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Numerical solution of Drinfel'd–Sokolov system with the Haar wavelets method

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Abstract

In this article, we use the Haar wavelets (HWs) method to numerically solve the nonlinear Drinfel'd–Sokolov (DS) system. For this purpose, we use an approximation of functions with the help of HWs, and we approximate spatial derivatives using this method. In this regard, to linearize the nonlinear terms of the equations, we use the quasilinearization technique. At the end, to show the effectiveness and accuracy of the method in solving this system one numerical example is provided.

Keywords. Drinfel'd–Sokolov system, Numerical solution, Haar wavelets method, Quasilinearization technique.

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1. Introduction

Nonlinear coupled partial differential equations are very significant in a type of scientific field, especially in fluid mechanics, solid-state physics, plasma waves, plasma physics, and chemical physics. Since many nonlinear physical phenomena can be explained by the exact and numerical solutions of nonlinear equations, the attempt for finding the exact and numerical solutions to these phenomena is important.

In this article, our main goal is to solve numerically the DS system. A generalized form of the DS system is given by:

$$\begin{cases} \varphi_t + (\psi^2)_x = 0, \\ \psi_t - \alpha\psi_{xxx} + 3\beta\varphi_x\psi + 3\delta\varphi\psi_x = 0, \end{cases} \tag{1.1}$$

where α, β, and δ are constants.

Drinfeld and Sokolov introduced system (1.1) as an example of a special form of a system of nonlinear equations possessing Lax pairs [8]. Many researchers have devoted considerable efforts by successfully implementing various methods to extract solitary wave solutions and other solutions of DS and DSW (Drinfel'd–Sokolov–Wilson) systems [2, 5, 14, 20–22].

In 1910, Alfred Haar, a Hungarian mathematician, introduced the HWs. Mathematically, HWs are among the simplest types of wavelets and are known as piecewise constant functions. The possibility to integrate analytically at arbitrary times of the HWs is a good property for them. Compared to other wavelets, HWs are only orthogonal wavelets that have a compact support and explicit formulas [6]. Due to its mathematical simplicity, the HW method is a suitable and optimal method for solving various differential and integral equations [12]. In addition, in solving various nonlinear systems in the fields of biology, physics, fluid mechanics, and chemical reactions, this method can be named as one of the effective methods [1, 3, 10, 11, 15, 17].

In the present article, we intend to use the HWs method to numerically solve the DS system (1.1) on $(0,1)\times(0,t_{fin})$, with the initial conditions

$$\varphi(x,0) = f_1(x), \qquad \psi(x,0) = f_2(x), \qquad x \in [0,1], \tag{1.2}$$

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and the boundary conditions

$$\begin{array}{lll} \varphi(0,t) = g_1(t), & \psi(0,t) = g_2(t), & t \in [0, t_{fin}], \\ \psi(1,t) = k_2(t), & \psi_x(0,t) = w_2(t), & t \in [0, t_{fin}], \end{array} \tag{1.3}$$

where t_{fin} represents the final time. The differentiable functions $f_1(x)$, $f_2(x)$, $g_1(t)$, $g_2(t)$, $k_2(t)$, and $w_2(t)$ are known.

The structure of this article is as follows: In section 2, the HWs family and their integrals are introduced. In the following, Haar matrices for the numerical solutions, the desired issue are described and the expanding functions into the HW series are discussed. The procedure of implementation of the HW method, for system (1.1) with specified initial and boundary conditions (1.2) and (1.3) is presented in section 3. The numerical performance of the method is given in section 4. Also, concluding remarks are made in section 5.

2. Haar Wavelets

2.1. **HWs family and their integrals.** Considering $\mathfrak{J}$ as the maximal level of resolution, we define $\mathtt{M} = 2^{\mathfrak{J}}$ and divide the interval $[a_0, b_0]$ into $2\mathtt{M}$ subintervals of equal length $\hbar_x = \frac{b_0 - a_0}{2\mathtt{M}}$. We get $s = 0:1:\mathfrak{J}$, $\kappa = 0:1:(2^s - 1)$, and define $\ell = 2^s + \kappa + 1$ as the wavelet number. So, the ℓ-th HW is defined ([13]):

$$h_\ell(x) = \begin{cases} 1, & x \in \big[\gamma_1(\ell), \gamma_2(\ell)\big), \\ -1, & x \in \big[\gamma_2(\ell), \gamma_3(\ell)\big), \\ 0, & \text{elsewhere}, \end{cases} \tag{2.1}$$

where

$$\begin{array}{ll} \gamma_1(\ell) = a_0 + 2\kappa\mu\hbar_x, & \gamma_2(\ell) = a_0 + (2\kappa+1)\mu\hbar_x, \\ \gamma_3(\ell) = a_0 + 2(\kappa+1)\mu\hbar_x, & \mu = \frac{\mathtt{M}}{2^s}. \end{array} \tag{2.2}$$

For $\ell > 2$, $\mathsf{h}_\ell(x)$ are valid and for $\ell = 1$, $\mathsf{h}_1(x) \cong 1$; for all $x \in [a_0, b_0]$. The HWs are orthogonal to each other; In other words from (2.1) we have:

$$\int_{a_0}^{b_0} \mathsf{h}_{\ell_1}(x)\mathsf{h}_{\ell_2}(x)\,\mathrm{d}x = \begin{cases} (b_0 - a_0)2^{-s}, & \ell_1 = \ell_2, \\ 0, & \ell_1 \neq \ell_2. \end{cases} \tag{2.3}$$

The integral of Haar functions can be calculated as follows:

$$p_{\vartheta,\ell}(x) = \underbrace{\int_{a_0}^{x}\int_{a_0}^{x}\dots\int_{a_0}^{x}}_{\vartheta-\text{times}} \mathsf{h}_\ell(t)\,\mathrm{d}t^\vartheta = \frac{1}{(\vartheta-1)!}\int_{a_0}^{x}(x-t)^{\vartheta-1}\mathsf{h}_\ell(t)\,\mathrm{d}t, \qquad \ell = 1:1:2\mathtt{M}. \tag{2.4}$$

Given the values of h_ℓ in (2.1), integrals (2.4) can be calculated as follows:

$$p_{\vartheta,\ell}(x) = \begin{cases} 0, & x < \gamma_1(\ell), \\ \frac{1}{\vartheta!}[x - \gamma_1(\ell)]^\vartheta, & x \in [\gamma_1(\ell), \gamma_2(\ell)], \\ \frac{1}{\vartheta!}\Big\{[x - \gamma_1(\ell)]^\vartheta - 2[x - \gamma_2(\ell)]^\vartheta\Big\}, & x \in [\gamma_2(\ell), \gamma_3(\ell)], \\ \frac{1}{\vartheta!}\Big\{[x - \gamma_1(\ell)]^\vartheta - 2[x - \gamma_2(\ell)]^\vartheta + [x - \gamma_3(\ell)]^\vartheta\Big\}, & x > \gamma_3(\ell). \end{cases} \tag{2.5}$$

The relation (2.5) for $\ell > 1$ is hold and for $\ell = 1$ we have:

$$p_{\vartheta,1}(x) = \frac{1}{\vartheta!}(x - a_0)^\vartheta. \tag{2.6}$$

2.2. **Haar matrices.** In this article, the collocation method is applied for the numerical solutions. So, we define the collocation points x_i as

$$x_i = a_0 + \Big(i - \frac{1}{2}\Big)\hbar_x, \qquad i = 1:1:2\mathtt{M}, \tag{2.7}$$

and replace $x \to x_i$ in equations (2.1), (2.5), and (2.6) for introducing the Haar matrices $\mathcal{H}(\ell, i) = \mathsf{h}_\ell(x_i)$ and $\mathcal{P}_\vartheta(\ell, i) = p_{\vartheta,\ell}(x_i)$ with dimension $2\mathtt{M} \times 2\mathtt{M}$.

2.3. **Expanding functions into the HW.** Using Haar functions, any integrable function $\Gamma(x) \in \mathcal{L}^2([a_0, b_0])$ can be approximated as an infinite series of these functions as follows ([16]):

$$\Gamma(x) = \sum_{\ell=1}^{\infty} c_\ell \mathsf{h}_\ell(x), \tag{2.8}$$

where the HWs coefficients c_ℓ are determined as

$$c_\ell = \frac{2^s}{b_0 - a_0} \int_{a_0}^{b_0} \Gamma(x) \mathsf{h}_\ell(x) \, \mathrm{d}x.$$

By truncating the infinite series (2.8), we obtain an approximate representation for $\Gamma(x)$ as

$$\Gamma_{\mathfrak{J}}(x) \cong \sum_{s=0}^{\mathfrak{J}} \sum_{\kappa=0}^{2^s-1} c_{2^s+\kappa+1} \mathsf{h}_{2^s+\kappa+1}(x) = \mathcal{C}^T \mathcal{H}(x). \tag{2.9}$$

3. Application of the method

In this section, we use the HWs method for finding the approximate solutions of system (1.1) with specified initial and boundary conditions (1.2) and (1.3). For this, we expand $\dot{\varphi}'$ and $\dot{\psi}'''$ in terms of HWs as,

$$\dot{\varphi}'(x,t) \cong \sum_{s=0}^{\mathfrak{J}} \sum_{\kappa=0}^{2^s-1} c_{2^s+\kappa+1} \mathsf{h}_{2^s+\kappa+1}(x) = \mathcal{C}^T \mathcal{H}(x), \tag{3.1}$$

$$\dot{\psi}'''(x,t) \cong \sum_{s=0}^{\mathfrak{J}} \sum_{\kappa=0}^{2^s-1} d_{2^s+\kappa+1} \mathsf{h}_{2^s+\kappa+1}(x) = \mathcal{D}^T \mathcal{H}(x), \tag{3.2}$$

where *prime* and *dot* mean differentiation concerning x and t, respectively. Dividing the interval $[0, t_{fin}]$ into $\mathtt{N}$ equal parts of length $\hbar_t = \frac{t_{fin}}{\mathtt{N}}$ and denoting $t_j = (j-1)\hbar_t$, $j = 1:1:(\mathtt{N}+1)$, in each subinterval $[t_j, t_{j+1}]$, $j = 1:1:\mathtt{N}$, the vectors $\mathcal{C}^T$ and $\mathcal{D}^T$ are constants.

Case 1: Considering the equation (3.1): By integrating this equation once concerning t from t_j to t and once concerning x from 0 to x, we have

$$\varphi'(x,t) = (t - t_j)\mathcal{C}^T \mathcal{H}(x) + \varphi'(x, t_j), \tag{3.3}$$

$$\dot{\varphi}(x,t) = \mathcal{C}^T \mathcal{P}_1(x) + g_1'(t). \tag{3.4}$$

Also, integrating equation (3.4) once concerning t from t_j to t, we obtain

$$\varphi(x,t) = (t - t_j)\mathcal{C}^T \mathcal{P}_1(x) + [g_1(t) - g_1(t_j)] + \varphi(x, t_j). \tag{3.5}$$

Case 2: Considering the equation (3.2): By integrating this equation once concerning t from t_j to t and three times concerning x from 0 to x, we have

$$\psi'''(x,t) = (t - t_j)\mathcal{D}^T \mathcal{H}(x) + \psi'''(x, t_j), \tag{3.6}$$

$$\psi'(x,t) = (t - t_j)\mathcal{D}^T \mathcal{P}_2(x) + \psi'(x, t_j) + [w_2(t) - w_2(t_j)] + x[\psi''(0,t) - \psi''(0,t_j)], \tag{3.7}$$

$$\psi(x,t) = (t - t_j)\mathcal{D}^T \mathcal{P}_3(x) + \psi(x, t_j) + [g_2(t) - g_2(t_j)] + x[w_2(t) - w_2(t_j)] + \frac{x^2}{2}[\psi''(0,t) - \psi''(0,t_j)], \tag{3.8}$$

$$\dot{\psi}(x,t) = \mathcal{D}^T \mathcal{P}_3(x) + g_2'(t) + x w_2'(t) + \frac{x^2}{2}\dot{\psi}''(0,t). \tag{3.9}$$

Using the boundary condition $\psi(1,t)=k_2(t)$ equations (3.7)-(3.9) are transformed into the following equations

$$\begin{aligned}\psi'(x,t)&=(t-t_j)\mathcal{D}^T[\mathcal{P}_2(x)-2x\mathcal{P}_3(1)]+\psi'(x,t_j)+(1-2x)[w_2(t)-w_2(t_j)]\\&\quad+2x[k_2(t)-k_2(t_j)]-2x[g_2(t)-g_2(t_j)],\end{aligned}\tag{3.10}$$

$$\begin{aligned}\psi(x,t)&=(t-t_j)\mathcal{D}^T[\mathcal{P}_3(x)-x^2\mathcal{P}_3(1)]+\psi(x,t_j)+(1-x^2)[g_2(t)-g_2(t_j)]\\&\quad+x(1-x)[w_2(t)-w_2(t_j)]+x^2[k_2(t)-k_2(t_j)],\end{aligned}\tag{3.11}$$

$$\dot\psi(x,t)=\mathcal{D}^T[\mathcal{P}_3(x)-x^2\mathcal{P}_3(1)]+(1-x^2)g_2'(t)+x(1-x)w_2'(t)+x^2k_2'(t),\tag{3.12}$$

where, $\mathcal{P}_1(x)=p_{1,\ell}(x)$, $\mathcal{P}_2(x)=p_{2,\ell}(x)$, and $\mathcal{P}_3(x)=p_{3,\ell}(x)$ are obtained from (2.4).

By placing $x\to x_i$ and $t\to t_{j+1}$ in equations (3.3)-(3.5) and (3.6) and (3.10)-(3.12), we get

$$\varphi'(x_i,t_{j+1})=\hbar_t\mathcal{C}^T\mathcal{H}(x_i)+\varphi'(x_i,t_j),\tag{3.13}$$

$$\dot\varphi(x_i,t_{j+1})=\mathcal{C}^T\mathcal{P}_1(x_i)+g_1'(t_{j+1}),\tag{3.14}$$

$$\varphi(x_i,t_{j+1})=\hbar_t\mathcal{C}^T\mathcal{P}_1(x_i)+[g_1(t_{j+1})-g_1(t_j)]+\varphi(x_i,t_j),\tag{3.15}$$

$$\psi'''(x_i,t_{j+1})=\hbar_t\mathcal{D}^T\mathcal{H}(x_i)+\psi'''(x_i,t_j),\tag{3.16}$$

$$\begin{aligned}\psi'(x_i,t_{j+1})&=\hbar_t\mathcal{D}^T[\mathcal{P}_2(x_i)-2x_i\mathcal{P}_3(1)]+\psi'(x_i,t_j)+(1-2x_i)[w_2(t_{j+1})-w_2(t_j)]\\&\quad+2x_i[k_2(t_{j+1})-k_2(t_j)]-2x_i[g_2(t_{j+1})-g_2(t_j)],\end{aligned}\tag{3.17}$$

$$\begin{aligned}\psi(x_i,t_{j+1})&=\hbar_t\mathcal{D}^T[\mathcal{P}_3(x_i)-x_i^2\mathcal{P}_3(1)]+\psi(x_i,t_j)+(1-x_i^2)[g_2(t_{j+1})-g_2(t_j)]\\&\quad+x_i(1-x_i)[w_2(t_{j+1})-w_2(t_j)]+x_i^2[k_2(t_{j+1})-k_2(t_j)],\end{aligned}\tag{3.18}$$

$$\dot\psi(x_i,t_{j+1})=\mathcal{D}^T[\mathcal{P}_3(x_i)-x_i^2\mathcal{P}_3(1)]+(1-x_i^2)g_2'(t_{j+1})+x_i(1-x_i)w_2'(t_{j+1})+x_i^2k_2'(t_{j+1}).\tag{3.19}$$

To linearized the nonlinear terms $\psi\psi_x$, $\varphi_x\psi$, and $\varphi\psi_x$ in system (1.1), we use the quasilinearization technique [4] as follows:

$$\psi\psi_x=\psi_x(x,t_j)\psi(x,t_{j+1})-\psi_x(x,t_j)\psi(x,t_j)+\psi(x,t_j)\psi_x(x,t_{j+1}),\tag{3.20}$$

$$\varphi_x\psi=\psi(x,t_j)\varphi_x(x,t_{j+1})-\psi(x,t_j)\varphi_x(x,t_j)+\varphi_x(x,t_j)\psi(x,t_{j+1}),\tag{3.21}$$

$$\varphi\psi_x=\psi_x(x,t_j)\varphi(x,t_{j+1})-\psi_x(x,t_j)\varphi(x,t_j)+\varphi(x,t_j)\psi_x(x,t_{j+1}).\tag{3.22}$$

Using linear expressions (3.20)-(3.22), the discrete form of system (1.1) considering x_i and t_{j+1} is as follows:

$$\begin{cases}\dot\varphi(x_i,t_{j+1})+2\psi'(x_i,t_j)\psi(x_i,t_{j+1})+2\psi(x_i,t_j)\psi'(x_i,t_{j+1})=2\psi'(x_i,t_j)\psi(x_i,t_j),\\ \dot\psi(x_i,t_{j+1})-\alpha\psi'''(x_i,t_{j+1})+3\beta\psi(x_i,t_j)\varphi'(x_i,t_{j+1})+3\beta\varphi'(x_i,t_j)\psi(x_i,t_{j+1})\\ \quad+3\delta\psi'(x_i,t_j)\varphi(x_i,t_{j+1})+3\delta\varphi(x_i,t_j)\psi'(x_i,t_{j+1})=3\beta\psi(x_i,t_j)\varphi'(x_i,t_j)+3\delta\psi'(x_i,t_j)\varphi(x_i,t_j).\end{cases}\tag{3.23}$$

Now, by using equations (3.13)-(3.19), system (3.23) leads to

$$\begin{cases}\mathcal{C}^T\mathcal{X}_1+\mathcal{D}^T\mathcal{X}_2=\mathcal{B}_1(x_i,t_j),\\ \mathcal{C}^T\mathcal{X}_3+\mathcal{D}^T\mathcal{X}_4=\mathcal{B}_2(x_i,t_j),\end{cases}\tag{3.24}$$

where

$$\begin{aligned}\mathcal{X}_1&=\mathcal{P}_1(x_i),\\ \mathcal{X}_2&=2\psi'(x_i,t_j)\hbar_t[\mathcal{P}_3(x_i)-x_i^2\mathcal{P}_3(1)]+2\psi(x_i,t_j)\hbar_t[\mathcal{P}_2(x_i)-2x_i\mathcal{P}_3(1)],\\ \mathcal{X}_3&=3\beta\psi(x_i,t_j)\hbar_t\mathcal{H}(x_i)+3\delta\psi'(x_i,t_j)\hbar_t\mathcal{P}_1(x_i),\\ \mathcal{X}_4&=\mathcal{P}_3(x_i)-x_i^2\mathcal{P}_3(1)-\alpha\hbar_t\mathcal{H}(x_i)\\&\quad+3\beta\varphi'(x_i,t_j)\hbar_t[\mathcal{P}_3(x_i)-x_i^2\mathcal{P}_3(1)]+3\delta\varphi(x_i,t_j)\hbar_t[\mathcal{P}_2(x_i)-2x_i\mathcal{P}_3(1)],\end{aligned}$$

$$\begin{aligned}\mathcal{B}_1(x_i,t_j) =& -2\psi(x_i,t_j)\psi'(x_i,t_j) - g_1'(t_{j+1})\\ &-2\psi'(x_i,t_j)\Big[(1-x_i^2)[g_2(t_{j+1})-g_2(t_j)] + x_i(1-x_i)[w_2(t_{j+1})-w_2(t_j)]\\ &+x_i^2[k_2(t_{j+1})-k_2(t_j)]\Big] - 2\psi(x_i,t_j)\Big[(1-2x_i)[w_2(t_{j+1})-w_2(t_j)]\\ &+2x_i[k_2(t_{j+1})-k_2(t_j)] - 2x_i[g_2(t_{j+1})-g_2(t_j)]\Big],\end{aligned}$$

$$\begin{aligned}\mathcal{B}_2(x_i,t_j) =&\alpha\psi'''(x_i,t_j) - 3\beta\psi(x_i,t_j)\varphi'(x_i,t_j) - \Big[(1-x_i^2)g_2'(t_{j+1}) + x_i(1-x_i)w_2'(t_{j+1})\\ &+x_i^2k_2'(t_{j+1})\Big] - 3\beta\varphi'(x_i,t_j)\Big[(1-x_i^2)[g_2(t_{j+1})-g_2(t_j)]\\ &+x_i(1-x_i)[w_2(t_{j+1})-w_2(t_j)] + x_i^2[k_2(t_{j+1})-k_2(t_j)]\Big]\\ &-3\delta\varphi(x_i,t_j)\Big[(1-2x_i)[w_2(t_{j+1})-w_2(t_j)] + 2x_i[k_2(t_{j+1})-k_2(t_j)]\\ &-2x_i[g_2(t_{j+1})-g_2(t_j)]\Big] - 3\delta\psi'(x_i,t_j)[g_1(t_{j+1})-g_1(t_j)+\varphi(x_i,t_j)].\end{aligned}$$

The matrix-vector form of system (3.24) is as follows:

$$\left[\begin{array}{c:c}(\mathcal{X}_1)_{2\mathtt{M}\times 2\mathtt{M}} & (\mathcal{X}_2)_{2\mathtt{M}\times 2\mathtt{M}}\\ \hdashline (\mathcal{X}_3)_{2\mathtt{M}\times 2\mathtt{M}} & (\mathcal{X}_4)_{2\mathtt{M}\times 2\mathtt{M}}\end{array}\right]_{4\mathtt{M}\times 4\mathtt{M}} \left[\begin{array}{c}(\mathcal{C})_{2\mathtt{M}\times 1}\\ \hdashline (\mathcal{D})_{2\mathtt{M}\times 1}\end{array}\right]_{4\mathtt{M}\times 1} = \left[\begin{array}{c}(\mathcal{B}_1)_{2\mathtt{M}\times 1}\\ \hdashline (\mathcal{B}_2)_{2\mathtt{M}\times 1}\end{array}\right]_{4\mathtt{M}\times 1} \tag{3.25}$$

From (3.25), the coefficients vectors $\mathcal{C}$ and $\mathcal{D}$ can be calculated. With these coefficients and using the equations (3.15) and (3.18), the approximate solutions are successively obtained.

Theorem 3.1. *(Convergence analysis) Suppose that $\varphi(x,t)$ and $\psi(x,t)$ satisfy the Lipschitz condition that is,*

$$\exists\ \iota_1 > 0,\ \forall\ x_1,x_2 \in [0,1]\ :\ |\varphi(x_1,t)-\varphi(x_2,t)| \le \iota_1|x_1-x_2|, \tag{3.26}$$

$$\exists\ \iota_2 > 0,\ \forall\ x_1,x_2 \in [0,1]\ :\ |\psi(x_1,t)-\psi(x_2,t)| \le \iota_2|x_1-x_2|. \tag{3.27}$$

Then the error bound for $\|(E_{\mathfrak{J}}^{\varphi}, E_{\mathfrak{J}}^{\psi})\|_2$ is obtained as

$$\|(E_{\mathfrak{J}}^{\varphi}, E_{\mathfrak{J}}^{\psi})\|_2 \le \sqrt{\frac{2}{3}}\frac{\iota}{\mathtt{M}},$$

where $E_{\mathfrak{J}}^{\varphi}(x,t) = \varphi(x,t) - \varphi_{\mathfrak{J}}(x,t)$ and $E_{\mathfrak{J}}^{\psi}(x,t) = \psi(x,t) - \psi_{\mathfrak{J}}(x,t)$ are the corresponding errors at $\mathfrak{J}$-th level, and $\iota = \max\{\iota_1,\iota_2\}$. Moreover, the convergence is of order one, that is,

$$\|(E_{\mathfrak{J}}^{\varphi}, E_{\mathfrak{J}}^{\psi})\|_2 = O\left(\frac{1}{\mathtt{M}}\right).$$

Proof. See [7]. □

4. Numerical experiment

In this section, we apply the HWs method to obtain the numerical solutions of the DS system (1.1). To compare the obtained numerical results, we use the following solutions obtained by the *Tanh method* ([21]):

$$\begin{cases}\varphi(x,t) = \frac{c}{2\beta+\delta}\tanh^2\Big(\sqrt{\frac{c}{2\alpha}}(x-ct)\Big),\\ \psi(x,t) = \frac{c}{\sqrt{2\beta+\delta}}\tanh\Big(\sqrt{\frac{c}{2\alpha}}(x-ct)\Big).\end{cases}$$

To show the effectiveness and accuracy of the proposed method, we considered an example with $\alpha = \beta = \delta = 1$, $t_{fin} = 1$, $\hbar_t = 0.01$. Numerical results are compared with the Legendre wavelets method (LWs) [18].

Remark 4.1. For describing the error, we introduce the root mean square (RMS) error norm as follows:

$$\mathrm{RMS}^{\varphi} = \left[\frac{1}{2\mathtt{M}} \sum_{i=1}^{2\mathtt{M}} \Big(\varphi(x_i,t) - \varphi^*(x_i,t) \Big)^2 \right]^{\frac{1}{2}}, \tag{4.1}$$

where φ^* is the approximate solution of φ. Similarly, the RMS^{ψ} is obtained according to formula (4.1).

The numerical results are shown in Tables 1-4. Also, the RMS errors (4.1) are stated in Table 5. The CPU times for various dimensions of HWs and LWs are reported in Table 6. The values of the absolute errors φ and ψ are plotted in Figures 1 and 2, respectively.

Table 1: The numerical results for $\varphi(x,t)$ at $t = 1$.

		HWs ($2\mathtt{M}=4$)		LWs ($k=2,\ \mathtt{M}=2$)	
x_i	$\varphi(x_i,t)$	$\varphi^*(x_i,t)$	$\lvert\varphi(x_i,t)-\varphi^*(x_i,t)\rvert$	$\varphi^*(x_i,t)$	$\lvert\varphi(x_i,t)-\varphi^*(x_i,t)\rvert$
0.125	0.000000	-0.000000	$2.386936e-11$	0.005036	$5.036477e-03$
0.375	0.000162	0.000162	$9.441407e-09$	0.005195	$5.033097e-03$
0.625	0.000644	0.000644	$3.087889e-08$	0.005679	$5.035138e-03$
0.875	0.001431	0.001431	$6.314510e-08$	0.006485	$5.053942e-03$
CPU time (s)		163.460017		731.121861	

Table 2: The numerical results for $\psi(x,t)$ at $t = 1$.

		HWs ($2\mathtt{M}=4$)		LWs ($k=2,\ \mathtt{M}=2$)	
x_i	$\psi(x_i,t)$	$\psi^*(x_i,t)$	$\lvert\psi(x_i,t)-\psi^*(x_i,t)\rvert$	$\psi^*(x_i,t)$	$\lvert\psi(x_i,t)-\psi^*(x_i,t)\rvert$
0.125	0.000000	0.000000	$1.280109e-09$	0.000013	$1.251707e-05$
0.375	0.004505	0.004505	$1.381351e-10$	0.004557	$5.267140e-05$
0.625	0.008974	0.008974	$1.278569e-09$	0.009053	$7.825935e-05$
0.875	0.013375	0.013375	$8.757810e-10$	0.013372	3.653990e-06
CPU time (s)		163.460017		731.121861	

Table 3: The numerical results for $\varphi(x,t)$ at collocation point $x = 0.375$.

		HWs $(2\mathtt{M}=4)$		LWs $(k=2,\ \mathtt{M}=2)$	
t_j	$\varphi(x,t_j)$	$\varphi^*(x,t_j)$	$\lvert\varphi(x,t_j)-\varphi^*(x,t_j)\rvert$	$\varphi^*(x,t_j)$	$\lvert\varphi(x,t_j)-\varphi^*(x,t_j)\rvert$
0.1	0.000340	0.000340	$1.368867e-09$	0.000876	$5.357247e-04$
0.2	0.000317	0.000317	$2.643632e-09$	0.001286	$9.683819e-04$
0.3	0.000295	0.000295	$3.824172e-09$	0.001854	$1.558918e-03$
0.4	0.000274	0.000274	$4.910369e-09$	0.002349	$2.075418e-03$
0.5	0.000253	0.000253	$5.902117e-09$	0.002755	$2.501692e-03$
0.6	0.000233	0.000233	$6.799325e-09$	0.003164	$2.930167e-03$
0.7	0.000215	0.000215	$7.601914e-09$	0.003726	$3.511928e-03$
0.8	0.000196	0.000196	$8.309823e-09$	0.004360	$4.163695e-03$
0.9	0.000179	0.000179	$8.922999e-09$	0.004887	$4.708348e-03$
1	0.000162	0.000162	$9.441407e-09$	0.005195	$5.033097e-03$
CPU time (s)		163.460017		731.121861	

Table 4: The numerical results for $\psi(x,t)$ at collocation point $x = 0.375$.

		HWs $(2\mathtt{M}=4)$		LWs $(k=2,\ \mathtt{M}=2)$	
t_j	$\psi(x,t_j)$	$\psi^*(x,t_j)$	$\lvert\psi(x,t_j)-\psi^*(x,t_j)\rvert$	$\psi^*(x,t_j)$	$\lvert\psi(x,t_j)-\psi^*(x,t_j)\rvert$
0.1	0.006522	0.006522	$1.215435e-11$	0.006584	$6.131815e-05$
0.2	0.006299	0.006299	$1.063830e-11$	0.006328	$2.906959e-05$
0.3	0.006075	0.006075	$5.389050e-12$	0.006131	$5.574840e-05$
0.4	0.005851	0.005851	$3.602231e-12$	0.005901	$4.968168e-05$
0.5	0.005627	0.005627	$1.637221e-11$	0.005690	$6.301798e-05$
0.6	0.005403	0.005403	$3.295689e-11$	0.005460	$5.704236e-05$
0.7	0.005178	0.005178	$5.339150e-11$	0.005240	$6.183361e-05$
0.8	0.004954	0.004954	$7.771047e-11$	0.005024	$6.992855e-05$
0.9	0.004729	0.004729	$1.059474e-10$	0.004784	$5.516922e-05$
1	0.004505	0.004505	$1.381351e-10$	0.004557	$5.267140e-05$
CPU time (s)		163.460017		731.121861	

5. Conclusion

In this article, using the HWs method and using the initial and boundary conditions (1.2) and (1.3), we solved the DS system (1.1) numerically. Numerical comparisons have been made between the implementations of the proposed method and the Legendre wavelet method. Due to the numerical solutions which are presented in the Tables and Figures, the obtained numerical solutions by the presented method are the most accurate in comparison with the LWs method and are in good agreement with the exact solutions. It can be concluded that the presented method for solving the DS system (1.1) is an efficient and high accuracy method. The strength of this method is the simplicity of calculations with low storage space.

Table 5: The RMS errors at different times.

t	HWs ($2\mathtt{M}=4$) $\varphi(x,t)$	HWs ($2\mathtt{M}=4$) $\psi(x,t)$	LWs ($k=2,\ \mathtt{M}=2$) $\varphi(x,t)$	LWs ($k=2,\ \mathtt{M}=2$) $\psi(x,t)$
0.2	$8.112137e-09$	$2.260094e-10$	$9.697420e-04$	$2.681494e-05$
0.4	$1.571690e-08$	$4.357983e-10$	$2.078281e-03$	$4.815645e-05$
0.6	$2.281139e-08$	$6.357693e-10$	$2.934143e-03$	$5.218810e-05$
0.8	$2.939332e-08$	$8.261827e-10$	$4.169215e-03$	$6.705956e-05$
1	$3.546107e-08$	$1.007409e-09$	$5.039670e-03$	$4.761520e-05$
t	HWs ($2\mathtt{M}=8$) $\varphi(x,t)$	HWs ($2\mathtt{M}=8$) $\psi(x,t)$	LWs ($k=3,\ \mathtt{M}=2$) $\varphi(x,t)$	LWs ($k=3,\ \mathtt{M}=2$) $\psi(x,t)$
0.2	$2.698335e-10$	$1.656756e-12$	$1.007237e-03$	$8.424354e-05$
0.4	$5.312738e-10$	$3.069774e-12$	$1.954106e-03$	$4.687166e-05$
0.6	$7.843147e-10$	$4.448526e-12$	$2.889904e-03$	$3.732068e-05$
0.8	$1.028952e-09$	$5.792596e-12$	$3.841224e-03$	$8.410049e-05$
1	$1.265184e-09$	$7.101738e-12$	$4.945043e-03$	$8.056653e-05$
t	HWs ($2\mathtt{M}=16$) $\varphi(x,t)$	HWs ($2\mathtt{M}=16$) $\psi(x,t)$	LWs ($k=4,\ \mathtt{M}=2$) $\varphi(x,t)$	LWs ($k=4,\ \mathtt{M}=2$) $\psi(x,t)$
0.2	$8.667650e-12$	$1.490447e-14$	$9.624309e-04$	$3.898140e-05$
0.4	$1.720039e-11$	$2.368515e-14$	$1.868619e-03$	$9.220079e-05$
0.6	$2.559825e-11$	$3.238861e-14$	$2.906207e-03$	$4.249411e-05$
0.8	$3.386126e-11$	$4.101297e-14$	$3.846596e-03$	$5.850635e-05$
1	$4.198946e-11$	$4.955633e-14$	$4.819426e-03$	$4.743753e-05$
t	HWs ($2\mathtt{M}=32$) $\varphi(x,t)$	HWs ($2\mathtt{M}=32$) $\psi(x,t)$	LWs ($k=5,\ \mathtt{M}=2$) $\varphi(x,t)$	LWs ($k=5,\ \mathtt{M}=2$) $\psi(x,t)$
0.2	$2.743417e-13$	$1.914177e-16$	$9.268936e-04$	$5.146389e-05$
0.4	$5.465452e-13$	$2.378363e-16$	$1.805207e-03$	$4.677735e-05$
0.6	$8.166110e-13$	$2.840893e-16$	$2.841064e-03$	$6.889437e-05$
0.8	$1.084540e-12$	$3.303265e-16$	$3.904342e-03$	$4.754377e-05$
1	$1.350332e-12$	$3.764246e-16$	$4.985977e-03$	$5.679880e-05$

Table 6: The CPU times in seconds for HWs and LWs.

HWs	$2\mathtt{M}=4$	$2\mathtt{M}=8$	$2\mathtt{M}=16$	$2\mathtt{M}=32$
	163.460017	414.718527	1307.473296	4529.120789
LWs	$k=2,\ \mathtt{M}=2$	$k=3,\ \mathtt{M}=2$	$k=4,\ \mathtt{M}=2$	$k=5,\ \mathtt{M}=2$
	731.121861	1441.642294	2734.317669	5585.482952

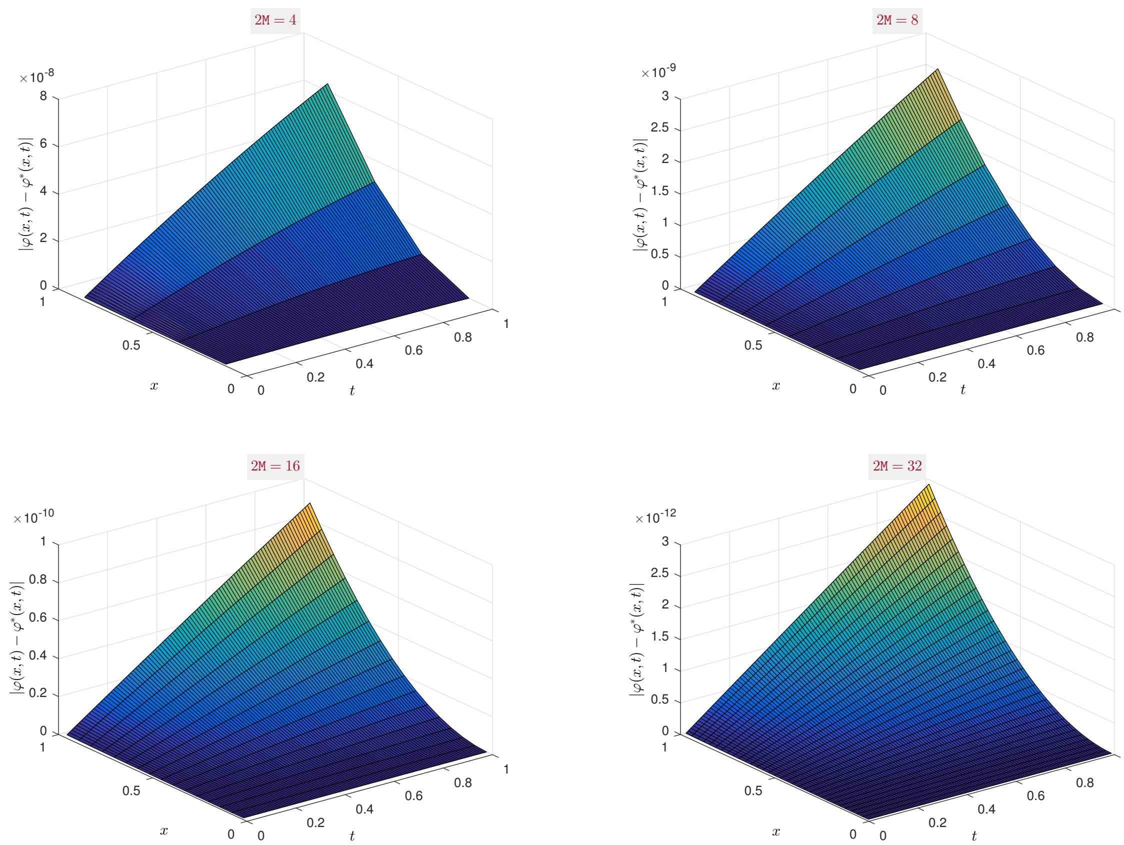

Figure 1: Plot of the absolute errors $\varphi(x,t)$.

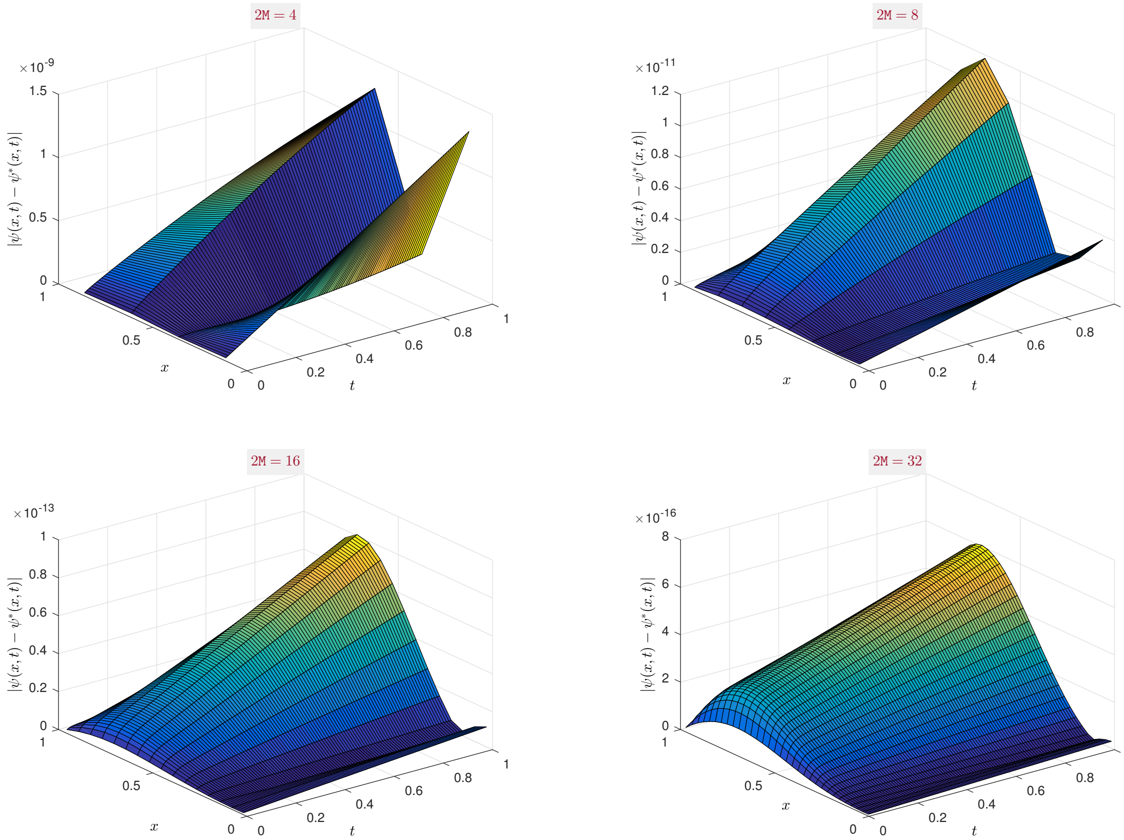

Figure 2: Plot of the absolute errors $\psi(x, t)$.

References

[1] G. Ahmadnezhad, N. Aghazadeh, and S. Rezapour, *Haar wavelet iteration method for solving time fractional Fisher's equation*, *8* (2020), 505–522.

[2] A. H. Arnous, M. Mirzazadeh, and M. Eslami, *Exact solutions of the Drinfel'd–Sokolov–Wilson equation using bäcklund transformation of riccati equation and trial function approach*, Pramana, *86* (2016), 1153–1160.

[3] I. Aziz, S. Islam, and F. Khan, *A new method based on Haar wavelet for the numerical solution of two-dimensional nonlinear integral equations*, Journal of Computational and Applied Mathematics, *272* (2014), 70–80.

[4] R. E. Bellman and R. E. Kalaba, *Quasilinearization And Nonlinear Boundary-Value Problems*, 1965.

[5] C. Cattani, T. A. Sulaiman, H. M. Baskonus, and H. Bulut, *On the soliton solutions to the Nizhnik-Novikov-Veselov and the Drinfel'd-Sokolov systems*, Optical and Quantum Electronics, *50* (2018), 1–11.

[6] I. Daubechies and W. Sweldens, *Factoring wavelet transforms into lifting steps*, Journal of Fourier analysis and applications, *4* (1998), 247–269.

[7] S. Foadian, R. Pourgholi, S. H. Tabasi, and J. Damirchi, *The inverse solution of the coupled nonlinear reaction–diffusion equations by the Haar wavelets*, International Journal of Computer Mathematics, *96* (2019), 105–125.

[8] Ü. Göktaş and W. Hereman, *Symbolic computation of conserved densities for systems of nonlinear evolution equations* Journal of Symbolic Computation, *24* (1997), 591–622.

[9] W. Härdle, G. Kerkyacharian, D. Picard, and A. Tsybakov, *Wavelets, Approximation, And Statistical Applications*, Springer Science & Business Media, 2012.

[10] G. Hariharan and K. Kannan, *A comparative study of Haar wavelet method and Homotopy perturbation method for solving one-dimensional reaction-diffusion equations*, International Journal of Applied Mathematics and Computation, *3* (2011), 21–34.

[11] M. Kumar and S. Pandit, *A composite numerical scheme for the numerical simulation of coupled Burgers' equation*, Computer Physics Communications, *185* (2014), 809–817.

[12] Ü. Lepik, *Solving integral and differential equations by the aid of non-uniform Haar wavelets*, Applied Mathematics and Computation, *198* (2008), 326–332.

[13] Ü. Lepik and H. Hein, *Haar Wavelets: With Applications*, Springer Science & Business Media, 2014.

[14] E. Misirli and Y. Gurefe, *Exact solutions of the Drinfel'd–Sokolov–Wilson equation using the exp-function method*, Applied Mathematics and Computation, *216* (2010), 2623–2627.

[15] A. Mohammadi, N. Aghazadeh, and S. Rezapour, *Wavelet-Picard iterative method for solving singular fractional nonlinear partial differential equations with initial and boundary conditions*, Computational Methods for Differential Equations, *8* (2020), 610–638.

[16] R. Pourgholi, N. Tavallaie, and S. Foadian, *Applications of Haar basis method for solving some ill-posed inverse problems*, Journal of Mathematical Chemistry, *50* (2012), 2317–2337.

[17] S. S. Ray and A. K. Gupta, *Comparative analysis of variational iteration method and Haar wavelet method for the numerical solutions of Burgers–Huxley and Huxley equations*, Journal of mathematical chemistry, *52* (2014), 1066–1080.

[18] L. Shen, S. Zhu, B. Liu, Z. Zhang, and Y. Cui, *Numerical implementation of nonlinear system of fractional Volterra integral–differential equations by Legendre wavelet method and error estimation*, Numerical Methods for Partial Differential Equations, *37* (2020), 1344–1360.

[19] P. Sliwinski and Z. Hasiewicz, *Computational algorithms for wavelet identification of nonlinearities in hammerstein systems with random inputs*, IEEE Transactions on Signal Processing, *56* (2008), 846–851.

[20] E. Sweet and R. A. V. Gorder, *Exponential-type solutions to a generalized Drinfel'd–Sokolov equation*, Physica Scripta, *82* (2010), 035006.

[21] A. M. Wazwaz, *Exact and explicit travelling wave solutions for the nonlinear Drinfeld–Sokolov system* Communications in nonlinear science and numerical simulation, *11* (2006), 311–325.

[22] N. Xiao-Xing and L. Qing-Ping, *Darboux transformation for Drinfel'd–Sokolov–Wilson equation*, Communications in Theoretical Physics, *64* (2015), 491.