



## On the numerical scheme for solving non-linear two-dimensional Hammerstein integral equations

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### Abstract

In this work, solving non-linear two-dimensional Hammerstein integral equations is considered by an iterative method of successive approximation. This method is an efficient approach based on a combination of the quadrature formula and the successive approximations method. Also, the convergence analysis and the numerical stability of the suggested method are studied. Finally, to survey the accuracy of the present method, some numerical experiments are given.

**Keywords.** Fixed point theorem, Hammerstein integral equations, Quadrature formula, Iterative method.

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### 1. INTRODUCTION

Non-linear two-dimensional Fredholm integral equations of the second kind appear in the mathematical modeling of various physical phenomena (such as electro-magnetic fluid dynamics), engineering problems and applied sciences [8, 18, 21, 31, 32, 34–36]. The numerical methods elaborated for two-dimensional non-linear integral equations involve various techniques: the Galerkin and collocations methods [13, 17], successive approximations and other iterative techniques [7, 12, 24, 25, 30, 38, 39], the Nyström type methods [16], the piecewise approximation by Chebyshev polynomials [15], the operational matrix operational matrix [11, 26, 33], wavelet method [9], the rationalized Haar functions [10], triangular and block-pulse functions [29], multi-step methods [37], Runge-Kutta method [27], neural network method [20], expansion method [28] and the regularization-homotopy method [3].

For our purpose here, we consider non-linear two-dimensional Hammerstein Fredholm integral equation of the second kind (2D-NHFIE) as follows

$$X(s, t) = r(s, t) + \lambda \int_c^d \int_a^b K(s, t, x, y) \psi(x, y, X(x, y)) dx dy, \quad (s, t) \in I, \quad (1.1)$$

and focus on a numerical iterative algorithm for solving it, where  $a, b, c, d \in \mathbb{R}$ ,  $r : I = [a, b] \times [c, d] \rightarrow \mathbb{R}$ ,  $K : I \times I \rightarrow \mathbb{R}$  and  $r, K$  are continuous. Having a strong physical background, the Hammerstein equations arise from electro-magnetic fluid dynamics. In particular, Eq. (1.1) arises from various reformulations of an elliptic partial differential equation with non-linear boundary conditions [4, 6]. Also, one-dimensional analogues of Eq. (1.1) are a reformulation of two-point boundary value problems with a certain non-linear boundary condition, [5, 14]. It should be noted that as a special case of the Hammerstein integral equations, the Fredholm integral equations of the second kind can be considered.

Here, by the combination of the fixed point technique and Simpson's rule, a numerical method is presented for solving Eq. (1.1). Our method is an iterative procedure using the successive approximations method.

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Some results about the existence and uniqueness of the solution of non-linear 2D integral equations can be studied in [1, 2, 19, 22, 23]. In this paper, the Banach fixed point theorem is used to prove the existence and uniqueness of the solution of Eq. (1.1).

We arranged the remainder of the paper as follows: In section 2, some required definitions and theorems are provided. In section 3, an iterative procedure based on quadrature formula is presented. Also, the numerical stability and the convergence are investigated. Numerical results are presented in section 4. Moreover, a comparison is made between the method [24] and our method, to show the efficiency of our method. Some conclusions are drawn in section 5.

## 2. PRELIMINARIES

In current section, some basic definitions and required theorems that will be needed are expressed.

**Definition 2.1.** For  $L \geq 0$ , the function  $f : I \rightarrow \mathbb{R}$  is  $L$ -Lipschitz if

$$|f(x, y) - f(x', y')| \leq L(|x - x'| + |y - y'|), \quad \forall x, x' \in [a, b] \text{ and } y, y' \in [c, d].$$

**Theorem 2.2.** ([24]) Let  $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ , be a  $L$ -Lipschitz function. Then, for any divisions  $a = x_0 < x_1 < \dots < x_n = b$ ,  $c = y_0 < y_1 < \dots < y_n = d$  and any points  $\xi_i \in [x_{i-1}, x_i]$ ,  $\eta_j \in [y_{j-1}, y_j]$ , we have

$$\left| \int_c^d \int_a^b f(s, t) ds dt - \sum_{j=1}^n \sum_{i=1}^n (x_i - x_{i-1})(y_j - y_{j-1}) f(\xi_i, \eta_j) \right| \leq \frac{L}{2} \sum_{j=1}^n \sum_{i=1}^n (y_j - y_{j-1})(x_i - x_{i-1})^2 + \frac{L}{2} \sum_{j=1}^n \sum_{i=1}^n (x_i - x_{i-1})(y_j - y_{j-1})^2.$$

**Corollary 2.3.** Let  $f$  be the function defined in Theorem 2.2. Then, we have

$$\begin{aligned} \left| \int_c^d \int_a^b f(s, t) ds dt - ((\gamma - \theta)(\alpha - a)f(u, p) + (\theta - c)(\alpha - a)f(u, r) \right. \\ - (d - \gamma)(\alpha - a)f(u, s) + (\gamma - \theta)(\beta - \alpha)f(v, p) - (\beta - \alpha)(\theta - c)f(v, r) \\ + (\beta - \alpha)(d - \gamma)f(v, s) - (b - \beta)((\gamma - \theta)f(w, p) \\ + (\theta - c)f(w, r) - f(w, s)(d - \gamma)) \left. \right| \\ \leq \frac{L(d - c)}{2} ((\beta - v)^2 + (b - \beta)^2 + (v - \alpha)^2 + (\alpha - a)^2) \\ + \frac{L(b - a)}{2} ((\theta - c)^2 + (p - \theta)^2 + (\gamma - p)^2 + (d - \gamma)^2) \end{aligned}$$

for any  $\alpha, \beta, \gamma, \theta, r, s, p, u, v, w$  where  $a \leq u < \alpha < v < \beta < w \leq b$  and  $c \leq r < \theta < p < \gamma < s \leq d$ .

*Proof.* In theorem 2.2, take  $n = 4$  and

$$\begin{aligned} x_0 = a, x_1 = \alpha, x_2 = v, x_3 = \beta, x_4 = b, \xi_1 = u, \xi_2 = \xi_3 = v, \xi_4 = w, \\ y_0 = a_2, y_1 = \theta, y_2 = p, y_3 = \gamma, y_4 = d, \eta_1 = r, \eta_2 = \eta_3 = p, \eta_4 = s. \end{aligned}$$

□



**Corollary 2.4.** For the same function  $f$ , we have

$$\begin{aligned} \left| \int_c^d \int_a^b f(s,t) ds dt - \frac{1}{36}(b-a)(d-c) \left( f(a,c) + f(a,d) + f(b,d) + f(b,c) \right. \right. \\ \left. \left. + 4\left(f\left(\frac{a+b}{2},c\right) + f\left(\frac{a+b}{2},d\right) + f\left(a,\frac{c+d}{2}\right) + f\left(b,\frac{c+d}{2}\right)\right) \right. \right. \\ \left. \left. + 16f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \right) \right| \\ \leq \frac{5}{36}L(b-a)(d-c)(b-a+d-c) \end{aligned}$$

for all  $x \in [a, b], u \in [a, x], v \in [x, b], y \in [c, d], \alpha \in [c, y]$  and  $\beta \in [y, d]$ .

*Proof.* In Corollary 2.3, take

$$\begin{aligned} \alpha &= \frac{5a+b}{6}, v = \frac{a+b}{2}, \beta = \frac{a+5b}{6}, u = a, w = b, \\ \theta &= \frac{5c+d}{6}, p = \frac{c+d}{2}, \gamma = \frac{c+5d}{6}, r = c, s = d. \end{aligned}$$

□

In fact, the recent corollary is the classical Simpson’s rule with a new error bound. Also, The Corollary 2.4, is extended for uniform partitions in the following Corollary.

**Corollary 2.5.** Let  $D_x : a = s_0 < s_1 < s_2 < \dots < s_{2n-1} < s_{2n} = b, D_y : c = t_0 < t_1 < t_2 < \dots < t_{2n-1} < t_{2n} = d$ , with  $s_i = a + ih_x, t_j = c + jh_y$ , where  $h_x = \frac{b-a}{2n}, h_y = \frac{d-c}{2n}, i, j = 0, 1, 2, \dots, 2n$ , then

$$\left| \int_c^d \int_a^b f(s,t) ds dt - S(f) \right| \leq \frac{5L}{18}(b-a)(d-c)(h_x + h_y)$$

where

$$\begin{aligned} S(f) &= \sum_{j=1}^n \sum_{i=1}^n \frac{1}{36}(s_{2i-2} - s_{2i})(t_{2j-2} - t_{2j}) \\ &\times \left[ \left( f(s_{2i-2}, t_{2j-2}) + f(s_{2i-2}, t_{2j}) + f(s_{2i}, t_{2j}) + f(s_{2i}, t_{2j-2}) \right) \right. \\ &+ 4 \left( f(s_{2i-1}, t_{2j-2}) + f(s_{2i-1}, t_{2j}) + f(s_{2i-2}, t_{2j-1}) + f(s_{2i}, t_{2j-1}) \right) \\ &\left. + 16f(s_{2i-1}, t_{2j-1}) \right] \end{aligned}$$



*Proof.* By previous corollary, we have

$$\begin{aligned}
 \left| \int_{t_{j-2}}^{t_j} \int_{s_{2i-2}}^{s_{2i}} f(s, t) ds dt - \frac{1}{36} (s_{2i} - s_{2i-2})(t_j - t_{j-2}) \right. \\
 \times \left[ \left( f(s_{2i-2}, t_{j-2}) + f(s_{2i-2}, t_j) + f(s_{2i}, t_j) + f(s_{2i}, t_{j-2}) \right) \right. \\
 + 4 \left( f(s_{2i-1}, t_{j-2}) + f(s_{2i-1}, t_j) + f(s_{2i-2}, t_{j-1}) + f(s_{2i}, t_{j-1}) \right) \\
 \left. \left. + 16f(s_{2i-1}, t_{j-1}) \right] \right| \\
 \leq \frac{5L}{36} (s_{2i} - s_{2i-2})(t_j - t_{j-2})((s_{2i} - s_{2i-2}) + (t_j - t_{j-2}))
 \end{aligned}$$

Where  $L$  is the Lipschitz constant of  $f$ . So, we have

$$\begin{aligned}
 \left| \int_c^d \int_a^b f(s, t) ds dt - S(f) \right| &= \left| \sum_{j=1}^n \sum_{i=1}^n \int_{t_{j-2}}^{t_j} \int_{s_{2i-2}}^{s_{2i}} f(s, t) ds dt - S(f) \right| \\
 &\leq \sum_{j=1}^n \sum_{i=1}^n \left| \int_{t_{j-2}}^{t_j} \int_{s_{2i-2}}^{s_{2i}} f(s, t) ds dt - \frac{1}{36} (s_{2i} - s_{2i-2})(t_j - t_{j-2}) \right. \\
 &\times \left[ \left( f(s_{2i-2}, t_{j-2}) + f(s_{2i-2}, t_j) + f(s_{2i}, t_j) + f(s_{2i}, t_{j-2}) \right) \right. \\
 &+ 4 \left( f(s_{2i-1}, t_{j-2}) + f(s_{2i-1}, t_j) + f(s_{2i-2}, t_{j-1}) + f(s_{2i}, t_{j-1}) \right) \\
 &\left. \left. + 16f(s_{2i-1}, t_{j-1}) \right] \right| \\
 &\leq \sum_{j=1}^n \sum_{i=1}^n \frac{5L}{36} (s_{2i} - s_{2i-2})(t_j - t_{j-2})((s_{2i} - s_{2i-2}) + (t_j - t_{j-2})) \\
 &\leq \sum_{j=1}^n \sum_{i=1}^n \frac{5L}{36} \left( \frac{b-a}{n} \right) \left( \frac{d-c}{n} \right) \left( \frac{b-a}{n} + \frac{d-c}{n} \right) \\
 &\leq \frac{5L}{18} (b-a)(d-c)(h_x + h_y).
 \end{aligned}$$

□

### 3. MAIN RESULTS

#### 3.1. The existence problem.

**Theorem 3.1.** ([24]) Under the conditions

- (i)  $r \in C(I, \mathbb{R})$ ,  $K \in C(I \times I, \mathbb{R})$ ,  $\psi \in C(I \times \mathbb{R}, \mathbb{R})$  where  $C(I, \mathbb{R}) = \{f : I \rightarrow \mathbb{R}; f \text{ is continuous}\}$ ,
- (ii) there exist  $\alpha, \beta > 0$ , such that  $|\psi(x, y, u) - \psi(x', y', u')| \leq \alpha(|x - x'| + |y - y'|) + \beta|u - u'|$ ,  $\forall (x, y) \in I, \forall u, u' \in \mathbb{R}$ .
- (iii)  $\Theta = \beta \lambda M_k (b-a)(d-c) < 1$ , where  $M_k > 0$  is such that  $|K(s, t, x, y)| \leq M_k$ ,  $\forall s, x \in [a, b], t, y \in [c, d]$ , according continuity of  $K$ ,



the Eq. (1.1) has a unique solution  $X^* \in \mathbf{X}$ , and the following sequence of successive approximations

$$\begin{aligned}
 X_0(s, t) &= r(s, t), \\
 X_m(s, t) &= r(s, t) + \lambda \int_c^d \int_a^b K(s, t, x, y) \psi(x, y, X_{m-1}(x, y)) dx dy, \quad m \geq 1,
 \end{aligned}
 \tag{3.1}$$

converges to the solution  $X^*$ . Also, the following error estimates hold.

$$\|X^* - X_m\| \leq \frac{\Theta^m}{1 - \Theta} \|X_0 - X_1\|,
 \tag{3.2}$$

$$\|X^* - X_m\| \leq \frac{\Theta}{1 - \Theta} \|X_{m-1} - X_m\|,
 \tag{3.3}$$

and choosing  $X_0 \in \mathbf{X}, X_0 = r$ , the inequality (3.2) becomes

$$\|X^* - X_m\| \leq \frac{\Theta^{m+1}}{\beta(1 - \Theta)} M_0,
 \tag{3.4}$$

where  $M_0$  is given in (3.9).

Now, for implementation of the method, we consider Eq. (1.1) where  $K(s, t, x, y)$  is a continuous kernel defined on  $[a, b] \times [c, d] \times [a, b] \times [c, d]$ . Also, let

$$\begin{aligned}
 D_x &: a = s_0 < s_1 < s_2 < \dots < s_{2n-1} < s_{2n} = b, \\
 D_y &: c = t_0 < t_1 < t_2 < \dots < t_{2n-1} < t_{2n} = d,
 \end{aligned}
 \tag{3.5}$$

are uniform partitions with  $s_i = a + ih_x, t_j = c + jh_y$ , where  $h_x = \frac{b-a}{2n}, h_y = \frac{d-c}{2n}, i, j = 0, 1, 2, \dots, 2n$  or  $i, j = \overline{0, 2n}$ . Then, the approximate solution of Eq. (1.1) in  $(s, t)$  is obtained by the following iterative procedure.

$$X_0(s, t) = r(s, t)
 \tag{3.6}$$

$$X_m(s, t) = r(s, t) + \frac{\lambda h_x h_y}{9} \sum_{i=1}^n \sum_{j=1}^n \sum_{l_2=0}^2 \sum_{l_1=0}^2 (C_2^{l_2})^2 (C_2^{l_1})^2
 \tag{3.7}$$

$$\times K(s, t, s_{2i-2+l_1}, t_{2j-2+l_2}) \psi(s_{2i-2+l_1}, t_{2j-2+l_2}, \bar{X}_{m-1}(s_{2i-2+l_1}, t_{2j-2+l_2})).
 \tag{3.8}$$

where

$$C_2^l = \binom{2}{l} = \frac{2!}{l!(2-l)!}, \quad l = 0, 1, 2$$

**Proposition 3.2.** Under the assumptions (i)-(iii) of Theorem (3.1), the sequence of successive approximations (3.1) is uniformly bounded. Moreover, let  $\Psi_m(x, y) = \psi(x, y, X_m(x, y)), m \in N$ , and suppose that

- (i) there exist  $\eta, \zeta > 0$ , such that
 
$$|K(s_1, t_1, x_1, y_1) - K(s_2, t_2, x_2, y_2)| \leq \zeta(|s_1 - s_2| + |t_1 - t_2|) + \eta(|x_1 - x_2| + |y_1 - y_2|),$$

$$\forall (s_1, t_1), (s_2, t_2), (x_1, y_1), (x_2, y_2) \in I.$$
- (ii) there exists  $\theta > 0$ , such that
 
$$|r(s_1, t_1) - r(s_2, t_2)| \leq \theta(|s_1 - s_2| + |t_1 - t_2|), \quad \forall (s_1, t_1), (s_2, t_2) \in I.$$

then the functions  $\Psi_m, m \in N$  is uniformly Lipschitz with constant  $L' = \alpha + \beta(\theta + \lambda(b-a)(d-c)M\zeta)$ , where  $M$  is given in (3.10).

*Proof.* Let  $\psi_0 : [a, b] \times [c, d] \rightarrow \mathbb{R}$  and  $\psi_0(x, y) = \psi(x, y, r(x, y))$ . Since  $\psi, r$  are continuous, it can be concluded that  $\psi_0$  is continuous on the compact set  $[a, b] \times [c, d]$ . Hence,  $M_0 \geq 0$  exists such that

$$|\psi_0(x, y)| \leq M_0 \quad \forall (x, y) \in [a, b] \times [c, d].
 \tag{3.9}$$



For  $(s, t) \in [a, b] \times [c, d]$ , we have

$$\begin{aligned} |X_m(s, t) - X_{m-1}(s, t)| &\leq \lambda |K(s, t, x, y)| \int_c^d \int_a^b |\psi(x, y, X_{m-1}(x, y)) - \psi(x, y, X_{m-2}(x, y))| dx dy \\ &\leq \lambda M_k \int_c^d \int_a^b |\psi(x, y, X_{m-1}(x, y)) - \psi(x, y, X_{m-2}(x, y))| dx dy \\ &\leq \beta \lambda M_k (b-a)(d-c) \max_{\substack{a \leq x \leq b \\ c \leq y \leq d}} |X_{m-1}(x, y) - X_{m-2}(x, y)| = \Theta \|X_{m-1} - X_{m-2}\|, \end{aligned}$$

and by induction,

$$|X_m(s, t) - X_{m-1}(s, t)| \leq \Theta^{m-1} \|X_1 - X_0\|.$$

So,

$$\begin{aligned} |X_m(s, t) - X_0(s, t)| &\leq |X_m(s, t) - X_{m-1}(s, t)| + \dots + |X_1(s, t) - X_0(s, t)| \\ &\leq (\Theta^{m-1} + \Theta^{m-2} + \dots + \Theta + 1) \|X_1 - X_0\| \\ &= \frac{1 - \Theta^m}{1 - \Theta} \|X_1 - X_0\| \leq \frac{\Theta M_0}{\beta(1 - \Theta)} \quad \forall (s, t) \in [a, b] \times [c, d]. \end{aligned}$$

Let  $M_r \geq 0$  such that  $|r(s, t)| \leq M_r$  for all  $(s, t) \in [a, b] \times [c, d]$ . Then

$$|X_m(s, t)| \leq |X_m(s, t) - X_0(s, t)| + |X_0(s, t)| \leq \frac{\Theta M_0}{\beta(1 - \Theta)} + M_r = L.$$

Moreover, by considering

$$M = \max(M_0, \max\{|\psi(s, t, u)| : (s, t) \in [a, b] \times [c, d], u \in [-L, L]\}), \quad (3.10)$$

we get

$$|\Psi_m(s, t)| = |\psi(s, t, X_m(s, t))| \leq M,$$

for all  $(s, t) \in [a, b] \times [c, d]$  and  $m \in \mathbf{N}$ . Let  $(s_1, t_1), (s_2, t_2) \in [a, b] \times [c, d]$ , we obtain

$$\begin{aligned} |X_0(s_1, t_1) - X_0(s_2, t_2)| &\leq \theta(|s_1 - s_2| + |t_1 - t_2|), \\ |X_m(s_1, t_1) - X_m(s_2, t_2)| &\leq |r(s_1, t_1) - r(s_2, t_2)| \\ &\quad + \lambda \int_c^d \int_a^b |K(s_1, t_1, x, y) - K(s_2, t_2, x, y)| |\psi(x, y, X_{m-1}(x, y))| dx dy \\ &\leq \theta(|s_1 - s_2| + |t_1 - t_2|) + \lambda(b-a)(d-c)M\zeta(|s_1 - s_2| + |t_1 - t_2|) \\ &= L_0(|s_1 - s_2| + |t_1 - t_2|), \end{aligned}$$

with  $L_0 = \theta + \lambda(b-a)(d-c)M\zeta$  and

$$\begin{aligned} |\Psi_0(s_1, t_1) - \Psi_0(s_2, t_2)| &\leq \alpha(|s_1 - s_2| + |t_1 - t_2|) + \beta |X_0(s_1, t_1) - X_0(s_2, t_2)| \\ &\leq (\alpha + \beta\theta)(|s_1 - s_2| + |t_1 - t_2|), \end{aligned}$$

$$\begin{aligned} |\Psi_m(s_1, t_1) - \Psi_m(s_2, t_2)| &\leq \alpha(|s_1 - s_2| + |t_1 - t_2|) + \beta |X_m(s_1, t_1) - X_m(s_2, t_2)| \\ &\leq \alpha(|s_1 - s_2| + |t_1 - t_2|) + \beta L_0(|s_1 - s_2| + |t_1 - t_2|) \\ &= (\alpha + \beta L_0)(|s_1 - s_2| + |t_1 - t_2|). \end{aligned}$$

□



**Corollary 3.3.** *The functions  $K(s_p, t_q, x, y)\psi(x, y, X_m(x, y)), p = \overline{0, 2n}, q = \overline{0, 2n}, m \in \mathbb{N}$  are uniformly Lipschitz with constant*

$$L = \eta M + M_k(\alpha + \beta(\theta + \lambda(b - a)(d - c)M\zeta)).$$

*Proof.* Let  $(s_1, t_1), (s_2, t_2) \in [a, b] \times [c, d]$ . Define the function  $\Psi_{m,p,q} : [a, b] \times [c, d] \rightarrow \mathbb{R}$ ,

$$\Psi_{m,p,q}(x, y) = K(s_p, t_q, x, y)\psi(x, y, X_m(x, y)), \quad p = \overline{0, 2n}, q = \overline{0, 2n}.$$

Then,

$$\begin{aligned} |\Psi_{m,p,q}(x_1, y_1) - \Psi_{m,p,q}(x_2, y_2)| &\leq |K(s_p, t_q, x_1, y_1)\psi(x_1, y_1, X_m(x_1, y_1)) \\ &\quad - K(s_p, t_q, x_2, y_2)\psi(x_1, y_1, X_m(x_1, y_1))| \\ &\quad + |K(s_p, t_q, x_2, y_2)\psi(x_1, y_1, X_m(x_1, y_1)) \\ &\quad - K(s_p, t_q, x_2, y_2)\psi(x_2, y_2, X_m(x_2, y_2))| \\ &\leq M|K(s_p, t_q, x_1, y_1) - K(s_p, t_q, x_2, y_2)| \\ &\quad + M_k|\psi(x_1, y_1, X_m(x_1, y_1)) - \psi(x_2, y_2, X_m(x_2, y_2))| \\ &\leq M\eta(|x_1 - x_2| + |y_1 - y_2|) + M_k(\alpha + \beta L_0)(|x_1 - x_2| + |y_1 - y_2|) \\ &\leq L(|x_1 - x_2| + |y_1 - y_2|), \end{aligned} \tag{3.11}$$

for  $m \in \mathbb{N}$ . □

**3.2. Algorithm of the approach.** Consider the uniform partitions 3.5 with  $s_p = a + p\frac{b-a}{2n}, p = \overline{0, 2n}$  and  $t_q = c + q\frac{d-c}{2n}, q = \overline{0, 2n}$ , on these knots, relation (3.1) becomes:

$$\begin{aligned} X_0(s_p, t_q) &= r(s_p, t_q), \\ X_m(s_p, t_q) &= r(s_p, t_q) + \lambda \int_c^d \int_a^b K(s_p, t_q, x, y)\psi(x, y, X_{m-1}(x, y))dxdy, \quad p = \overline{0, 2n}, q = \overline{0, 2n}, m \geq 1 \end{aligned} \tag{3.12}$$

and applying the quadrature (2.1) to relation (3.12), we obtain:

$$\begin{aligned} X_0(s_p, t_q) &= r(s_p, t_q), \\ X_m(s_p, t_q) &= r(s_p, t_q) + \frac{\lambda h_x h_y}{9} \sum_{i=1}^n \sum_{j=1}^n \sum_{l_2=0}^2 \sum_{l_1=0}^2 (C_2^{l_2})^2 (C_2^{l_1})^2 \\ &\quad K(s_p, t_q, s_{2i-2+l_1}, t_{2j-2+l_2})\psi(s_{2i-2+l_1}, t_{2j-2+l_2}, X_{m-1}(s_{2i-2+l_1}, t_{2j-2+l_2})) + R_{m,p,q}, \end{aligned} \tag{3.13}$$

with the following remainder estimate

$$|R_{m,p,q}| \leq \frac{5L(b-a)(d-c)}{18}(h_x + h_y), \quad \forall p, q = \overline{0, 2n}, \forall m \in \mathbb{N}, \tag{3.14}$$

according (3.11) and (2.1). It obtains the following iterative algorithm:

- Step 1: Choose the functions  $r, K, \psi$  and the values  $a, b, c, d, \lambda, \varepsilon', n$ .
- Step 2: Set  $h_x = \frac{b-a}{2n}$  and  $h_y = \frac{d-c}{2n}$ .
- Step 3: Choose  $\varepsilon' > 0$ . For  $p = \overline{0, 2n}, q = \overline{0, 2n}$ , set  $\bar{X}_0(s_p, t_q) = r(s_p, t_q)$ .
- Step 4: For  $m = 1$ , and for all  $p = \overline{0, 2n}, q = \overline{0, 2n}$ , Compute

$$\begin{aligned} \bar{X}_1(s_p, t_q) &= r(s_p, t_q) + \frac{\lambda h_x h_y}{9} \sum_{i=1}^n \sum_{j=1}^n \sum_{l_2=0}^2 \sum_{l_1=0}^2 (C_2^{l_2})^2 (C_2^{l_1})^2 \\ &\quad \times K(s_p, t_q, s_{2i-2+l_1}, t_{2j-2+l_2})\psi(s_{2i-2+l_1}, t_{2j-2+l_2}, \bar{X}_0(s_{2i-2+l_1}, t_{2j-2+l_2})). \end{aligned} \tag{3.15}$$



Step 5: For  $m \geq 2$ , and for all  $p = \overline{0, 2n}$ ,  $q = \overline{0, 2n}$ , Compute

$$\begin{aligned} \bar{X}_m(s_p, t_q) &= r(s_p, t_q) + \frac{\lambda h_x h_y}{9} \sum_{i=1}^n \sum_{j=1}^n \sum_{l_2=0}^2 \sum_{l_1=0}^2 (C_2^{l_2})^2 (C_2^{l_1})^2 \\ &\times K(s_p, t_q, s_{2i-2+l_1}, t_{2j-2+l_2}) \psi(s_{2i-2+l_1}, t_{2j-2+l_2}, \bar{X}_{m-1}(s_{2i-2+l_1}, t_{2j-2+l_2})). \end{aligned} \tag{3.16}$$

Step 6: Compute  $|\bar{X}_m(s_p, t_q) - \bar{X}_{m-1}(s_p, t_q)|$ , for  $p = \overline{0, 2n}$ ,  $q = \overline{0, 2n}$ .

Step 7: If  $|\bar{X}_m(s_p, t_q) - \bar{X}_{m-1}(s_p, t_q)| < \varepsilon$ , print  $m$  and print  $\bar{X}_m(s_p, t_q)$ , for all  $p = \overline{0, 2n}$ ,  $q = \overline{0, 2n}$ ; if not set  $m = m + 1$  and go to Step 5.

A practical criterion for this algorithm is presented below in Remark (3.7).

**3.3. The convergence analysis.** In this section, an upper bound of the error for the present method is obtained.

**Theorem 3.4.** Consider the assumptions of Theorem 3.1. Then, the iterative procedure (3.6) converges to  $X^*$  (the unique solution of Eq. (1.1)). Also, the upper bound of the error is as follows

$$d(X^*, X_m) \leq \frac{\Theta^{m+1}}{\beta(1-\Theta)} M_0 + \frac{5L(b-a)(d-c)}{1-\Theta} (h_x + h_y),$$

**Proof.** Using (3.4) we have

$$d(X^*, \bar{X}_m) \leq d(X^*, X_m) + d(X_m, \bar{X}_m) \leq \frac{\Theta^{m+1}}{\beta(1-\Theta)} M_0 + d(X_m, \bar{X}_m). \tag{3.17}$$

Therefore, it is necessary to estimate  $|X_m(s, t, r) - \bar{X}_m(s, t, r)|$ .

Form (3.12) for  $m = 1$ , (3.15) and (3.14) we obtain

$$|X_1(s_p, t_q) - \bar{X}_1(s_p, t_q)| \leq |R_{1,p,q}| \leq \frac{5L(b-a)(d-c)}{18} (h_x + h_y).$$

Using (3.13) and (3.16) we get

$$\begin{aligned} |X_m(s_p, t_q) - \bar{X}_m(s_p, t_q)| &\leq |R_{m,p,q}| + \frac{\lambda h_x h_y}{9} \sum_{i=1}^n \sum_{j=1}^n \sum_{l_2=0}^2 \sum_{l_1=0}^2 (C_2^{l_2})^2 (C_2^{l_1})^2 |K(s_p, t_q, s_{2i-2+l_1}, t_{2j-2+l_2})| \\ &|\psi(s_{2i-2}, t_{2j-2}, X_{m-1}(s_{2i-2+l_1}, t_{2j-2+l_2})) - \psi(s_{2i-2}, t_{2j-2}, \bar{X}_{m-1}(s_{2i-2+l_1}, t_{2j-2+l_2}))| \\ &\leq \frac{5L(b-a)(d-c)}{18} (h_x + h_y) + \frac{5M_k \beta L(b-a)(d-c)}{36} \sum_{l_2=0}^2 \sum_{l_1=0}^2 (C_2^{l_2})^2 (C_2^{l_1})^2 \|X_{m-1} - \bar{X}_{m-1}\|. \end{aligned} \tag{3.18}$$

Now, from (3.18) for  $m = 2$  it follows that

$$\begin{aligned} |X_2(s_p, t_q) - \bar{X}_2(s_p, t_q)| &\leq \frac{5L(b-a)(d-c)}{18} (h_x + h_y) + \frac{5\lambda M_k \beta L(b-a)(d-c)}{36} \sum_{l_2=0}^2 \sum_{l_1=0}^2 (C_2^{l_2})^2 (C_2^{l_1})^2 \|X_1 - \bar{X}_1\| \\ &\leq \frac{5L(b-a)(d-c)}{18} (h_x + h_y) + 5\lambda M_k \beta L(b-a)(d-c) \|X_1 - \bar{X}_1\| \\ &\leq (1 + \lambda M_k \beta L(b-a)(d-c)) \frac{5L(b-a)(d-c)}{18} (h_x + h_y). \end{aligned}$$





For  $m \in N, m \geq 3$  and by using induction, we get

$$\begin{aligned} |X_m(s_p, t_q) - \bar{X}_m(s_p, t_q)| &\leq [1 + \lambda\beta M_K(b-a)(d-c) + \dots + (\lambda\beta M_K(b-a)(d-c))^{m-1}] \frac{5L(b-a)(d-c)}{18} (h_x + h_y) \\ &= \frac{1 - (\lambda\beta M_K(b-a)(d-c))^m}{1 - \lambda\beta M_K(b-a)(d-c)} \frac{5L(b-a)(d-c)}{18} (h_x + h_y) \\ &\leq \frac{1}{1 - \lambda\beta M_K(b-a)(d-c)} \frac{5L(b-a)(d-c)}{18} (h_x + h_y) \\ &= \frac{5L(b-a)(d-c)}{18(1 - \Theta)} (h_x + h_y). \end{aligned}$$

Therefore,

$$d(X_m, \bar{X}_m) \leq \frac{5L(b-a)(d-c)}{18(1 - \Theta)} (h_x + h_y). \tag{3.19}$$

Hence, from (3.4) and (3.19) we conclude that

$$d(X^*, \bar{X}_m) \leq \left( \frac{\Theta^{m+1}}{\beta(1 - \Theta)} \right) M_0 + \frac{5L(b-a)(d-c)}{18(1 - \Theta)} (h_x + h_y).$$

Since  $\Theta < 1$ , it is easy to see that

$$\lim_{\substack{m \rightarrow \infty \\ h_x, h_y \rightarrow 0}} d(X^*, \bar{X}_m) = 0,$$

This means that the proposed method is convergent.

**3.4. The stability analysis.** Here, the numerical stability of the suggested method is studied. To do this, we consider the term  $X_0(s, t) = f(s, t) \in C([a, b] \times [c, d], R)$  such that for an  $\varepsilon > 0$  we have  $|r(s, t) - f(s, t)| < \varepsilon, \forall (s, t) \in [a, b] \times [c, d]$ . Consider  $\theta'$  and  $M'_0$  such that

$$\begin{aligned} |f(s, t) - f(s', t')| &\leq \theta' (|s - s'| + |t - t'|) \quad \forall (s, t), (s', t') \in I, \\ M'_0 &= \max\{|\psi(x, y, r(x, y))| : (s, t) \in I\} \end{aligned}$$

and  $L'$  the Lipschitz constant obtained similar as in Proposition (3.2) and Corollary (3.3).

By replacing  $r(s, t) = f(s, t)$  in Eq. (1.1) and applying the iterative method to the obtained equation, we have

$$Y(s, t) = f(s, t) + \lambda \int_c^d \int_a^b K(s, t, x, y) \psi(x, y, Y(x, y)) dx dy, \quad (s, t) \in I, \tag{3.20}$$

The following sequence of successive approximations on the knots  $s_p = a + p \frac{b-a}{2n}, p = 0, 2n$  and  $t_q = c + q \frac{d-c}{2n}, q = 0, 2n$  is obtained.

$$Y_0(s_p, t_p) = f(s_p, t_p), \tag{3.21}$$

$$Y_m(s_p, t_p) = f(s_p, t_p) + \lambda \int_c^d \int_a^b K(s_p, t_p, x, y) \psi(x, y, Y_{m-1}(x, y)) dx dy, \quad m \geq 1. \tag{3.22}$$

Now, by using the iterative procedure (3.6), the following values are obtained.

$$\begin{aligned} \bar{Y}_0(s, t) &= f(s, t). \\ \bar{Y}_m(s, t) &= f(s, t) + \frac{\lambda h_x h_y}{9} \sum_{i=1}^n \sum_{j=1}^n \sum_{l_2=0}^2 \sum_{l_1=0}^2 (C_2^{l_2})^2 (C_2^{l_1})^2 \\ &\quad K(s, t, s_{2i-2+l_1}, t_{2j-2+l_2}) \psi(s_{2i-2+l_1}, t_{2j-2+l_2}, \bar{Y}_{m-1}(s_{2i-2+l_1}, t_{2j-2+l_2})). \end{aligned} \tag{3.23}$$

**Theorem 3.5.** Under assumptions of theorem 3.4 and with respect to the first iteration, the iterative procedure (3.6) is numerically stable.



*Proof.* We have

$$d(Y_m, \bar{Y}_m) \leq \frac{5L'(b-a)(d-c)}{36(1-\Theta)}(h_x + h_y).$$

Clearly, we have

$$\begin{aligned} \|\bar{X}_m - \bar{Y}_m\| &\leq \|\bar{X}_m - X_m\| + \|X_m - Y_m\| + \|Y_m - \bar{Y}_m\| \\ &\leq \frac{5L(b-a)(d-c)}{36(1-\Theta)}(h_x + h_y) + \frac{5L'(b-a)(d-c)}{36(1-\Theta)}(h_x + h_y) + \|X_m - Y_m\|. \end{aligned}$$

Now, we can write

$|X_0(s, t) - Y_0(s, t)| < \varepsilon$ , for all  $(s, t) \in I$ ,  
and

$$\begin{aligned} |X_1(s, t) - Y_1(s, t)| &\leq |X_0(s, t) + \lambda \int_c^d \int_a^b K(s, t, x, y) \Psi(x, y, X_0(x, y)) dx dy \\ &\quad - Y_0(s, t) - \lambda \int_c^d \int_a^b K(s, t, x, y) \Psi(x, y, Y_0(x, y)) dx dy| \\ &\leq \varepsilon + \beta \lambda M_k \int_c^d \int_a^b |X_0(s, t) - Y_0(s, t)| dx dy \\ &\leq \varepsilon + \beta \lambda M_k (b-a)(d-c) \varepsilon, \end{aligned}$$

By considering the condition  $\beta \lambda M_k (b-a)(d-c) < 1$ , and using induction for  $m \geq 2$ , we obtain

$$\|X_m - Y_m\| \leq \frac{1}{1-\Theta} \varepsilon,$$

for  $m \geq 0$ . Therefore, we can write

$$\|X_m - Y_m\| \leq \frac{1}{1-\Theta} \varepsilon + \frac{5L(b-a)(d-c)}{18(1-\Theta)}(h_x + h_y) + \frac{5L'(b-a)(d-c)}{36(1-\Theta)}(h_x + h_y).$$

□

**Remark 3.6.** Since  $\Theta < 1$ , it can be concluded that

$$\lim_{h_x, h_y, \varepsilon \rightarrow 0} \|X_m - Y_m\| = 0.$$

**Remark 3.7.** The relation (3.3) can be used to get the stopping criterion as follows:

Consider previously chosen  $\varepsilon' > 0$ . The first integer positive number  $m$  is determined such that

$$|\bar{X}_m(s_p, t_q) - \bar{X}_{m-1}(s_p, t_q)| < \varepsilon'.$$

We can write

$$\begin{aligned} \|X^* - \bar{X}_m\| &\leq \|X^* - X_m\| + \|X_m - \bar{X}_m\| \\ &\leq \frac{\Theta}{1-\Theta} \|X_m - X_{m-1}\| + \frac{5L(b-a)(d-c)}{18(1-\Theta)}(h_x + h_y), \end{aligned}$$



and

$$\begin{aligned} \|X_m - X_{m-1}\| &\leq \|X_m - \bar{X}_m\| + \|\bar{X}_m - \bar{X}_{m-1}\| + \|\bar{X}_{m-1} - X_{m-1}\| \\ &\leq \frac{5L(b-a)(d-c)}{9(1-\Theta)}(h_x + h_y) + \|\bar{X}_m - \bar{X}_{m-1}\|. \end{aligned}$$

So,

$$\|X^* - \bar{X}_m\| \leq \frac{\Theta}{1-\Theta} \|\bar{X}_m - \bar{X}_{m-1}\| + \frac{5\Theta L(b-a)(d-c)}{9(1-\Theta)^2}(h_x + h_y) + \frac{5L(b-a)(d-c)}{18(1-\Theta)}(h_x + h_y).$$

In order to get  $|X^*(s, t) - \bar{X}_m(s, t)| < \varepsilon$ , it is necessary that

$$\frac{5L(b-a)(d-c)(1+\Theta)}{18(1-\Theta)^2}(h_x + h_y) < \frac{\varepsilon}{2} \tag{3.24}$$

and

$$\frac{\Theta}{1-\Theta} \|\bar{X}_m - \bar{X}_{m-1}\| < \frac{\varepsilon}{2}.$$

Now, to hold inequality (3.24), the smallest integer positive number  $n$  can be selected. Finally, the smallest number  $m \in \mathbb{N}$  (as the last iterative step) is obtained and

$$\|\bar{X}_m - \bar{X}_{m-1}\| < \frac{\varepsilon}{2} \cdot \frac{1-\Theta}{\Theta} = \varepsilon'.$$

With these, the inequality  $|\bar{X}_m(s_p, t_q) - \bar{X}_{m-1}(s_p, t_q)| < \varepsilon'$  leads to  $|X^*(s_p, t_q) - \bar{X}_m(s_p, t_q)| < \varepsilon$ .

#### 4. NUMERICAL EXAMPLES

In this section, some examples are studied to show the applicability, efficiency and accuracy of the proposed method. We introduce the notations

$$E_{p,q} := |X^*(s_p, t_q) - \bar{X}_m(s_p, t_q)|, \tag{4.1}$$

and

$$\|E_n\|_\infty := \max\{E_{p,q} | p, q = 0, 1, \dots, n\}, \tag{4.2}$$

where  $X^*$  and  $\bar{X}_m$  are the exact and approximate solutions of the integral equation, respectively. The numerical implementation is carried out in Maple 17.

**Example 1.** As the first example, a 2D non-linear Fredholm integral equation is considered as

$$X(s, t) = s^2 + t^2 + 1 + \frac{0.00568783}{(s+1)(t+3)} + \int_0^1 \int_0^1 \frac{xy}{(s+1)(t+3)} \sin(x^2 + y + X(x, y)) dx dy, \quad (s, t) \in [0, 1]^2, \tag{4.3}$$

with the exact solution

$$X(s, t) = s^2 + t^2 + 1.$$

In order to ensure that there is an unique solution for equation (4.3) with above  $r$  and  $k$  and  $\psi$ , we test conditions of Theorem (3.1).

It is obvious that  $r \in C([0, 1]^2, \mathbb{R})$ ,  $K \in C([0, 1]^4, \mathbb{R})$ ,  $\psi \in C([0, 1]^2 \times \mathbb{R}, \mathbb{R})$ . We show that Conditions (ii) and (iii) also holds. Since  $x, x', y, y' \in [0, 1]$  and  $X, X' : [0, 1]^2 \rightarrow [1, 3]$ , we have

$$\begin{aligned} |\psi(x, y, X) - \psi(x', y', X')| &= |\sin(x^2 + y + X) - \sin(x'^2 + y' + X')| \\ &\leq |x^2 - x'^2| + |y - y'| + |X - X'| \\ &\leq 2(|x - x'| + |y - y'|) + |X - X'|. \end{aligned}$$

So,  $\alpha = 2, \beta = 1$ . Also  $|k(s, t, x, y)| \leq \frac{1}{3} = M_k$  and  $\lambda = 1$ , therefore we have  $\Theta = (1)(1)(\frac{1}{3})(1-0)(1-0) < 1$ . Now, by using the algorithm for  $n = 10$ ,  $\varepsilon' = 10^{-15}$ , we obtain  $m = 9$  ( $m$  is the number of iterations).



For more details, see Table 1. For testing the numerical stability, we put  $\varepsilon = 0.1$ . The computed values  $D_{p,q} = |\overline{X}_9(s_p, t_q) - \overline{Y}_9(s_p, t_q)|, p, q = \overline{0, 10}$  in Table 1 show that the algorithm has the numerical stability. In order to study the convergence, by taking  $n = 100$ ,  $\varepsilon' = 10^{-15}$ , we have  $m = 11$ . Now, one can see how  $E_{p,q}, p, q = \overline{0, n}$  decreases when  $h_x$  and  $h_y$  decrease. For this case, the results are presented in Table 2. Also, for  $n = 1000, \varepsilon' = 10^{-15}$  with  $m = 15$ , the results are listed in 3. The values  $\|E_n\|_\infty$  for  $\varepsilon' = 10^{-15}$  and  $n \in \{10, 100, 1000\}$  are  $8.6950 \times 10^{-7}, 6.8423 \times 10^{-11}$  and  $4.5911 \times 10^{-14}$ , respectively. The reported results in Tables 1-3 show that the numerical method is convergent, that is  $E_{p,q} \rightarrow 0$  as  $h_x, h_y \rightarrow 0$ .

TABLE 1. Numerical results for Example 1 for  $n = 10$ 

$(s_p, t_q)$	$X^*(s_p, t_q)$	$\overline{X}_9(s_p, t_q)$	$E_{p,q}$	$D_{p,q}$
(0.0,0.0)	1.00	1.0000008198280153642	$8.1982801 \times 10^{-7}$	0.108
(0.1,0.1)	1.02	1.0200007212563185023	$7.2125632 \times 10^{-7}$	0.106
(0.2,0.2)	1.08	1.0800006404906370033	$6.4049064 \times 10^{-7}$	0.104
(0.3,0.3)	1.18	1.1800005733063044505	$5.7330630 \times 10^{-7}$	0.103
(0.4,0.4)	1.32	1.3200005166983290111	$5.1669832 \times 10^{-7}$	0.102
(0.5,0.5)	1.50	1.5000004684731516367	$4.6847315 \times 10^{-7}$	0.101
(0.6,0.6)	1.72	1.7200004269937580022	$4.2699376 \times 10^{-7}$	0.101
(0.7,0.7)	1.98	1.9800003910149516841	$3.9101495 \times 10^{-7}$	0.101
(0.8,0.8)	2.28	2.2800003595736909492	$3.5957369 \times 10^{-7}$	0.100
(0.9,0.9)	2.62	2.6200003319141762608	$3.3191418 \times 10^{-7}$	0.100
(1.0,1.0)	3.00	3.0000003074355057616	$3.0743551 \times 10^{-7}$	0.100

TABLE 2. Numerical results for Example 1 for  $n = 100$ 

$(s_p, t_q)$	$X^*(s_p, t_q)$	$\overline{X}_{11}(s_p, t_q)$	$E_{p,q}$
(0.0,0.0)	1.00	1.0000000000823828075	$8.2382807 \times 10^{-11}$
(0.1,0.1)	1.02	1.0200000000724775433	$7.2477543 \times 10^{-11}$
(0.2,0.2)	1.08	1.0800000000643615684	$6.4361568 \times 10^{-11}$
(0.3,0.3)	1.18	1.1800000000576103549	$5.7610355 \times 10^{-11}$
(0.4,0.4)	1.32	1.3200000000519219375	$5.1921937 \times 10^{-11}$
(0.5,0.5)	1.50	1.5000000000349954371	$4.7075890 \times 10^{-11}$
(0.6,0.6)	1.72	1.7200000000429077122	$4.2907712 \times 10^{-11}$
(0.7,0.7)	1.98	1.9800000000292092282	$3.9292277 \times 10^{-11}$
(0.8,0.8)	2.28	2.2800000000361328103	$3.6132810 \times 10^{-11}$
(0.9,0.9)	2.62	2.6200000000333533634	$3.3353363 \times 10^{-11}$
(1.0,1.0)	3.00	3.0000000000308935528	$3.0893553 \times 10^{-11}$

**Example 2.** Here, we consider the following 2D non-linear Fredholm integral equation [24]

$$X(s, t) = r(s, t) + \int_0^1 \int_0^1 K(s, t, x, y) \Psi(X(x, y)) dx dy, \quad (s, t) \in [0, 1] \times [0, 1], \quad (4.4)$$

where

$$\begin{aligned} r(s, t) &= \sin(t) - \frac{1}{18} st^2 (1 - \cos(1) (\frac{1}{2} \sin^2(1) + 1)), \\ K(s, t, x, y) &= \frac{1}{6} xst^2, \\ \Psi(X) &= X^3, \end{aligned}$$



TABLE 3. Numerical results for Example 1 for  $n = 1000$

$(s_p, t_q)$	$X^*(s_p, t_q)$	$\bar{X}_{15}(s_p, t_q)$	$E_{p,q}$
(0.0,0.0)	1.00	1.000000000000004010262	$4.0102061 \times 10^{-14}$
(0.1,0.1)	1.02	1.020000000000003528040	$3.5280405 \times 10^{-14}$
(0.2,0.2)	1.08	1.080000000000003132973	$3.1329735 \times 10^{-14}$
(0.3,0.3)	1.18	1.180000000000002804340	$2.8043399 \times 10^{-14}$
(0.4,0.4)	1.32	1.320000000000002527441	$2.5274408 \times 10^{-14}$
(0.5,0.5)	1.50	1.500000000000002291546	$2.2915463 \times 10^{-14}$
(0.6,0.6)	1.72	1.720000000000002088649	$2.0886490 \times 10^{-14}$
(0.7,0.7)	1.98	1.980000000000001912658	$1.9126579 \times 10^{-14}$
(0.8,0.8)	2.28	2.280000000000001758862	$1.7588623 \times 10^{-14}$
(0.9,0.9)	2.62	2.620000000000001623565	$1.6235652 \times 10^{-14}$
(1.0,1.0)	3.00	3.000000000000001503827	$1.5038273 \times 10^{-14}$

TABLE 4. Comparison of errors of numerical results for Example 2 for  $n = 10$

$(s_p, t_q)$	$X^*(s_p, t_q)$	Present method	Method of [24]
(0.0,0.0)	0.	0.	0.
(0.1,0.1)	0.09983341664682815230	$3.899410 \times 10^{-11}$	$8.203942576330200 \times 10^{-8}$
(0.2,0.2)	0.19866933079506121546	$3.119528 \times 10^{-10}$	$6.563154061064100 \times 10^{-7}$
(0.3,0.3)	0.29552020666133957511	$1.052848 \times 10^{-9}$	$2.215064495609150 \times 10^{-6}$
(0.4,0.4)	0.38941834230865049167	$2.495623 \times 10^{-9}$	$5.250523248851320 \times 10^{-6}$
(0.5,0.5)	0.47942553860420300027	$4.874263 \times 10^{-9}$	$1.025492822041272 \times 10^{-5}$
(0.6,0.6)	0.56464247339503535720	$8.422726 \times 10^{-9}$	$1.772051596487319 \times 10^{-5}$
(0.7,0.7)	0.64421768723769105367	$1.337498 \times 10^{-8}$	$2.813952303681252 \times 10^{-5}$
(0.8,0.8)	0.71735609089952276163	$1.996498 \times 10^{-8}$	$4.200418599081052 \times 10^{-5}$
(0.9,0.9)	0.78332690962748338846	$2.842670 \times 10^{-8}$	$5.980674138144701 \times 10^{-5}$
(1.0,1.0)	0.84147098480789650665	$3.899410 \times 10^{-8}$	$8.203942576330180 \times 10^{-5}$

with the exact solution

$$X(s, t) = \sin(t).$$

In this example, we have  $a = c = 0, b = d = 1, \lambda = 1, M_k = \frac{1}{6}$ . Also

$$|\psi(x, y, X) - \psi(x', y', X')| = |X^3 - X'^3| \leq 3|X - X'|,$$

for  $X, X' : [0, 1]^2 \rightarrow [0, 1]$ . Therefore,  $\beta = 3$  and we have  $\Theta = (1)(3)(\frac{1}{6})(1-0)(1-0) = \frac{1}{2} < 1$ . Because of all conditions hold, Theorem (3.1) implies the existence of unique solution for equation.

Comparisons between the obtained errors by the present method and the method in [24] are shown in Tables 4-6.

**Example 3.** [17, 24] As the last example, we consider the integral equation (4.3) with

$$r(s, t) = -\frac{s}{6(1+t)} + \frac{1}{(1+s+t)^2},$$

$$K(s, t, x, y) = \frac{s}{1+t}(1+y+x),$$

$$\Psi(X) = X^2,$$

where the exact solution is

$$X(s, t) = \frac{1}{(1+s+t)^2}.$$



TABLE 5. Comparison of errors of numerical results for Example 2 for  $n = 100$

$(s_p, t_q)$	$X^*(s_p, t_q)$	Present method	Method of [24]
(0.0,0.0)	0.	0.	0.
(0.1,0.1)	0.09983341664682815230	$3.888832 \times 10^{-15}$	$8.1846966058500 \times 10^{-10}$
(0.2,0.2)	0.19866933079506121546	$3.111065 \times 10^{-14}$	$6.5477572846800 \times 10^{-9}$
(0.3,0.3)	0.29552020666133957511	$1.049985 \times 10^{-13}$	$2.2098680835790 \times 10^{-8}$
(0.4,0.4)	0.38941834230865049167	$2.488852 \times 10^{-13}$	$5.2382058277440 \times 10^{-8}$
(0.5,0.5)	0.47942553860420300027	$4.861039 \times 10^{-13}$	$1.0230870757312 \times 10^{-7}$
(0.6,0.6)	0.56464247339503535720	$8.399876 \times 10^{-13}$	$1.7678944668635 \times 10^{-7}$
(0.7,0.7)	0.64421768723769105367	$1.333869 \times 10^{-12}$	$2.8073509358064 \times 10^{-7}$
(0.8,0.8)	0.71735609089952276163	$1.991082 \times 10^{-12}$	$4.1905646621950 \times 10^{-7}$
(0.9,0.9)	0.78332690962748338846	$2.834958 \times 10^{-12}$	$5.9666438256644 \times 10^{-7}$
(1.0,1.0)	0.84147098480789650665	$3.888832 \times 10^{-12}$	$8.1846966058496 \times 10^{-7}$

TABLE 6. Comparison of errors of numerical results for Example 2 for  $n = 1000$

$(s_p, t_q)$	$X^*(s_p, t_q)$	Present method	Method of [24]
(0.0,0.0)	0.	0.	0.
(0.1,0.1)	0.09983341664682815230	$6.2226423 \times 10^{-18}$	$9.093915900300 \times 10^{-12}$
(0.2,0.2)	0.19866933079506121546	$4.9781138 \times 10^{-17}$	$7.275132720200 \times 10^{-11}$
(0.3,0.3)	0.29552020666133957511	$1.6801134 \times 10^{-16}$	$2.455357293070 \times 10^{-10}$
(0.4,0.4)	0.38941834230865049167	$3.9824911 \times 10^{-16}$	$5.820106176170 \times 10^{-10}$
(0.5,0.5)	0.47942553860420300027	$7.7783029 \times 10^{-16}$	$1.136739487534 \times 10^{-9}$
(0.6,0.6)	0.56464247339503535720	$1.3440907 \times 10^{-15}$	$1.964285834459 \times 10^{-9}$
(0.7,0.7)	0.64421768723769105367	$2.1343663 \times 10^{-15}$	$3.119213153794 \times 10^{-9}$
(0.8,0.8)	0.71735609089952276163	$3.1859928 \times 10^{-15}$	$4.656084940940 \times 10^{-9}$
(0.9,0.9)	0.78332690962748338846	$4.5363062 \times 10^{-15}$	$6.629464691299 \times 10^{-9}$
(1.0,1.0)	0.84147098480789650665	$6.2226423 \times 10^{-15}$	$9.093915900273 \times 10^{-9}$

TABLE 7. Comparison of errors of the numerical results for Example 3 for  $n = 10, n = 100, n = 1000$

$(s, t)$	Method of [24]			Present Method		
	n=10	n=100	n=1000	n=10	n=100	n=1000
(0.0,0.0)	0	0	0	0	0	0
(0.1,0.1)	$1.381 \times 10^{-6}$	$1.380 \times 10^{-8}$	$2.839 \times 10^{-10}$	$1.071 \times 10^{-7}$	$1.078 \times 10^{-11}$	$8.302 \times 10^{-16}$
(0.2,0.2)	$5.525 \times 10^{-6}$	$5.520 \times 10^{-8}$	$6.389 \times 10^{-10}$	$1.963 \times 10^{-7}$	$1.977 \times 10^{-11}$	$1.530 \times 10^{-15}$
(0.3,0.3)	$1.243 \times 10^{-5}$	$1.242 \times 10^{-7}$	$1.775 \times 10^{-9}$	$2.718 \times 10^{-7}$	$2.737 \times 10^{-11}$	$2.112 \times 10^{-15}$
(0.4,0.4)	$2.210 \times 10^{-5}$	$2.208 \times 10^{-7}$	$2.555 \times 10^{-9}$	$3.365 \times 10^{-7}$	$3.388 \times 10^{-11}$	$2.624 \times 10^{-15}$
(0.4,0.5)	$3.453 \times 10^{-5}$	$3.450 \times 10^{-7}$	$3.478 \times 10^{-9}$	$3.925 \times 10^{-7}$	$3.953 \times 10^{-11}$	$3.061 \times 10^{-15}$
(0.6,0.6)	$4.972 \times 10^{-5}$	$4.968 \times 10^{-7}$	$4.543 \times 10^{-9}$	$4.416 \times 10^{-7}$	$4.447 \times 10^{-11}$	$3.436 \times 10^{-15}$
(0.7,0.7)	$6.768 \times 10^{-5}$	$6.762 \times 10^{-7}$	$5.750 \times 10^{-9}$	$4.850 \times 10^{-7}$	$4.883 \times 10^{-11}$	$3.771 \times 10^{-15}$
(0.8,0.8)	$8.836 \times 10^{-5}$	$8.832 \times 10^{-7}$	$7.099 \times 10^{-9}$	$5.234 \times 10^{-7}$	$5.280 \times 10^{-11}$	$4.077 \times 10^{-15}$
(0.9,0.9)	$1.119 \times 10^{-4}$	$1.118 \times 10^{-6}$	$1.147 \times 10^{-8}$	$5.578 \times 10^{-7}$	$5.618 \times 10^{-11}$	$4.343 \times 10^{-15}$
(1.0,1.0)	$1.381 \times 10^{-4}$	$1.380 \times 10^{-6}$	$1.394 \times 10^{-8}$	$5.888 \times 10^{-7}$	$5.930 \times 10^{-11}$	$4.582 \times 10^{-15}$

Table 7 shows comparisons between the obtained errors by the present method and the method in [24]. According to the reported results, our proposed method has a reasonable convergence rate. Also, the absolute errors in the solutions by our method are accurate in comparison with [24].



## 5. CONCLUSIONS

This paper applied an iterative method of successive approximation for solving the 2D non-linear Hammerstein integral equations. The basic results of this paper are presented in Theorems 3.1, 3.4, and 3.5 which survey the numerical stability and convergence of the proposed method. These two aspects of the method were studied and confirmed by three numerical examples. The numerical results show that to obtain a good approximate solution a small number of iterations is required.

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