Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 10, No. 4, 2022, pp. 905-913
DOI:10.22034/CMDE.2021.36045.1624

Optical Solitons and Rogue wave solutions of NLSE with variables coefficients and modulation instability analysis

Ebru Cavlak Aslan[1,∗] **and Mustafa Inc**[2,3]

[1]Fırat University, Science Faculty, Department of Mathematics, 23119, Elazig, Turkey.

[2]Department of Computer Engineering, Biruni University, Istanbul, Turkey.

[3]Department of Medical Research, China Medical University Hospital, China Medical University,Taichung, Taiwan.

Abstract

In this work, we investigate soliton solutions of the generalized variable coefficients nonlinear Schrödinger equation. The Jacobi elliptic ansatz method is applied to obtain the optical soliton solutions. The necessary conditions that warrant the presence of these solutions are determined. We consider the Lie symmetry analysis of governing equation. Also, the stability of this equation is analyzed by the modulation instability.

Keywords. NLSE, Modified Jacobi elliptic functions, Optical soliton, Rogue wave, The exp-function approach.

2010 Mathematics Subject Classification. 35Q55, 35J10, 33E05.

1. Introduction

The Rogue waves are unusual high-amplitude phenomenons that are characterized by so-called extreme values statical dispersion. The optical rogue waves were firstly studied by Solli and his friends [22]. They have indicated ”the large probability of encountering an extremely great rogue wave in the open ocean” [14, 16, 22, 23]. Following these studies, optical rogue waves have been studied extensively in various fields; one-dimensional optical systems, photonic crystal fibers, hydrodynamics, acoustic, finance [3, 5, 13, 17–19, 21, 27, 31].

We study the properties of solitons in nonlinear optic which can be described by the following variable-coefficients NLS equation (vcNLSE) [17].

$$i\frac{\partial q}{\partial x}+i\alpha(x)\frac{\partial q}{\partial t}+\beta(x)\frac{\partial^2 q}{\partial t^2}+\gamma(x)\left|q\right|^2 q=0, \tag{1.1}$$

where $q(x,t)$is the temporal envelope of solitons. $\alpha(x)$, $\beta(x)$ represent different GVD (group velocity dispersion) coefficients and $\gamma(x)$ represent nonlinearity coefficients [17].

Our aim is to investigate the new solitary wave solutions to the vcNLSE by using the modified Jacobi elliptic functions. In section 3, we perform Lie symmetry analysis for vcNLSE [10]-[32]. Finally, in section 4, we consider the phenomenon of modulation instability.

2. Soliton Solutions

To solve Eq.(1.1) by the modified Jacobi elliptic functions, the initial assumption is

$$q(x,t)=P(x,t)e^{i\phi(x,t)}, \tag{2.1}$$

where

$$\phi(x,t)=-\kappa x+\omega t+\theta. \tag{2.2}$$

Received: 09 October 2019 ; Accepted: 30 January 2022.
∗ Corresponding author. Email: ebrucavlak@hotmail.com.

In Eq. (2.2), κ is the frequency, ω is the wave number and θ is the phase constant of soliton, [1, 2, 6–8, 11, 12, 24–26, 29, 30].

We replace (2.1) by (1.1), real and imaginary parts are as follows, respectively,

$$\kappa P - \omega\alpha P + \beta\frac{\partial^2 P}{\partial t^2} - \omega^2\beta P + \gamma P^3 = 0, \tag{2.3}$$

and

$$\frac{\partial P}{\partial x} + \alpha\frac{\partial P}{\partial t} + 2\omega\beta\frac{\partial P}{\partial t} = 0. \tag{2.4}$$

The imaginary part yields

$$v = \frac{1}{\alpha + 2\omega\beta}, \tag{2.5}$$

and where $\alpha + 2\omega\beta \neq 0$. From the real part of Eq. (2.3), we obtain the following soliton solutions.

2.1. **Rogue Dark-Soliton Solutions of vcNLSE.** We suppose that P as follows

$$P(x,t) = \mu_0 + \mu_1 sn^{p_1}(\xi, l) \tag{2.6}$$

with

$$\xi = B_1(x - vt), \tag{2.7}$$

where B_1 is the inverse width of soliton and l is the modulus. By substituting Eq. (2.6) into Eq. (2.3), we get

$$\begin{aligned} &\beta\mu_1 B_1^2 v^2 p_1\{(p_1-1)sn^{p_1-2} + (p_1-2)(l^2+1)sn^{p_1} + (p_1+1)l^2 sn^{p_1+2}\} \\ &+ \gamma\mu_0^3 + (\kappa - \omega\alpha - \omega^2\beta)\mu_0 + (3\gamma\mu_0^2\mu_1 + (\kappa - \omega\alpha - \omega^2\beta)\mu_1)sn^{p_1} \\ &3\gamma\mu_0\mu_1^2 sn^{2p_1} + \gamma\mu_1^3 sn^{3p_1} = 0. \end{aligned} \tag{2.8}$$

From this place, equating of the coefficients of sn $(p_1+2, 3p_1)$ leads to

$$p_1 = 1. \tag{2.9}$$

So, the coefficients sn^{p_i+j} equal to zero, we obtain

$$\omega = \frac{-\alpha \pm \sqrt{\alpha^2 + 4\beta(\kappa + \gamma\mu_0^2)}}{2\beta}, \tag{2.10}$$

and

$$B_1 = \sqrt{\frac{-\gamma}{2\beta}\frac{\mu_1}{vl}}. \tag{2.11}$$

So, we get

$$q(x,t) = (\mu_0 + \mu_1 sn^{p_1}(\xi, l))e^{i\phi(x,t)}. \tag{2.12}$$

In the case of $l \to 1$, the dark optical solitary wave of the vcNLSE is given

$$q(x,t) = (\mu_0 + \mu_1 \tanh[B_1(x - vt), l])e^{i\phi(x,t)}, \tag{2.13}$$

where

$$B_1 = \sqrt{-\frac{\gamma}{2\beta}\frac{\mu_1}{v}}. \tag{2.14}$$

The solitary waves will exist provided $\gamma\beta < 0$.

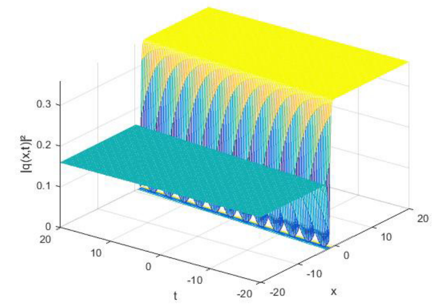

FIGURE 1. The rogue-dark solution of $|q(x,t)|^2$ with $\alpha = 2, \beta = 5, \gamma = -0.1, A_0 = 0.1, A_1 = 0.5$

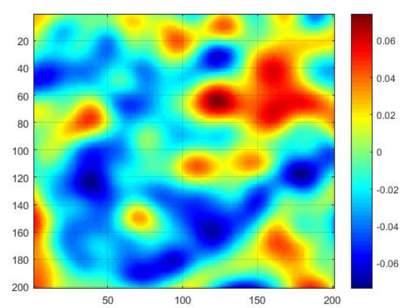

FIGURE 2. The contour plot of rogue-dark solution

2.2. **Rogue Bright-Soliton Solutions of vcNLSE.** Another modified Jacobi elliptic function solution is

$$P(x,t) = \mu_0 + \mu_2 cn^{p_2}(\xi, l), \tag{2.15}$$

with

$$\xi = B_2(x - vt). \tag{2.16}$$

Similarly, if Eq. (2.15) is taken into account in Eq. (2.3)

$$\begin{aligned} &\beta\mu_2 B_2^2 v^2 p_2\{-(p_2-1)(l^2-1)cn^{p_2-2} - (p_2+1)l^2 cn^{p_2+2} + p_2(2l^2-1)cn^{p_2}\} \\ &+\gamma\mu_0^3 + (\kappa - \omega\alpha - \omega^2\beta)\mu_0 + (3\gamma\mu_0^2\mu_2 + (\kappa - \omega\alpha - \omega^2\beta)\mu_2)cn^{p_2} \\ &3\gamma\mu_0\mu_2^2 cn^{2p_2} + \gamma\mu_2^3 cn^{3p_2} = 0, \end{aligned} \tag{2.17}$$

and here

$$p_2 = 1. \tag{2.18}$$

By operations similar to case 1, w and b are obtained as following

$$\omega = \frac{-\alpha \pm \sqrt{\alpha^2 + 4\beta(\kappa + \gamma\mu_0^2)}}{2\beta}, \tag{2.19}$$

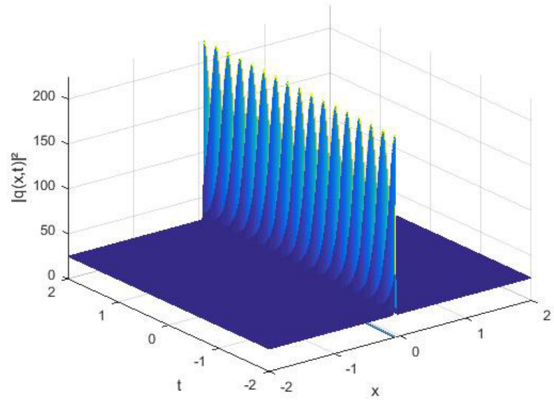

FIGURE 3. The rogue-bright solution of $|q(x,t)|^2$ with $\alpha = 2, \beta = 5, \gamma = -0.1, A_0 = 0.1, A_1 = 0.5$

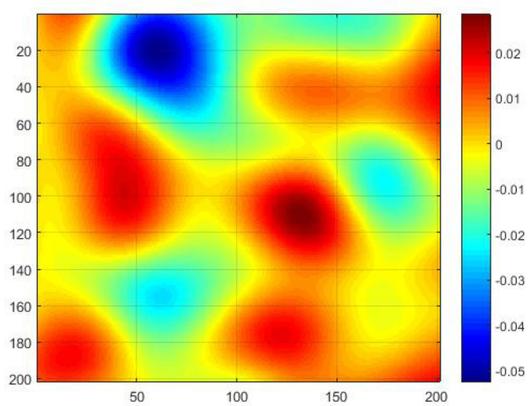

FIGURE 4. The contour plot of rogue-bright solution

and

$$B_2 = \sqrt{\frac{\gamma}{2\beta}\frac{\mu_2}{vl}}. \tag{2.20}$$

Under these conditions, we get

$$q(x,t) = (\mu_0 + \mu_2 cn^{p_2}(\xi, l))e^{i\phi(x,t)}. \tag{2.21}$$

In case of $l \to 1$, the bright optical soliton is given

$$q(x,t) = (\mu_0 + \mu_2 \sec h[B_2(x - vt), l])e^{i\phi(x,t)}, \tag{2.22}$$

where

$$B_2 = \sqrt{\frac{\gamma}{2\beta}\frac{\mu_2}{v}}. \tag{2.23}$$

Here, the necessary condition for soliton presence is $\gamma\beta > 0$.

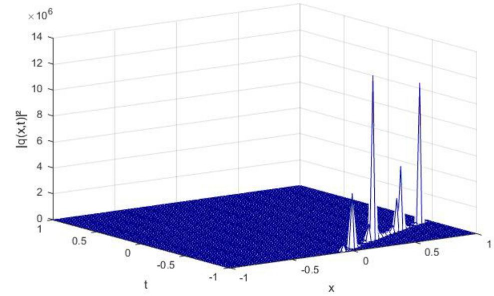

FIGURE 5. The Rogue wave solutions given by Eq. (2.27) with $\alpha = 0.1, \beta = 1, \omega = 0.5, a_1 = 1, 5, b_0 = 1, \gamma = 0.5xe^{(-0.1x^2)}$.

2.3. **Rogue Wave Solution of vcNLSE.** Considering the exp-function approach, we present a transformation such that

$$q(x,t) = U(\eta)e^{i\varphi}, \ \eta = x - vt, \ \varphi = -\kappa x - wt + \theta \tag{2.24}$$

[6, 10–12, 29, 33]. The reduced equation is as follows

$$(\kappa - wx - w^2\beta)U + \beta v^2 U'' + \gamma U^3 = 0. \tag{2.25}$$

The solution form of Eq. (2.25) is

$$U(\eta) = \frac{a_{-1}e^{-\eta} + a_0 + a_1 e^{\eta}}{b_{-1}e^{-\eta} + b_0 + b_1 e^{\eta}}, \tag{2.26}$$

$$q(x,t) = \left(\frac{8\beta a_1 b_0 e^{\eta}}{(\alpha + 2\beta w)^2 \gamma a_1^2 + 8\beta b_0^2 e^{2\eta}} \right) e^{i\varphi}. \tag{2.27}$$

We replace Eq. (2.26) by Eq. (2.25), and from the coefficients of $\exp(\eta)$, we have the exact solution of the governing

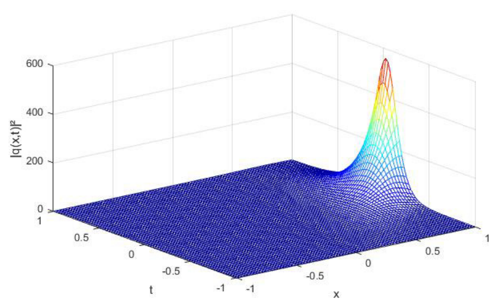

FIGURE 6. The Rogue wave solutions given by Eq. (2.27) with $\alpha = 0.01, \beta = 0.5, \omega = 0.5, a_1 = 0, 5, b_0 = 1, \gamma = 5x^2e^{x^2}$.

model

$$a_0 = 0, a_2 = 0, b_1 = 0, b_2 = \frac{\gamma a_1^2}{8v^2\beta b_0}, v = \frac{1}{\alpha + 2\beta w}.$$

3. Lie Symmetry Analysis

For the complex-valued function, the division into real and imaginary parts yield as

$$q(x,t) = u(x,t)e^{iv(x,t)} \tag{3.1}$$

[33]-[35]. So, Eq. (1.1) decompose into the following system of equations

$$\begin{aligned} uv_x &= \gamma u^3 - \alpha u v_t - \beta u v_t^2 + \beta u_{tt}, \\ u_x &= -\alpha u_t - 2\beta u_t v_t - \beta u v_{tt}. \end{aligned} \tag{3.2}$$

Considering the Lie group of point transformations

$$\begin{aligned} x^* &= x + \epsilon\sigma_1(x,t,u,v) + O(\epsilon^2), \\ t^* &= t + \epsilon\sigma_2(x,t,u,v) + O(\epsilon^2), \\ u^* &= u + \epsilon\eta_1(x,t,u,v) + O(\epsilon^2), \\ v^* &= v + \epsilon\eta_2(x,t,u,v) + O(\epsilon^2), \end{aligned} \tag{3.3}$$

where $\epsilon << 1$[28]-[9].

The vector field of group transformations is follows

$$\Gamma = \sigma_1(x,t,u,v)\frac{\partial}{\partial x} + \sigma_2(x,t,u,v)\frac{\partial}{\partial t} + \eta_1(x,t,u,v)\frac{\partial}{\partial u} + \eta_2(x,t,u,v)\frac{\partial}{\partial v}. \tag{3.4}$$

Admits the following infinitesimals

$$\begin{aligned} &\sigma_1(x,t,u,v) = 2C_2x + C_5, \\ &\sigma_2(x,t,u,v) = 2C_1\beta x + C_2\alpha x + C_2 t + C_4, \\ &\eta_1(x,t,u,v) = -C_2u, \\ &\eta_2(x,t,u,v) = -C_1\alpha x + C_1 t + C_3, \end{aligned} \tag{3.5}$$

where C_1, C_2, C_3, C_4 and C_5 are arbitrary constants. The Lie point symmetries of Eq. (1.1) is generated by five vector fields

$$\begin{aligned} V_1 &= 2\beta x\frac{\partial}{\partial t} + (-\alpha x + t)\frac{\partial}{\partial x}, \\ V_2 &= -u\frac{\partial}{\partial u} + 2x\frac{\partial}{\partial x} + (\alpha x + t)\frac{\partial}{\partial t}, \\ V_3 &= \frac{\partial}{\partial v}, \\ V_4 &= \frac{\partial}{\partial t}, \\ V_5 &= \frac{\partial}{\partial x}. \end{aligned} \tag{3.6}$$

4. Modulation Instability

Here, we explored modulation instability of Eq. (1.1). The modulation instability is considered to accent the importance of the inter plays between the dispersive and nonlinear effects that can occur in the anomalous dispersion of optical fibers. The Eq. (1.1) has the steady state solution

$$q(x,t) = \sqrt{P_0}e^{i\phi_{NL}}, \tag{4.1}$$

where P_0 is the optical power. ϕ_{NL} is the nonlinear phase shift induced by the Self-phase modulation (SPM) [2]. So, we consider the development of perturbation with Eq. (4.1)

$$q(x,t) = (\sqrt{P_0} + \Psi(x,t))e^{i\phi_{NL}}. \tag{4.2}$$

Substituting Eq. (4.2) into Eq. (1.1) and linearizing $\Psi(x,t)$, we obtain the following equation

$$i\frac{\partial\Psi}{\partial x}+i\alpha\frac{\partial\Psi}{\partial t}+\beta\frac{\partial^2\Psi}{\partial t^2}-\gamma P_0(\sqrt{P_0}+\Psi)+\gamma(P_0+\Psi)^3=0, \tag{4.3}$$

and considering the solution of Eq. (4.3) in the form

$$\Psi(x,t)=\Psi_1(x,t)e^{i\lambda}+\Psi_2(x,t)e^{-i\lambda}, \tag{4.4}$$

where $\lambda=Wx-Kt$. W and K are the wave number and the frequency of perturbation, respectively. From Eq. (4.3) and Eq. (4.4), acquire two homogeneous equation for Ψ_1 and Ψ_2. So

$$W=\alpha K\pm\beta K^2\pm(-2P_0\gamma). \tag{4.5}$$

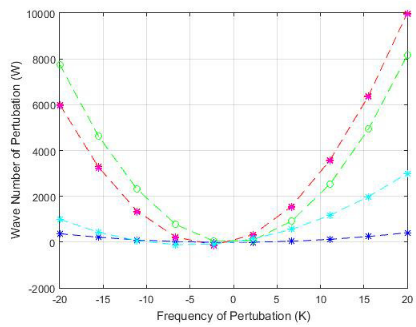

FIGURE 7. The relationship between frequency K and wave numbers W.

In case of the normal GVD (Group Velocity Dispersion), the wave number W is real for all K and the steady state is stable. On the contrary, in case of the anomalous GVD, W becomes imaginary for K. So the continous wave solution Eq.(4.1) is unstable by anomalous of GVD and this unstable is called modulation stability.

5. Conclusion

To conclude, the modified Jacobi elliptic functions are used to the exact solutions of generalized variable coefficients NLS equation and obtained the new dark- bright optical solitons. We have also found the Rogue wave solutions by using exp-function approach. The mentioned cases of rogue-dark and rogue-bright optical solitons Eq. (2.13) and Eq. (2.22) are shown in Figures 1-4. For the cases to the Rogue wave solutions (2.27) are presented in Figures 5-6. We have considered the vcNLSE by using the Lie symmetry analysis. Moreover, we wanted to demonstrate the modulation instability of the vcNLSE. In Figure 7, we give the relation between K frequency and W wave numbers of Eq. (4.5) for different values of α,β,P_0 and $\gamma=0.5xe^{-0.1x^2}$.

References

[1] S. S. Afzal, M. Younis, and S. T. R.Rizvi, *Optical dark and dark-singular solitons with anti-cubic nonlinearity*, Optik, *147* (2017), 27-31.
[2] G. P. Agrawal, Nonlinear Fiber Optics, Academic Press, USA, 2007.
[3] S. Ali, M. Younis, M. O. Ahmad, and S. T. R. Rizvi, *Rogue wave solutions in nonlinear optics with coupled Schrödinger equations*, Optical and Quantum Electronics, *50* (2018), 266.
DOI.10.1007/s11082-018-1526-9.

[4] M. M. Alipour, G. Domairy, and A. G. Dovadi, *An Application of Exp-Function Method to Approximate General and Explicit Solutions for Nonlinear Schrödinger Equations*, Numerical Methods for Partial Differential Equations, *27* (2011), 1016-1025.

[5] A. Ankiewicz and N. Akhmediev, *Rogue wave-type solutions of the mKdV equation and their relation to known NLSE rogue wave solutions*, Nonlinear Dyn., *91* (2018), 1931-1938.

[6] E. C. Aslan and M. Inc, *Soliton Solutions of NLSE with Quadratic-Cubic Nonlinearity and Stability Analysis*, Waves in Random and Complex Media, *27* (2017), 594-601.

[7] E. C. Aslan, M. Inc, and D. Baleanu, *Optical solitons and stability analysis of the NLSE with anti-cubic nonlinearity*, Superlattices and Microstructures, *109* (2017), 784-793.

[8] M. Asma, W. A. M. Othman, B. R. Wong, and A. Biswas, *Optical Soliton Perturbation with Quadratic-Cubic Nonlinearity by Semi-Inverse Variational Principle*, Proceedings of The Romanian Academy, Series A, *18* (2017), 331-336.

[9] A. Bansal, A. Biswas, Q. Zhou, and M. M. Babatin, *Lie symmetry analysis for cubic–quartic nonlinear Schrödinger's equation*, Optik, *169* (2018), 12-15.

[10] A. Bansal, A. H. Kara, A. Biswas, S. P. Moshokoa, and M. Belic, *Optical soliton perturbation, group invariants and conservation laws of perturb e d Fokas–Lenells equation*, Chaos, Solitons and Fractals, *114* (2018), 275-280.

[11] A. Biswas, M. Ekici, A. Sonmezoğlu, and M. R. Belic, *Optical solitons in birefringent fibers having anti-cubic nonlinearity with extended trial function*, Optik, *185* (2019), 456-463.

[12] A. Biswas, *Optical soliton perturbation with Radhakrishnan–Kundu–Lakshmanan equation by traveling wave hypothesis*, Optik, *171* (2018), 217-220.

[13] M. Dehghan, J. Manafian, and H. A Saadatmandi, *Application of semi-analytic methods for theFitzhugh–Nagumo equation*, which modelsthe transmission of nerve impulses, Math. Methods Appl. Sci., *33* (2010), 1384–1398.

[14] H. I. A. Gawad, M. Tantawy, and R. E. A. Elkhair, *On the extension of solutions of the real to complex KdV equation and a mechanism for the construction of rogue waves*, Waves in Random and Complex Media, *26* (2016), 397-406.

[15] K. Hosseini, J. Manafian, F. Samadani, M. Foroutan, M. Mirzazadeh, and Q. Zhou, *Resonant optical solitons with perturbation terms andfractional temporal evolution using improved* $\tan(\phi(\eta)/2)$*-expansion method and exp function approach*, Optik, *158* (2018), 933-939.

[16] S. L. Jia, Y. T. Gao, C. Zhao, J. W. Yang, and Y. J. Feng, *Breathers and rogue waves for an eighth-order variable-coefficient nonlinear Schrödinger equation in an ocean or optical fiber Waves*, Random and Complex Media, *27*(2017), 544–561.

[17] B. Q. Li and Y. L. Man, *Rogue waves for the optics fiber system with variable coefficients*, Optik, *158* (2018), 177-187.

[18] J. Manafian, *Optical soliton solutions for Schrödinger type nonlinear evolution equations by the tan*$(\phi/2)$*-expansion method*, Optik-Int. J. Elec. Opt., *127* (2016), 4222-4245.

[19] J. Manafian and M. Lakestani, *Dispersive dark optical soliton with Tzitze ica type nonlinear evolution equations arising in nonlinear optics*, Opt. Quantum Electron., *48* (2016), 1-32.

[20] J. Manafian, *On the complex structures of the Biswas-Milovic equation for power, parabolic and dual parabolic law nonlinearities*, The Eur. Phys. J. Plus, *130* (2015), 1-20.

[21] C. Y. Qin, S. F. Tian, X. B. Wang, T. T. Zhang, and J. Li, *Rogue waves, bright–dark solitons and traveling wave solutions of the (3 + 1)-dimensional generalized Kadomtsev–Petviashvili equation*, Computers and Mathematics with Applications, *75* (2018), 4221-4231.

[22] D. R. Solli, C. Ropers, P. Koonath, and B. Jalali, *Optical rogue waves*, Nature, *450* (2007), 1054-1057.

[23] D. R. Solli, C. Ropers, P. Koonath, and B. Jalali, *Active control of rogue waves for simulated super continuum generation*, Phys. Rev. Lett., *101* (2008), 275–278.

[24] K. U. Tariq and M. Younis, *Bright, dark and other optical solitons with second order spatiotemporal dispersion*, Optik, *142* (2017), 446-450.

[25] K. U.Tariq, M. Younis, and S. T. R. Rizvi, *Optical solitons in monomode fibers with higher order nonlinear Schrödinger equation*, Optik, *154* (2018), 360-371.

[26] F. Tchier, E. Cavlak Aslan, and M. Inc, *Optical solitons for cascased system: Jacobi elliptic functions*, J. Modern Optics, *63* (2016), 2298-2307.
[27] Y. Yang, X. Wang, and Z. Yan, *Optical temporal rogue waves in the generalized inhomogeneous nonlinear Schrödinger equation with varying higher-order even and odd terms*, Nonlinear Dyn., *81* (2015), 833-842.
[28] Z. Yang and A. F. Cheviakov, *Some relations between symmetries of nonlocally related systems*, Journal of Mathematical Physics, *55* (2014), 083514.
[29] M. Younis, U.Younas, S. Rehman, M. Bilal, and A. Waheed, *Optical bright–dark and Gaussian soliton with third order dispersion*, Optik, *134* (2017), 233-238.
[30] M. Younis, S. Ali, and S. A. Mahmood, *Solitons for compound KdV–Burgers equation with variable coefficients and power law nonlinearity*, Nonlinear Dyn, *81*(2015), 1191–1196.
[31] X. Zhang and Y. Chen, *Deformation rogue wave to the (2+1)-dimensional KdV equation*, Nonlinear Dyn., *90*(2017), 755-763.
[32] Z. Küçükarslan Yüzbaşı E. Cavlak Aslan, M. Inc, and D. Baleanu, *On Exact Solutions for New Coupled Non-Linear Models Getting Evolution of Curves in Galilean Space*, Thermal Science, *23* (2019), 227-233.
[33] Z. Küçükarslan Yüzbaşı E. Cavlak Aslan, and M. Inc, *Lie Symmetry Analysis and Exact Solutions of Tzitzeica Surfaces PDE in Galilean Space*, Journal of Advanced Physics, *7* (2018), 88-91.
[34] Z. Küçükarslan Yüzbaşı E. Cavlak Aslan, and M. Inc, *Exact Solutions with Lie Symmetry Analysis for Nano-Ionic Currents along Microtubules*, ITM Web of Conferences, *22* (2018), 01017.
[35] Z. Küçükarslan Yüzbaşı E. Cavlak Aslan, D. Baleanu, and M. Inc, *Evolution of Plane Curves via Lie Symmetry Analysis in the Galilean Plane. Numerical Solutions of Realistic Nonlinear Phenomena*, Springer International Publishing, Switzerland, Chapter 12, 2020. DOI: 10.1007/978-3-030-37141-8.