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Extremal solutions for multi-term nonlinear fractional differential equations with nonlinear boundary conditions

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Abstract

This paper is devoted to prove the existence of extremal solutions for multi-term nonlinear fractional differential equations with nonlinear boundary conditions. The fractional derivative is of Caputo type and the inhomogeneous term depends on the fractional derivatives of lower orders. By establishing a new comparison theorem and applying the monotone iterative technique, we show the existence of extremal solutions. The method is a constructive method that yields monotone sequences that converge to the extremal solutions. As an application, some examples are presented to illustrate the main results.

Keywords. Caputo fractional derivative, Extremal solutions, Existence, Approximation, Nonlinear boundary conditions.

2010 Mathematics Subject Classification. 26A33, 34A08, 34A12.

1. Introduction

Fractional differential equations appear naturally in a number of fields such as physics, geophysics, polymer rheology, viscoelasticity, capacitor theory, electrical circuits, electron-analytical chemistry, biology, etc. For more details and applications, we refer the reader to the books [11, 23, 27] and references therein. The fractional operators are nonlocal, therefore they are suitable for constructing models possessing memory and hereditary properties of various materials and processes [25]. The presence of memory terms in such models not only takes into account the history of the process involved but also carries its impact to present and future development of the process. Fractional differential equations are also regarded as an alternative model to nonlinear differential equations [7] and the references [3, 4, 22, 32].

Recently, there are many papers dealing with the existence of solutions for nonlinear fractional differential equation using various methods. We refer the interested readers to the valuable monographs of Kilbas et al. [23] as well to [11, 27] and the references [1, 6, 15, 18, 20, 21, 29]. Among the most important of all fractional differential equations are undoubtedly multi-term fractional differential equations. These problems are of interest because of their appearance in mathematical models of several important physical phenomena. A classical example is Bagley-Torvik equation which arises in the modeling of the motion of a rigid plate immersed in a Newtonian fluid [16]. Other examples are The Langevin equation which is widely used to describe the evolution of physical phenomena in fluctuating environments and the steady nonlinear fractional advection-dispersion equation [14, 17, 19]. Analysis of multi-term fractional differential equations has been carried out by various researchers. Recently, Kosmatov [24] considered the existence of unique continuously differentiable solutions for multi-term fractional differential equations with initial conditions. In [9, 10] Deng et al. improved the result in [24], and obtained some new sufficient conditions for the existence and uniqueness results. Recently, multi-point boundary value problems for multi-term fractional differential equations have been studied in [28]. Some more recent work on multi-term fractional differential equations can be found in [5, 12, 26, 31].

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This paper deals with the existence of extremal solutions of the following nonlinear multi-term fractional equation with nonlinear conditions

$$\begin{cases} D^{\alpha}u(t) = f\left(t, u(t), D^{\alpha_1}u(t), D^{\alpha_2}u(t), \cdots, D^{\alpha_m}u(t)\right), \quad 0 < t < 1, \\ g_i\left(u^{(i)}(0), u^{(i)}(1)\right) = 0, \quad i = 0, 1, \cdots, n, \end{cases} \tag{1.1}$$

where $m, n \in \mathbb{N}$, $\alpha \in (n, n+1)$ and $\max\{\alpha_1, \alpha_2, \cdots, \alpha_m\} \leq n$, and D^{α} is the Caputo fractional derivative of order α. The nonlinear functions f and g_i are assumed to satisfy certain conditions, which will be specified later.

In this study, the technique used is in the basis of monotone iteration scheme which is an interesting and powerful mechanism that offers theoretical existence results for nonlinear problems. The advantage and importance of this method arises from the fact that it is a constructive method that yields monotone sequences that converge to the extremal solutions of the main problem, see [13, 33]. As a first step, the comparison result associated to the main problem is obtained. This step was the main advantage of our approach compared to the other studies which help us to construct some arbitrarily closely approximate solutions by using lower and upper solutions. On the other hand, to the best of our knowledge, the existence of extremal solutions for nonlinear multi-term fractional equations with nonlinear boundary conditions have not been studied previously.

These conditions are of significance because they have applications in the problems of physics and other areas of applied mathematics. Conditions of this type can be applied in the theory of elasticity with better effect than the initial or boundary conditions.

For the importance of nonlinear boundary conditions in different fields we refer to [8, 30] and the references cited therein.

The paper is organized as follows. We give a brief review of the fractional calculus theory in section 2. In section 3, we establish a new comparison principle and obtain the existence of extremal solutions for (1.1) by utilizing the monotone iterative technique and the method of lower and upper solutions. Finally, some examples are given to illustrate our results.

2. Preliminaries

For the convenience of the reader, we recall the necessary definitions of fractional calculus theory and Lemmas. For more detailed references, see [11, 23, 27]. Throughout the paper $AC^n[0,1], n \in \mathbb{N}$, denotes the set of functions having absolutely continuous nth derivative on $[0,1]$, and $AC[0,1]$ is the set of absolutely continuous functions on $[0,1]$.

Definition 2.1. The fractional order integral of order $\alpha > 0$ of a function $u : [0,1] \to \mathbb{R}$ is defined as

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau) d\tau,$$

provided the integral exists on $[0,1]$.

Definition 2.2. The Caputo fractional derivative of order $\alpha > 0$ of a function $u \in AC^{n-1}[0,1]$ is defined as

$$D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} u^{(n)}(\tau) d\tau = I^{n-\alpha}u^{(n)}(t),$$

where $n-1 < \alpha \leq n$.

Lemma 2.3. [11]. *Let $\alpha \in (n, n+1)$. If $u \in AC^n[0,1]$, then $I^{\alpha}D^{\alpha}u(t) = u(t) - \sum_{i=0}^{n} \frac{u^{(i)}(0)}{i!} t^i$.*

Lemma 2.4. [24]. *Let $u \in C^n[0,1]$ and $\alpha, \varepsilon \geq 0$ be such that $\alpha, \alpha+\varepsilon \in [n-1, n]$. Then $D^{\varepsilon}D^{\alpha}u(t) = D^{\alpha+\varepsilon}u(t)$.*

Lemma 2.5. [2]. *Let $\alpha \in (n, n+1)$ and $h \in AC[0,1]$. Then the linear fractional initial value problem*

$$D^{\alpha}u(t) = h(t), \qquad u^{(i)}(0) = \eta_i, \quad i = 0, 1, \cdots, n,$$

has a unique solution $u \in AC^n[0,1]$ with the following integral form

$$u(t) = \sum_{i=0}^{n} \frac{\eta_i}{i!} t^i + I^{\alpha}h(t).$$

3. Main result

In this section, we present an important comparison result about fractional differential equations and use it to construct two monotone iterative sequences which converge to extremal solutions for the problem (1.1).

Definition 3.1. We define the following order relation for $AC^n[0,1]$,

$$u \preceq v \quad \Longleftrightarrow \quad u^{(i)}(t) \leq v^{(i)}(t), \quad t \in [0,1], \quad i = 0, 1, \cdots, n.$$

Remark 3.2. In view of $D^\gamma u(t) = I^{\lceil \gamma \rceil - \gamma} D^{\lceil \gamma \rceil} u(t)$ where $\lceil \gamma \rceil = \min\{k \in \mathbb{N} : \gamma \leq k\}$ together with the fact that the fractional integral operator is a monotone operator, we see that $u \preceq v$ if and only if $D^\gamma u(t) \leq D^\gamma v(t)$ on $[0,1]$ for every $\gamma \in [0,n]$.

Lemma 3.3. (Comparison result). *Let $\alpha \in (n, n+1), n \in \mathbb{N}$ and $u \in AC^n[0,1]$ satisfies*

$$\begin{cases} D^\alpha u(t) \leq 0, & t \in (0,1], \\ u^{(i)}(0) \leq 0, & i = 0, 1, \cdots, n, \end{cases}$$

then $u \preceq 0$ on $[0,1]$.

Proof. Since $u \in AC^n[0,1]$, we have $D^\alpha u(t) = I^{(n+1)-\alpha} u^{(n+1)}(t) = D^{\alpha - n} u^{(n)}(t)$. Therefore

$$\begin{cases} D^{\alpha-n} u^{(n)}(t) \leq 0, \\ u^{(n)}(0) \leq 0. \end{cases} \tag{3.1}$$

Now by applying the fractional integral operator $I^{\alpha-n}$ on both sides of (3.1), and using Lemma 2.3 together with the fact that the integral operator is a monotone operator, we have $u^{(n)}(t) \leq u^{(n)}(0) \leq 0$ on $[0,1]$. In fact, we have

$$\begin{cases} u^{(n)}(t) \leq 0, \\ u^{(n-1)}(0) \leq 0. \end{cases} \tag{3.2}$$

Similarly, by applying the integral operator I^1 on both sides of (3.2), we have $u^{(n-1)}(t) \leq u^{(n-1)}(0) \leq 0$ on $[0,1]$. By repeating this process $n-1$ times, we deduce $u^{(i)}(t) \leq 0$ on $[0,1]$ for every $i = 0, 1, \cdots, n$. On the other hand, since $u \in C^n[0,1]$, we obtain from Lemma 2.4,

$$\begin{cases} D^{\lceil \gamma \rceil - \gamma} D^\gamma u(t) \leq 0, \\ D^\gamma u(0) = 0, \end{cases} \tag{3.3}$$

for $\gamma \in (0,n), \gamma \notin \mathbb{N}$. Note that $D^\gamma u(0) = I^{\lceil \gamma \rceil - \gamma} u^{(\lceil \gamma \rceil)}(0) = 0$. Therefore, by applying the integral operator $I^{\lceil \gamma \rceil - \gamma}$ on both sides of (3.3), we deduce $D^\gamma u(t) \leq 0$. $\square$

Definition 3.4. We say that $u \in AC^n[0,1]$ is called a lower solution of (1.1) if

$$\begin{cases} D^\alpha u(t) \leq f\left(t, u(t), D^{\alpha_1} u(t), D^{\alpha_2} u(t), \cdots, D^{\alpha_m} u(t)\right), & 0 < t < 1, \\ g_i\left(u^{(i)}(0), u^{(i)}(1)\right) \leq 0, & i = 0, 1, \cdots, n. \end{cases}$$

and it is an upper solution of (1.1) if the above inequalities are reverted.

We list the following assumptions for convenience.

(H1) Assume that $\underline{u}, \overline{u} \in AC^n[0,1]$ are lower and upper solutions of the problem (1.1), respectively, and $\underline{u} \preceq \overline{u}$.

(H2) $f : [0,1] \times \mathbb{R}^{m+1} \to \mathbb{R}$ be a function such that $f\left(t, u(t), D^{\alpha_1} u(t), D^{\alpha_2} u(t), \cdots, D^{\alpha_m} u(t)\right) \in AC[0,1]$ for every $u \in AC^n[0,1]$.

(H3) The function f satisfies

$$f\left(t, u(t), D^{\alpha_1} u(t), D^{\alpha_2} u(t), \cdots, D^{\alpha_m} u(t)\right) \leq f\left(t, v(t), D^{\alpha_1} v(t), D^{\alpha_2} v(t), \cdots, D^{\alpha_m} v(t)\right),$$

for $\underline{u} \preceq u \preceq v \preceq \overline{u}$.

(H4) For every $i = 0, 1, \cdots, n$, $g_i \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, there exist constants $\lambda_i > 0$ and $\mu_i \geq 0$, such that

$$g_i(\bar{x}, \bar{y}) - g_i(x, y) \leq \lambda_i(\bar{x} - x) - \mu_i(\bar{y} - y), \quad \underline{u}^{(i)}(0) \leq x \leq \bar{x} \leq \overline{u}^{(i)}(0), \quad \underline{u}^{(i)}(1) \leq y \leq \bar{y} \leq \overline{u}^{(i)}(1).$$

Theorem 3.5. *Suppose that conditions (H1)-(H4) hold. Then, there is an extremal solution $(u^*, v^*) \in [\underline{u}, \overline{u}] \times [\underline{u}, \overline{u}]$ of the problem (1.1). Moreover, there exist monotone iterative sequences $\{u_j\}, \{v_j\} \subseteq [\underline{u}, \overline{u}]$ such that for every $\gamma \in [0, n]$, $D^\gamma u_j \to D^\gamma u^*$ and $D^\gamma v_j \to D^\gamma v^*$ uniformly on $[0, 1]$.*

Proof. The proof is divided into four steps:
Step 1. Set $u_0 = \underline{u}$ and $v_0 = \overline{u}$ and then given $\{u_j\}_{j=0}^\infty$ and $\{v_j\}_{j=0}^\infty$ inductively define $u_{j+1} \in AC^n[0,1]$ and $v_{j+1} \in AC^n[0,1]$ to be the unique solutions of the linear problem

$$\begin{cases} D^\alpha u_{j+1}(t) = f\left(t, u_j(t), D^{\alpha_1}u_j(t), D^{\alpha_2}u_j(t), \cdots, D^{\alpha_m}u_j(t)\right), \quad j \geq 0 \\ u_{j+1}^{(i)}(0) = \eta_j^i, \quad i = 0, 1, \cdots, n, \end{cases} \tag{3.4}$$

and

$$\begin{cases} D^\alpha v_{j+1}(t) = f\left(t, v_j(t), D^{\alpha_1}v_j(t), D^{\alpha_2}v_j(t), \cdots, D^{\alpha_m}v_j(t)\right), \quad j \geq 0 \\ v_{j+1}^{(i)}(0) = \overline{\eta}_j^i, \quad i = 0, 1, \cdots, n, \end{cases} \tag{3.5}$$

where $\eta_j^i = u_j^{(i)}(0) - \frac{1}{\lambda_i} g_i\left(u_j^{(i)}(0), u_j^{(i)}(1)\right)$ and $\overline{\eta}_j^i = v_j(0) - \frac{1}{\lambda_i} g_i\left(v_j^{(i)}(0), v_j^{(i)}(1)\right)$. From Lemma 2.5, we know that (3.4) and (3.5) have a unique solutions in $AC^n[0,1]$.
Step 2. We claim

$$\underline{u} = u_0 \preceq u_1 \preceq \cdots \preceq u_j \preceq u_{j+1} \preceq \cdots \preceq v_{j+1} \preceq v_j \preceq \cdots \preceq v_1 \preceq v_0 = \overline{u}. \tag{3.6}$$

To confirm this, first note from (3.4) for $j = 0$ that

$$\begin{cases} D^\alpha u_1(t) = f\left(t, u_0(t), D^{\alpha_1}u_0(t), D^{\alpha_2}u_0(t), \cdots, D^{\alpha_m}u_0(t)\right), \\ u_1^{(i)}(0) = u_0^{(i)}(0) - \frac{1}{\lambda_i} g_i\left(u_0^{(i)}(0), u_0^{(i)}(1)\right), \quad i = 0, 1, \cdots, n. \end{cases} \tag{3.7}$$

Recalling the definition of lower solution $u_0 = \underline{u}$ and setting $w = u_0 - u_1$, we find

$$\begin{cases} D^\alpha w(t) \leq 0, \\ w^{(i)}(0) = \frac{1}{\lambda_i} g_i\left(u_0^{(i)}(0), u_0^{(i)}(1)\right) \leq 0, \quad i = 0, 1, \cdots, n. \end{cases}$$

Consequently Lemma 3.3 implies $w \preceq 0$, so that $u_0 \preceq u_1$. Now, from (3.7) and using assumptions (H3) and (H4), we infer

$$\begin{aligned} D^\alpha u_1(t) &= f\left(t, u_0(t), D^{\alpha_1}u_0(t), D^{\alpha_2}u_0(t), \cdots, D^{\alpha_m}u_0(t)\right) \\ &\leq f\left(t, u_1(t), D^{\alpha_1}u_1(t), D^{\alpha_2}u_1(t), \cdots, D^{\alpha_m}u_1(t)\right), \end{aligned}$$

and

$$\begin{aligned} g_i\left(u_1^{(i)}(0), u_1^{(i)}(1)\right) &\leq g_i\left(u_0^{(i)}(0), u_0^{(i)}(1)\right) + \lambda_i\left(u_1^{(i)}(0) - u_0^{(i)}(0)\right) - \mu_i\left(u_1^{(i)}(1) - u_0^{(i)}(1)\right) \\ &= -\mu_i\left(u_1^{(i)}(1) - u_0^{(i)}(1)\right) \\ &\leq 0. \end{aligned}$$

Therefore, u_1 is lower solution of problem (1.1). We can now repeat the argument above to deduce $u_1 \preceq u_2$ and then an induction verifies that $u_j \preceq u_{j+1}$ for $j \geq 2$. Assertion $v_j \preceq v_{j-1}$ for $j \in \mathbb{N}$ follows similarly. Now, we put $w = u_1 - v_1$. From (H3) and (H4), we have

$$\begin{cases} D^\alpha w(t) \leq 0, \\ w^{(i)}(0) \leq \frac{\mu_i}{\lambda_i}\left(u_0^{(i)}(1) - v_0^{(i)}(1)\right) \leq 0, \quad i = 0, 1, \cdots, n. \end{cases}$$

Consequently, $w \preceq 0$, so that $u_1 \preceq v_1$. Using mathematical induction, we see that $u_j \preceq v_j$ for $j \geq 2$.
Step 3. In light of (3.6), it is easy to show $\{u_j\}$ and $\{v_j\}$ are uniformly bounded and equicontinuous in $[\underline{u}, \overline{u}]$. By the Arzela-Ascoli Theorem, for every $\gamma \in [0, n]$, we have

$$\lim_{n \to \infty} D^\gamma u_j = u^*, \qquad \lim_{n \to \infty} D^\gamma v_j = D^\gamma v^*,$$

uniformly on $[0, 1]$, and the limit functions u^*, v^* satisfy (1.1). Moreover, $u^*, v^* \in [\underline{u}, \overline{u}]$.

Step 4. Finally, we prove u^* and v^* are the extremal solutions of (1.1) in $[\underline{u}, \overline{u}]$. Let $u \in [\underline{u}, \overline{u}]$ be any solution of (1.1). We suppose that $u_j \preceq u \preceq v_j$ for some $j \in \mathbb{N}$. Then, by assumption (H3), we see that

$$\begin{aligned} f\left(t, u_j(t), D^{\alpha_1} u_j(t), D^{\alpha_2} u_j(t), \cdots, D^{\alpha_m} u_j(t)\right) &\leq f\left(t, u(t), D^{\alpha_1} u(t), D^{\alpha_2} u(t), \cdots, D^{\alpha_m} u(t)\right) \\ &\leq f\left(t, v_j(t), D^{\alpha_1} v_j(t), D^{\alpha_2} v_j(t), \cdots, D^{\alpha_m} v_j(t)\right), \end{aligned}$$

and

$$\begin{aligned} u_{j+1}^{(i)}(0) &= u_j^{(i)}(0) - \frac{1}{\lambda_i} g_i\left(u_j^{(i)}(0), u_j^{(i)}(1)\right) \\ &= u_j^{(i)}(0) + \frac{1}{\lambda_i} g\left(u^{(i)}(0), u^{(i)}(1)\right) - \frac{1}{\lambda_i} g\left(u_j^{(i)}(0), u_j^{(i)}(1)\right) \\ &\leq u^{(i)}(0) - \frac{\mu_i}{\lambda_i}\left(u^{(i)}(1) - u_j^{(i)}(1)\right) \\ &\leq u^{(i)}(0). \end{aligned}$$

Similarly, we have $u^{(i)}(0) \leq v_{j+1}^{(i)}(0)$. Hence

$$\begin{cases} D^\alpha u_{j+1}(t) \leq D^\alpha u(t) \leq D^\alpha v_{j+1}(t), \\ u_{j+1}^{(i)}(0) \leq u^{(i)}(0) \leq v_{j+1}^{(i)}(0). \end{cases}$$

Consequently, $u_{j+1} \preceq u \preceq v_{j+1}$. Therefore, we have

$$u_j \preceq u \preceq v_j, \qquad j = 0, 1, 2, \cdots. \tag{3.8}$$

Thus, taking limit in (3.8) as $j \to \infty$, we have $u^* \preceq u \preceq v^*$. That is, u^* and v^* are the extremal solutions of (1.1) in $[\underline{u}, \overline{u}]$. □

Remark 3.6. In a similar way, we can deal with the existence results of solutions for problem (1.1) with more general nonlinear nonlocal conditions

$$g_i\left(u^{(i)}(t_0), u^{(i)}(t_1), \cdots, u^{(i)}(t_r)\right) = 0, \quad i = 0, 1, \cdots, n,$$

where $0 = t_0 < t_1 < \cdots < t_r = 1$, under some conditions.

4. Examples

Example 4.1. Let us consider the following problem

$$\begin{cases} D^{\frac{3}{2}} u(t) = \frac{3(1+t^2)}{10} + \frac{3}{10} D^{\frac{1}{2}} u(t) + \frac{1}{2} u'(t), \quad 0 < t < 1, \\ u^2(0) - u(0) = 0, \quad u'(0) = 0. \end{cases} \tag{4.1}$$

A relatively simple calculus, with the help of Maple, shows that $\underline{u}(t) = 0$ and $\overline{u}(t) = 1 + 0.1\,t + t^{1.5}$ be lower and upper solutions of (4.1), respectively, and $\underline{u} \preceq \overline{u}$. In addition, it is easy to verify that the assumptions (H2) - (H4) hold. Therefore, all the assumption of Theorem 3.5 hold and consequently, there exist monotone iterative sequences $\{u_j\}, \{v_j\}$, which converge uniformly on $[0,1]$ to the extremal solutions u^* and v^* of (4.1) in $[0, 1+0.1\,t+t^{1.5}]$. Moreover, for every $\gamma \in [0,1]$, the sequences $\{D^\gamma u_j\}$ and $\{D^\gamma v_j\}$ converge uniformly $D^\gamma u^*$ and $D^\gamma v^*$ on $[0,1]$, respectively. On the other hand, from Theorem 3.5 and Lemma 2.5, the sequences $\{u_j\}$ and $\{v_j\}$ can be obtained as

$$u_{j+1}(t) = u_j(0) - \left(u_j^2(0) - u_j(0)\right) + \frac{1}{\Gamma(\frac{3}{2})} \int_0^t (t-s)^{\frac{1}{2}} \left(\frac{3}{10}\left(1 + s^2 + D^{\frac{1}{2}} u_j(s)\right) + \frac{1}{2} u_j'(s)\right) ds, \qquad j \geq 0,$$

and

$$v_{j+1}(t) = v_j(0) - \left(v_j^2(0) - v_j(0)\right) + \frac{1}{\Gamma(\frac{3}{2})} \int_0^t (t-s)^{\frac{1}{2}} \left(\frac{3}{10}\left(1 + s^2 + D^{\frac{1}{2}} v_j(s)\right) + \frac{1}{2} v_j'(s)\right) ds, \qquad j \geq 0.$$

The graphs of u_j and v_j, for $j = 0, 1, \cdots, 6$ are shown in Figure 1.

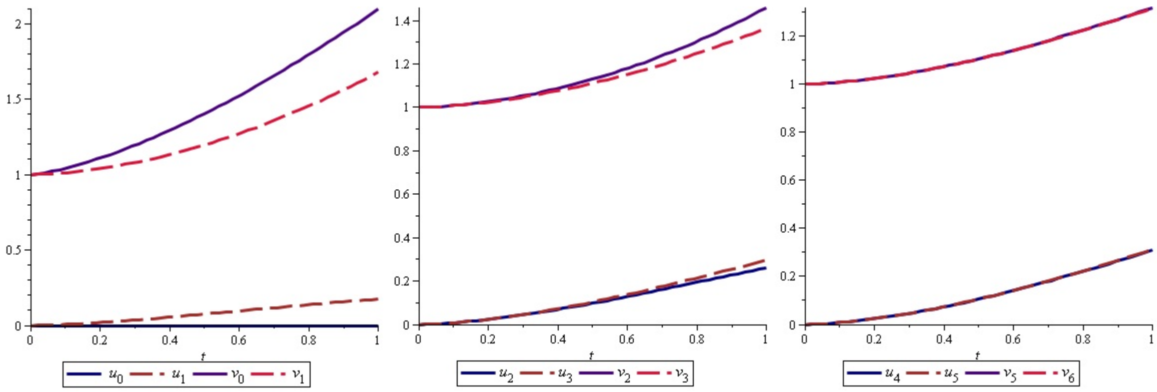

FIGURE 1. Graphs of u_j and v_j

As shown in Figure 1, we see that u_5 and v_5 can be a suitable approximations of the minimal and maximal solutions of (4.1), respectively. Furthermore, the graphs of some derivatives of u_j and v_j, for $j = 0, 1, \cdots, 8$ are shown in Figures 2 and 3.

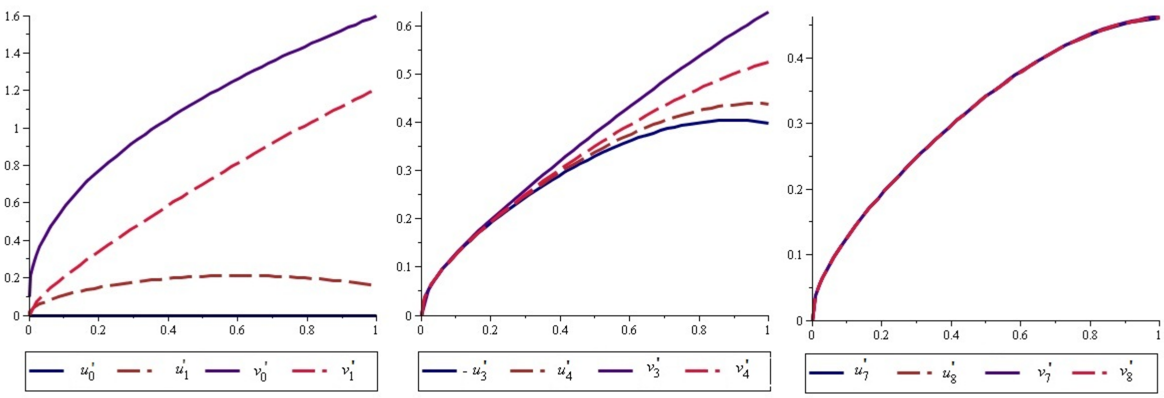

FIGURE 2. Graphs of u_j' and v_j'

Example 4.2. Let us consider the following problem

$$\begin{cases} D^{\frac{5}{2}}u(t) = 1 + t^2 + \frac{1}{4}u(t)u'(t) + \frac{1}{5}D^{\frac{1}{4}}u(t), \quad 0 < t < 1, \\ u^2(0) - u(1) + 4u(0) = 1, \quad 8u'(0) - \frac{1}{10}u'(1) = \frac{1}{5}, \quad 4u''(0) - \frac{1}{2}u''(0)u''(1) = 0. \end{cases} \tag{4.2}$$

Here $f(t, u(t), D^{\alpha_1}u(t), u'(t)) = 1 + t^2 + \frac{1}{4}u(t)u'(t) + \frac{1}{5}D^{\frac{1}{4}}u(t)$, $g_0(x, y) = x^2 - y + 4x - 1$, $g_1(x, y) = 8x - \frac{1}{10}y - \frac{1}{5}$ and $g_2(x, y) = 4x - \frac{1}{2}xy$. A relatively simple calculus, with the help of Maple, shows that $\underline{u}(t) = 0$ and $\overline{u}(t) = 1 + 0.1\,t + 0.05\,t^2 + 2\,t^{\frac{5}{2}}$ be lower and upper solutions of (4.2), respectively, and $\underline{u} \preceq \overline{u}$. In addition, it is easy to verify that the assumptions (H2) and (H3) hold and $g_i \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $i = 0, 1, 2$ and we have

$$\begin{aligned} g_0(\bar{x}, \bar{y}) - g_0(x, y) &\leq 6(\bar{x} - x) - (\bar{y} - y), && 0 \leq x \leq \bar{x} \leq 1, \quad 0 \leq y \leq \bar{y} \leq \frac{315}{100}, \\ g_1(\bar{x}, \bar{y}) - g_1(x, y) &\leq 8(\bar{x} - x) - \frac{1}{10}(\bar{y} - y), && 0 \leq x \leq \bar{x} \leq \frac{1}{10}, \quad 0 \leq y \leq \bar{y} \leq \frac{52}{10}, \\ g_2(\bar{x}, \bar{y}) - g_2(x, y) &\leq 4(\bar{x} - x), && 0 \leq x \leq \bar{x} \leq \frac{1}{10}, \quad 0 \leq y \leq \bar{y} \leq \frac{76}{10}. \end{aligned}$$

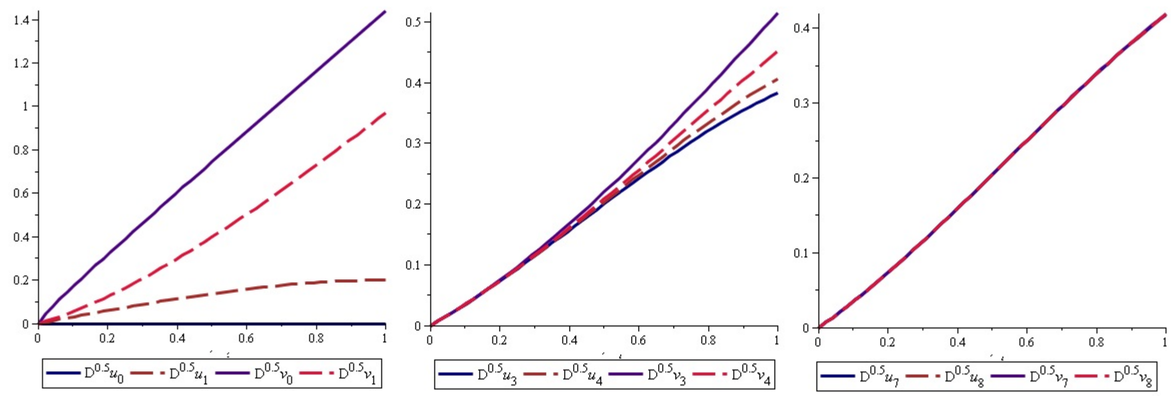

FIGURE 3. Graphs of $D^{\frac{1}{2}}u_j$ and $D^{\frac{1}{2}}v_j$

Therefore, all the assumption of Theorem 3.5 hold and consequently, there exist monotone iterative sequences $\{u_j\}, \{v_j\}$, which converge uniformly on $[0,1]$ to the extremal solutions (u^*, v^*) of (4.2) in $[0, 1+0.1\,t+0.05\,t^2+2\,t^{\frac{5}{2}}]$. Moreover, for every $\gamma \in [0,2]$, the sequences $\{D^\gamma u_j\}$ and $\{D^\gamma v_j\}$ converge uniformly $D^\gamma u^*$ and $D^\gamma v^*$ on $[0,1]$, respectively. On the other hand, from Theorem 3.5 and Lemma 2.5, the sequences $\{u_j\}$ and $\{v_j\}$ can be obtained as

$$\begin{aligned} u_{j+1}(t) &= u_j(0) - \frac{1}{6}\left(u_j^2(0) - u_j(1) + 4u_j(0) - 1\right) + \frac{1}{8}\left(\frac{1}{10}u_j'(1) + \frac{1}{5}\right)t + \frac{1}{8}u_j''(0)u_j''(1)t^2 \\ &\quad + \frac{1}{\Gamma(\frac{5}{2})}\int_0^t (t-s)^{\frac{3}{2}}\left(1 + s^2 + \frac{1}{4}u_j(s)u_j'(s) + \frac{1}{5}D^{\frac{1}{4}}u_j(s)\right)\, ds, \qquad j \geq 0, \end{aligned}$$

and

$$\begin{aligned} v_{j+1}(t) &= v_j(0) - \frac{1}{6}\left(v_j^2(0) - v_j(1) + 4v_j(0) - 1\right) + \frac{1}{8}\left(\frac{1}{10}v_j'(1) + \frac{1}{5}\right)t + \frac{1}{8}v_j''(0)v_j''(1)t^2 \\ &\quad + \frac{1}{\Gamma(\frac{5}{2})}\int_0^t (t-s)^{\frac{3}{2}}\left(1 + s^2 + \frac{1}{4}v_j(s)v_j'(s) + \frac{1}{5}D^{\frac{1}{4}}v_j(s)\right)\, ds, \qquad j \geq 0, \end{aligned}$$

The graphs of u_j and v_j, for $j = 0, 4, 8$ are shown in Figure 4.

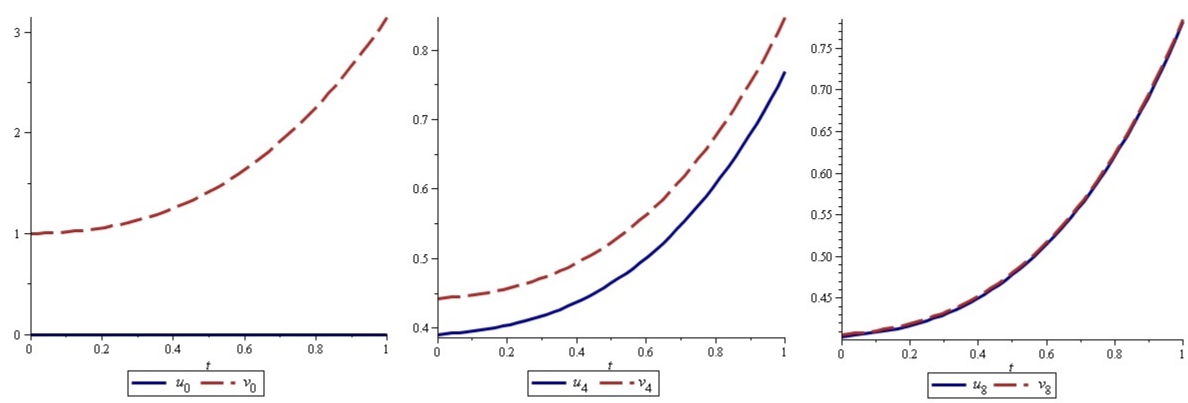

FIGURE 4. Graphs of u_j and v_j

This figure shows that the minimal and maximal solutions in the interval $[\underline{u}, \overline{u}]$ are equal. Therefore, from Theorem 3.5 , we deduce the problem (4.2) has unique solution in the interval $[\underline{u}, \overline{u}]$. As shown in Figure 4, we see that u_8 as well as v_8 can be a suitable approximations of the uniqe solution of (4.2). Furthermore, the graphs of some derivatives of u_j and v_j, for $j = 0, 4, 8$ are shown in Figures 5,6 and 7.

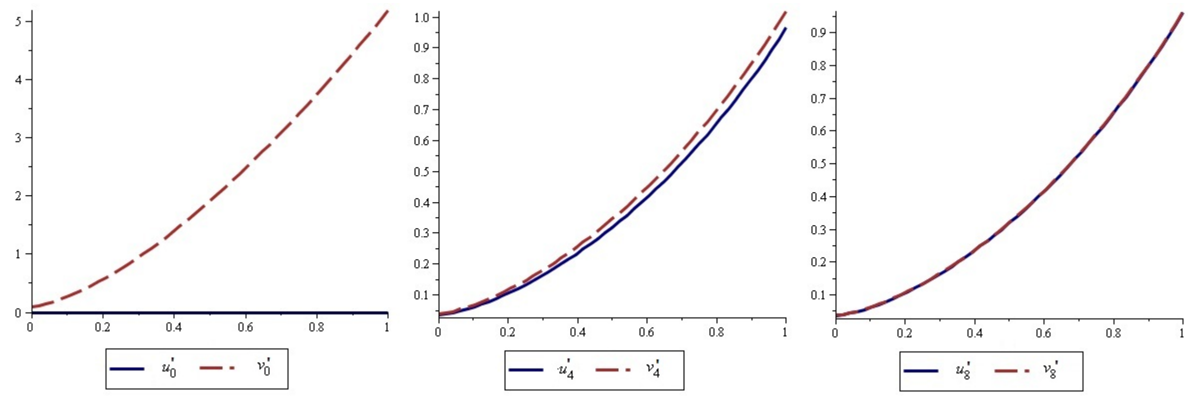

FIGURE 5. Graphs of u_j' and v_j'

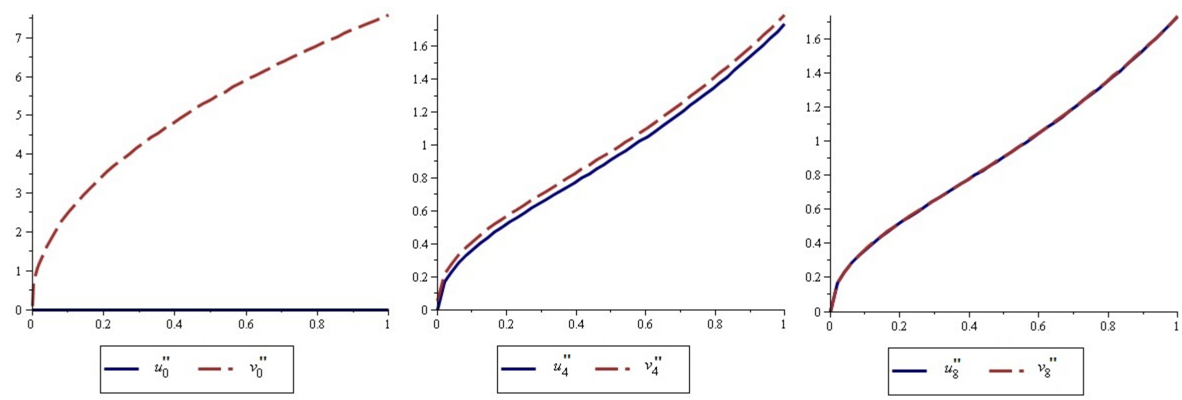

FIGURE 6. Graphs of u_j'' and v_j''

5. Conclusion

We have presented some results dealing with the existence of extremal solutions for multi-term nonlinear fractional differential equations with nonlinear boundary conditions. As a first step, we have established a new comparison result by applying the tools of fractional calculus. Then, the existence results are established by dint of the monotone iterative technique. The method is a constructive method that yields monotone sequences that converge to the extremal solutions. Two numerical examples have been carried out to verify the effectiveness and reliability of the method.

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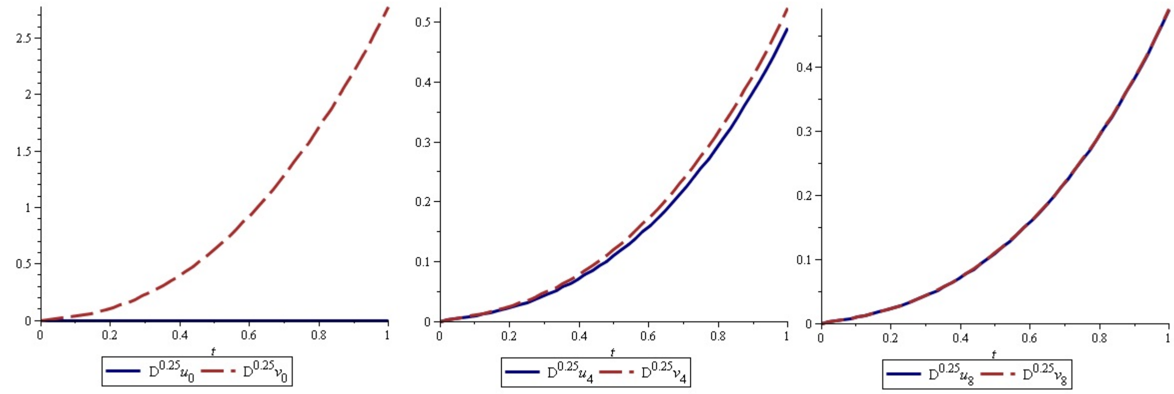

FIGURE 7. Graphs of $D^{\frac{1}{4}}u_j$ and $D^{\frac{1}{4}}v_j$

REFERENCES

[1] P. Agarwal, R. P. Agarwal, and M. Ruzhansky, (Eds.), *Special Functions and Analysis of Differential Equations (1st ed.)*, Chapman and Hall/CRC, 2020.

[2] R. P. Agarwal, M. Benchohra, and S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Appl. Math., *109* (2010), 973-1033.

[3] P. Agarwal, A. Berdyshev, and E. Karimov, *Solvability of a Non-local Problem with Integral Transmitting Condition for Mixed Type Equation with Caputo Fractional Derivative*, Results Math., *71* (2017), 1235-1257.

[4] P. Agarwal, S. Jain, and T. Mansour, *Further extended Caputo fractional derivative operator and its applications*, Russ. J. Math. Phys., *24* (2017), 415-425.

[5] A. Aghajani, Y. Jalilian, and J. J. Trujillo, *On the existence of solutions of fractional integro-differential equations*, Fract. Calc. Appl. Anal., *15* (2012), 44-69.

[6] F. Bahrami, H. Fazli, and A. Jodayree Akbarfam, *A new approach on fractional variational problems and Euler-Lagrange equations*, Commun. Nonlinear Sci. Numer. Simul., *23* (2015), 39-50.

[7] B. Bonilla, M. Rivero, L. Rodriguez-Germa, and J. J. Trujillo, *Fractional differential equations as alternative models to nonlinear differential equations*, Appl. Math. Comput., *187* (2007), 79-88.

[8] L. Byszewski, *Theorems about the existence and uniqueness of solution of semilinear evolution nonlocal Cauchy problem*, J. Math. Anal. Appl., *162* (1991), 494-505.

[9] J. Deng and L. Ma, *Existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations*, Appl. Math. Lett., *23* (2010), 676-680

[10] J. Deng and Z. Deng, *Existence of solutions of initial value problems for nonlinear fractional differential equations*, Appl. Math. Lett., *32* (2014), 6-12

[11] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer-Verlag, Berlin, 2010.

[12] K. Diethelm and N. J. Ford, *Multi-order fractional differential equations and their numerical solution*, Appl. Comput. Math., *154* (2004), 621-640.

[13] Y. Ding, Z. Wei, J. Xu, and D. ÓRegan, *Extremal solutions for nonlinear fractional boundary value problems with p-Laplacian*, J. Comput. Appl. Math., *288* (2015), 151-158.

[14] H. Fazli and F. Bahrami, *On the steady solutions of fractional reaction-diffusion equations*, Filomat, *31* (2017), 1655-1664.

[15] H. Fazli, J. J. Nieto, and F. Bahrami, *On the existence and uniqueness results for nonlinear sequential fractional differential equations*, Appl. Comput. Math., *17* (2018), 36-47.

[16] H. Fazli and J. J. Nieto, *An investigation of fractional Bagley-Torvikequation*, Open Math., *17* (2019), 499-512.

[17] H. Fazli and J. J. Nieto, *Fractional Langevin Equation with Anti-Periodic Boundary Conditions*, Chaos, Solitons and Fractals, *114* (2018), 332-337.

[18] H. Fazli and J. J. Nieto, *Nonlinear sequential fractional differential equations in partially ordered spaces*, Filomat, *32* (2018), 4577–4586.

[19] H. Fazli, H. Sun, and S.Agchi, *Existence of extremal solutions of fractional Langevin equation involving nonlinear boundary conditions*, Int. J. Comput. Math., (2020), 1720662

[20] H. Fazli, H. Sun, and J. J. Nieto, *New existence and stability results for fractional Langevin equation with three-point boundary conditions*, Comp. Appl. Math., *40* (2021), 48.

[21] H. Fazli, H. Sun, and J. J. Nieto, *Fréchet-Kolmogorov compactness of Prabhakar integral operator*, RACSAM, *115* (2021), 165.

[22] H. A Hammad, P. Agarwal, S. Momani, and F. Alsharari, *Solving a Fractional-Order Differential Equation Using Rational Symmetric Contraction Mappings*. Fractal Fract., *5* (2021), 159.

[23] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, in: North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam, 2006.

[24] N. Kosmatov, *Integral equations and initial value problems for nonlinear differential equations of fractional order*, Nonlinear Anal., *70* (2009), 2521-2529.

[25] M. P. Lazarevic and A. M. Spasic, *Finite-time stability analysis of fractional order time delay systems: Gronwall's approach*, Math. Comput. Modelling, *49* (2009), 475-481.

[26] Q. Li, C. Hou, L. Sun, and Z. Han, *Existence and uniqueness for a class of multi-term fractional differential equations*, J. Appl. Math. Comput., *53* (2017), 383-395.

[27] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[28] M. Rehman and R. A. Khan, *Existence and uniqueness of solutions for multi-point boundary value problems for fractional differential equations*, Appl. Math. Lett., *23* (2010), 1038-1044

[29] M. Ruzhansky, Y. J. Cho, P. Agarwal, I. Area, (Eds.), *Advances in Real and Complex Analysis with Applications* (Trends in Mathematics), Birkhuser, 2017.

[30] L. Schwan, O. Umnova, C. Boutin, and J. Groby, *Nonlocal boundary conditions for corrugated acoustic metasurface with strong near-field interactions* , Journal of Applied Physics, *123* (2018), 091712, doi.org/10.1063/1.5011385

[31] X. Su, *Boundary value problem for a coupled systerm of nonlinear fractional differential equations*, Appl. Math. Lett., *22* (2009), 64-69.

[32] A. Sunarto, P. Agarwal, J. Sulaiman, and J. Vui, *Computational Approach via Half-Sweep and Preconditioned AOR for Fractional Diffusion*. Intelligent Automation & Soft Computing, *31*(2) (2022), 1173-1184.

[33] G. T. Wang, R. P. Agarwal, and A. Cabada, *Existence results and the monotone iterative technique for systems of nonlinear fractional differential equations*, Appl. Math. Lett., *25* (2012), 1019-1024.