4}(f^2 f' z^2 + f')$.
$V_2 + V_4$	$\{x - \varepsilon t, u\}$	$f'\varepsilon = 6f^2 f' + f''' + \varepsilon f'' + \varepsilon f''''$
$-V_2 + V_4$	$\{x + \varepsilon t, u\}$	$-f'\varepsilon = 6f^2 f' + f''' + \varepsilon f'' + \varepsilon f''''$
$V_4 + V_5$	$\{x - t, u\}$	$f' = 6f^2 f' + f''' + \varepsilon f'' + \varepsilon f''''$
$V_4 - V_5$	$\{x + t, u\}$	$-f' = 6f^2 f' + f''' + \varepsilon f'' + \varepsilon f''''$
V_4	$\{t, u\}$	$f' = 0$
$aV_1 + V_2 + V_5$	$\{x - at + \varepsilon x, u\}$	$f'a = 6f^2 f'(\varepsilon + 1) +$ $f'''(3\varepsilon + 1) - \varepsilon(f'' + f''')$
$aV_1 + V_2 - V_5$	$\{x - at - \varepsilon x, u\}$	$f'a = 6f^2 f'(-\varepsilon + 1) +$ $f'''(-3\varepsilon + 1) + \varepsilon(f'' + f'''')$
$aV_1 + V_2 + bV_6$	$\{x - at + \varepsilon(\frac{2bx^2}{a} - 3btx),$ $u - \varepsilon btu\}$	$f' = 2a^4 b\varepsilon t(1 - b\varepsilon t) f'''$
$aV_1 + V_5$	$\{t - \varepsilon \frac{x}{a}, u\}$	$f' = \frac{-6\varepsilon}{az^4}(f^2 f' z^2 + f')$
$aV_1 + bV_3 + cV_6$	$\{\frac{(a-bx)^3}{t} + *, tu^3\}$	$b^4 x^3 f' = -36a^2 \ln(-a + bx) f' f^3 +$ $ab^2 c\varepsilon t f''' + ab\varepsilon t x^4 f f^2 f''$

In the Table 3, * is considered as

$$* = \frac{-3c}{bt}(a - bx)^2 \left(2(a - bx)\ln(-a + bx) + 3a\right).$$

6. Conclusions

In the present work, we studied the approximate symmetry group of the perturbed modified Korteweg-de Vries Kuramoto-Sivashinsky (mKdV-KS) equation and analyzed its Lie algebra. Moreover, the optimal system of one-dimensional Lie algebras of this equation were computed. Also, approximately differential invariants and approximate invariant solutions of perturbed mKdV-KS equation were obtained.

Availability of data and materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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