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An epidemic model for drug addiction

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Abstract

The two most common ways to prevent spreading drug addiction are counseling and imprisonment. In this paper, we propose and study a model for the spread of drug addiction incorporating the effect of consultation and incarceration of addicted individuals. We extract the basic reproductive ratio and study the occurrence of backward bifurcation. Also, we study the local and global stability of drug-free and endemic equilibria under suitable conditions. Finally, we use numerical simulations to illustrate the obtained analytical results.

Keywords. Epidemic model, Drug addiction, Backward bifurcation, Global Stability.
2010 Mathematics Subject Classification. 92D30; 34D23; 34C23.

1. Introduction

The issue of opioid drug addiction is one of the complex problems of human societies, which has become a social problem in most countries today. Predicting and analyzing addiction and quantifying the factors involved in it, is very useful for decision-makers in societies, so experts in various disciplines, including mathematics and statistics, have been modeled the addiction and studied some of the factors involved in epidemic or control of it.

According to [4, 34], "dynamic modeling complements indicators and direct data analysis in drug epidemiology at the macro the level. Instead of the usual inductive or empirical method of data collection and interpretation, it can be used to enhance the understanding of drug processes by simulating experiments that are difficult or impossible to perform in real life. Dynamic drug models can help in understanding a phenomenon via scenario analysis, thereby providing a tool to simulate experiments that are not possible in real life due to practical or ethical reasons".

There are three general approaches modeling the dynamics of the spread of drug use. Authors of [15] believe: "anyone could be a 'prey' to illicit drugs". They applied the predator-prey paradigm for the modeling of illicit drug consumption, see also [4, 7, 12]. On the other hand, drugs have been considered as an epidemic problem like an infectious disease, because most drug initiations start through contact with users, not through contact with drug sellers, see [20]. Also modeling with the optimal control method has been performed, see the monograph [17].

Among illicit drugs, heroin is one of the world's most dangerous opioids which is highly addictive. In the United States in the time interval of 2002 to 2014, the number of heroin users increased from about 404,000 to 914,000 and the number of addicted cases increased from about 214,000 to 586,000, see [13].

White and Comiskey assumed that the spread of heroin addiction has a mechanism like the spread of infectious diseases and introduced the first compartmental model with ordinary differential equations for the heroin use, see [35]. Compartmental models are powerful tools for the study and analysis of infectious diseases. Such models which are generally expressed by ordinary differential equations were first introduced by Kermack and McKendrick. These models have been used in modeling many diseases such as AIDS, tuberculosis, and influenza, see [24]. After White and Comiskey's work, Mulone and Straughan revisit their work, [27]. Nyabadza and Hove-Muskava, in [31], modified

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the White and Comiskey model and studied the dynamics of methamphetamine. For the study of the epidemiology of crystal and the effect of rehabilitation, relapse and information, see [1, 25, 26, 29, 32].

Despite increasing evidence that addiction is a treatable disease of the brain, most individuals do not receive treatment. Mostly, treating illnesses are more costly than preventing them. For example treatment of heroin users and a variety of drugs is a costly procedure and is a major burden on the health system of any country. Treating drug-involved offenders provides a unique opportunity to decrease substance abuse and reduce associated criminal behavior.

In this manuscript, we modify White-Comiskey's model and propose a compartmental model which incorporates the effect of consultation and incarceration of addicted individuals.

In section 2, we present the model and compute the basic reproduction number and study the boundedness and positivity of solutions. In section 3, we prove the existence of endemic equilibria and show that the system may have up to two endemic equilibrium points when $R_0 \leq 1$ and up to three endemic equilibrium points when $R_0 > 1$. In section 4, we show that backward bifurcation occurs leading to bistability. In section 5, we study the global stability of equilibrium points by using Lyapunov functions and the geometric stability method. Finally in section 6, we present numerical simulations to illustrate the analytical results.

2. Model formulation and basic properties

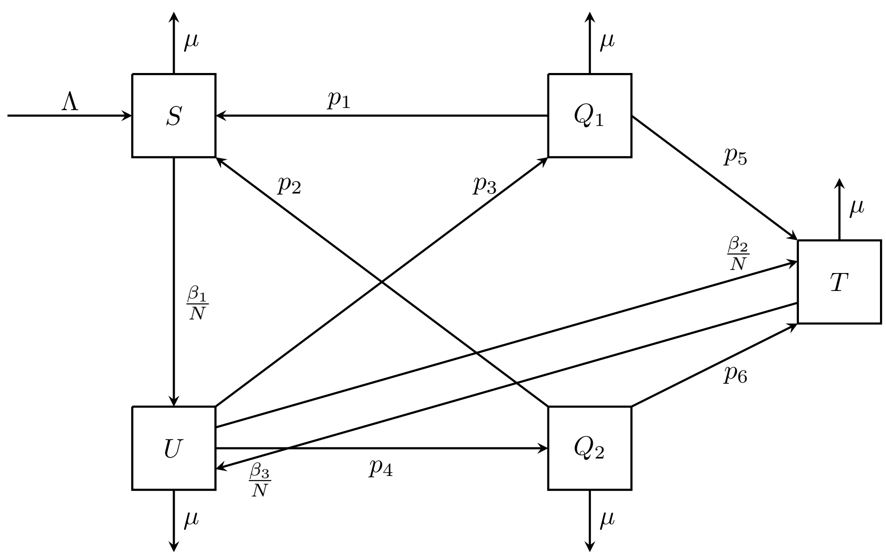


Figure 1. The flowchart of the model

In our model, the community is divided into five compartments: S susceptible individuals, i.e., the individuals at risk of drug use, U drug users which are not in treatment, incarceration or consultation/rehabilitation, T drug users under treatment, Q_1 incarcerated drug users and Q_2 drug users under consultation. The number of these compartments is $S(t)$, $U(t)$, $T(t)$, $Q_1(t)$, and $Q_2(t)$ respectively. We use $N(t)$ for the total population, i.e., $N(t) = S(t) + U(t) + Q_1(t) + Q_2(t) + T(t)$.

Table 1: Parameters of the model

Symbol	Description
Λ	The rate of recruitment of susceptible individuals.
μ	The rate of Natural death.
β_1	The rate of drug use.
β_2	The treatment rate.
β_3	The relapse rate.
p_1	The rate of return of prisoners to the susceptibles.
p_2	The rate at which individuals under consultation/rehabilitation back to susceptibles.
p_3	The rate of imprisonment of drug users.
p_4	The rate at which drug users turn to counsel.
p_5	The rate at which prisoners are referred to treatment.
p_6	The rate at which individuals under consultation are referred to treatment.

Based on the flow diagram of the model depicted in the above figure, we obtain the following ODE system:

$$\begin{cases} \frac{dS}{dt} = \Lambda - \frac{\beta_1 SU}{N} - \mu S + p_1 Q_1 + p_2 Q_2 \\ \frac{dU}{dt} = \frac{\beta_1 SU}{N} + \frac{\beta_3 UT}{N} - (\mu + \beta_2 + p_3 + p_4)U \\ \frac{dT}{dt} = -\frac{\beta_3 UT}{N} + \beta_2 U + p_5 Q_1 + p_6 Q_2 - \mu T \\ \frac{dQ_1}{dt} = p_3 U - (\mu + p_1 + p_5)Q_1 \\ \frac{dQ_2}{dt} = p_4 U - (\mu + p_2 + p_6)Q_2 \end{cases} \tag{2.1}$$

At first, we study the nonnegativity and boundedness of the solutions.

Lemma 2.1. *The variables of the trajectory $(S(t), U(t), Q_1(t), Q_2(t), T(t))$ of model are nonnegative for all $t \geq 0$, when the initial values are nonnegative, i.e., $S(0) \geq 0, U(0) \geq 0, Q_1(0) \geq 0, Q_1(0) \geq 0$ and $T(0) \geq 0$.*

Proof. All solutions of this model are smooth. Furthermore, if all of the components of the system have nonnegative initial conditions and that if any of the compartments are zero at time $t = t_i \geq 0$, then the derivatives are nonnegative. For example if $S(t_1) = 0$, $U(t_1) \geq 0$, $Q_1(t_1) \geq 0$, $Q_2(t_1) \geq 0$ and $T(t_1) \geq 0$, we get

$$\frac{dS(t_1)}{dt} = \Lambda + Q_1(t_1) + Q_2(t_1) \geq 0,$$

that implies $S(t_1^+) \geq 0$ and hence, $S(t)$ is nonnegative for all times $t \geq 0$. Next, assume that $T(t_2) = 0$, $S(t_2) \geq 0$, $U(t_2) \geq 0$,$Q_1(t_2) \geq 0$ and $Q_2(t_2) \geq 0$. Hence we have

$$\frac{dT(t_2)}{dt} = \beta_2 U(t_2) + p_5 Q_1(t_2) + p_6 Q_2(t_2) \geq 0,$$

that implies $T(t_2^+) \geq 0$ and hence, $T(t)$ is nonnegative for all times $t \geq 0$. A similar result can be obtained for the other components of the solution and as mentioned in [30], it can be concluded that all compartments are nonnegative at all times $t \geq 0$. The boundedness of the solutions is also of interest. We prove it in the following lemma.

Lemma 2.2. *The total population $N(t) = S(t)+U(t)+Q_1(t)+Q_2(t)+T(t)$ is bounded from above, for all nonnegative initial values.*

Proof. The equations of the system yield $\dot{N} = \Lambda - \mu N$, and by integration we have,

$$N(t) = N(0)e^{-\mu t} + \frac{\Lambda}{\mu}(1 - e^{-\mu t}) \leq \max(N(0), \frac{\Lambda}{\mu}) = M$$

for all $t \geq 0$. □

As in the White-Comiskey model, we consider the total population of the community, i.e., N, to be constant. Hence $\Lambda = \mu S + \mu U + \mu T + \mu Q_1 + \mu Q_2$. Now we replace Λ in (2.1) and then use the substitutions

$$s = \frac{S}{N}, \quad u = \frac{U}{N}, \quad \tau = \frac{T}{N}, \quad q_1 = \frac{Q_1}{N} \quad \text{and} \quad q_2 = \frac{Q_2}{N} = 1 - s - u - \tau - q_1, \tag{2.2}$$

which yields the following final form of our system:

$$\begin{cases} \dfrac{ds}{dt} = \mu - \beta_1 su - \mu s + p_1 q_1 + p_2\,(1 - s - u - \tau - q_1), \\ \dfrac{du}{dt} = \beta_1 su + \beta_3 u\tau - (\mu + \beta_2 + p_3 + p_4)\,u, \\ \dfrac{d\tau}{dt} = -\beta_3 u\tau + \beta_2 u + p_5 q_1 + p_6\,(1 - s - u - \tau - q_1) - \mu\tau, \\ \dfrac{dq_1}{dt} = p_3 u - (\mu + p_1 + p_5) q_1. \end{cases} \tag{2.3}$$

We study (2.3) in the region,

$$\Omega = \{(s, u, \tau, q_1) \in \mathbb{R}^4_+ \,:\, s \geq 0,\, u \geq 0,\, \tau \geq 0,\, q_1 \geq 0,\, s + u + \tau + q_1 \leq 1\}.$$

This region is positively invariant under (2.3).
This system has a unique drug-free equilibrium $P_0 = (s^*,\, u^*,\, \tau^*, q_1^*) = (1,\, 0\,, 0,\, 0)$ and the Jacobian matrix of P_0 has the following form:

$$J(P_0) = \begin{pmatrix} -\mu - p_2 & -\beta_1 - p_2 & -p_2 & p_1 - p_2 \\ 0 & \beta_1 - (\mu + \beta_2 + p_3 + p_4) & 0 & 0 \\ -p_6 & +\beta_2 - p_6 & -\mu - p_6 & p_5 - p_6 \\ 0 & p_3 & 0 & -(\mu + p_1 + p_5) \end{pmatrix}.$$

This matrix has the eigenvalues, $\lambda_1 = \beta_1 - (\mu + \beta_2 + p_3 + p_4)$, $\lambda_2 = -(\mu + p_1 + p_5)$ and the eigenvalues of the following submatrix:

$$\mathbb{E} = \begin{pmatrix} -\mu - p_2 & -p_2 \\ -p_6 & -\mu - p_6 \end{pmatrix}.$$

Furthermore, since $tr\mathbb{E} < 0$ and $det\mathbb{E} > 0$, the real part of eigenvalues are negative. The relation $\lambda_1 < 0$ can be written as $R_0 < 1$ in which,

$$R_0 = \frac{\beta_1}{\mu + \beta_2 + p_3 + p_4}.$$

Theorem 2.3. *The drug-free equilibrium P_0 is asymptotically stable when $R_0 < 1$ and unstable when $R_0 > 1$.*

3. Endemic equilibrium points

The endemic equilibrium points of (2.3) satisfy the following system,

$$\begin{cases} \mu - \beta_1 s^* u^* - \mu s^* + p_1 q_1^* + p_2 \left(1 - s^* - u^* - \tau^* - q_1^*\right) = 0, \\ \beta_1 s^* u^* + \beta_3 u^* \tau^* - \left(\mu + \beta_2 + p_3 + p_4\right) u^* = 0, \\ -\beta_3 u^* \tau^* + \beta_2 u^* + p_5 q_1^* + p_6 \left(1 - s^* - u^* - \tau^* - q_1^*\right) - \mu \tau^* = 0, \\ p_3 u^* - (\mu + p_1 + p_5) q_1^* = 0. \end{cases} \tag{3.1}$$

This yields that u^* is the positive root of:

$$F(u^*) = A(u^*)^3 + B(u^*)^2 + Cu^* + D = 0, \tag{3.2}$$

where

$$\begin{aligned} A &= -(\mu + p_2 + p_6)(1 + \frac{p_3}{\mu + p_1 + p_5})\, \beta_1 \beta_3, \\ B &= (\mu + p_2 + p_6) \Big[(1 - \frac{1}{R_0})\, \beta_1 \beta_3 + (\frac{\beta_3}{\beta_1} - 1)\, \beta_1\, (\frac{p_3}{\mu + p_1 + p_5}\, (p_5 - p_6) + \beta_2) \\ &\quad -(1 + \frac{p_3}{\mu + p_1 + p_5})\big(\beta_3(\mu + p_2) + \beta_1(\mu + p_6)\big)\Big], \\ C &= (\mu + p_2 + p_6) \Big[(1 - \frac{1}{R_0})\, \big(\beta_3(\mu + p_2 + \beta_1(\mu + p_6)\big) + (\frac{\beta_3}{\beta_1} - 1)\, \frac{p_3}{\mu + p_1 + p_5} \\ &\quad \big(p_6(\mu - p_1) + p_2 p_5\big)\ - (1 + \frac{p_3}{\mu + p_1 + p_5})\, \mu\, (\mu + p_2 + p_6)\Big], \\ D &= (\mu + p_2 + p_6)^2\, (1 - \frac{1}{R_0}). \end{aligned}$$

Now, we consider $f(u) = Au^3 + Bu^2 + Cu + D$, with $f'(u) = 3Au^2 + 2Bu + C$ and $f"(u) = 6Au + 2B$. The negativity of A implies $\lim_{u \to +\infty} f(u) = \infty$ and $\lim_{u \to -\infty} f(u) = +\infty$. The following cases can occur,

Case I: $R_0 > 1$.
In this case $D > 0$. Some subcases can occur and we study them:
$I_0 : \Delta = 4B^2 - 12AC < 0$.
The negativity of Δ implies that $f'(u) < 0$ for all u, so it has one positive root.
$I_1 : \Delta > 0,\ B > 0$ and $C > 0$.
From these relations and by using the second derivative, we see that $f(u)$ has a local minimum at $u_1 = \frac{-2B+\sqrt{\Delta}}{6A} < 0$ and a local maximum at $u_2 = \frac{-2B-\sqrt{\Delta}}{6A} > 0$. Therefore, $f(u)$ has one positive real solution.
$I_2 : \Delta > 0,\ B < 0$ and $C < 0$.
From these relations and by using the second derivative, we see that $u_1 = \frac{-2B+\sqrt{\Delta}}{6A} < 0$ is a local minimum and $u_2 = \frac{-2B-\sqrt{\Delta}}{6A} < 0$ is a local maximum with $u_1 < u_2$. Hence $f(u)$ has one positive real solution.
$I_3 : \Delta > 0,\ B > 0$ and $C < 0$.
From these relations and by using the second derivative, we see that $u_1 = \frac{-2B+\sqrt{\Delta}}{6A} > 0$ is a local minimum and $u_2 = \frac{-2B-\sqrt{\Delta}}{6A} > 0$ is a local maximum with $u_1 < u_2$. Hence, $f(u)$ has three solutions.
$I_4 : \Delta > 0,\ B < 0$ and $C > 0$.
From these relations, we see that $f(u)$ has a local minimum at $u_1 = \frac{-2B+\sqrt{\Delta}}{6A} < 0$ and a local maximum at $u_2 = \frac{-2B-\sqrt{\Delta}}{6A} > 0$. Hence, it has one positive solution.
$I_5 : \Delta > 0$ and $B = 0$.
These relations imply $C > 0$ hence, $f(u)$ has a local minimum at $u_1 = \frac{\sqrt{\Delta}}{6A} < 0$ and a local maximum at $u_2 = \frac{-\sqrt{\Delta}}{6A} > 0$. Therefore, there is a unique positive solution.
$I_6 : \Delta > 0$ and $C = 0$.
The relations $\Delta > 0$ and $C = 0$ imply $B \neq 0$. When $B > 0$, $f(u)$ has a local minimum at $u_1 = 0$ and a local maximum

at $u_2 = \frac{-2}{3}\frac{B}{A} > 0$. Hence, $f(u)$ has one positive real root. When $B < 0$, by a similar argument we see that $f(u)$ has one positive solution.
$I_7 : \Delta = 0$ and $C \neq 0$.
The relation $\Delta = 0$ implies $C = \frac{B^2}{3A} < 0$. When $B > 0$, $u = \frac{-B}{3A} > 0$ is a horizontal tangent of $f(u)$, and when $B < 0$, $u = \frac{-B}{3A} < 0$ is a horizontal tangent of $f(u)$. Hence, for sufficiently small $\epsilon > 0$ we have $f'(\frac{-B}{3A}+\epsilon) = \frac{-B^2}{3A}+3A\epsilon^2+C = 3A\epsilon^2 < 0$ and $f'(\frac{-B}{3A}-\epsilon) = \frac{-B^2}{3A}+3A\epsilon^2+C = 3A\epsilon^2 < 0$. So it has only one solution which is real.
$I_8 : \Delta = B = 0$.
These relations imply $C = 0$ and $u^* = \sqrt[3]{\frac{-D}{A}} > 0$ is the root of $f(u)$.
$I_9 : \Delta = C = 0$.
These relations imply $B = 0$ and $u^* = \sqrt[3]{\frac{-D}{A}} > 0$ is the root of $f(u)$.

Case II: $R_0 < 1$. In this case $D < 0$. Some subcases can occur and we study them:
$II_0 : \Delta < 0$.
The negativity of Δ implies that $f(u)$ is strictly decreasing without any positive root.
$II_1 : \Delta > 0,\ B > 0$ and $C > 0$.
From these relations and by using the second derivative, we see that $f(u)$ has a local minimum at $u_1 = \frac{-2B+\sqrt{\Delta}}{6A} < 0$, and a local maximum at $u_2 = \frac{-2B-\sqrt{\Delta}}{6A} > 0$. And $f(u)$ has two positive roots, a unique positive root and has no positive root, when $f(u_2) > 0$, $f(u_2) = 0$ and $f(u_2) < 0$ respectively.
$II_2 : \Delta > 0,\ B > 0$ and $C < 0$.
From these relations and by using the second derivative, we see that $f(u)$ has a local minimum at $u_1 = \frac{-2B+\sqrt{\Delta}}{6A} > 0$ and a local maximum at $u_2 = \frac{-2B-\sqrt{\Delta}}{6A} > 0$ with $u_1 < u_2$. And $f(u)$ has two positive roots, a unique positive root and has no positive root, when $f(u_2) > 0$, $f(u_2) = 0$ and $f(u_2) < 0$, respectively.
$II_3 : \Delta > 0,\ B < 0$ and $C > 0$.
From these relations and by using the second derivative, we see that $f(u)$ has a local minimum at $u_1 = \frac{-2B+\sqrt{\Delta}}{6A} < 0$ and a local maximum at $u_2 = \frac{-2B-\sqrt{\Delta}}{6A} > 0$. And $f(u)$ has two positive roots, a unique positive root and has no positive root, when $f(u_2) > 0$, $f(u_2) = 0$ and $f(u_2) < 0$ respectively.
$II_4 : \Delta > 0,\ B < 0$ and $C < 0$.
From these relations and by using the second derivative, we see that $f(u)$ has a local minimum at $u_1 = \frac{-2B+\sqrt{\Delta}}{6A} < 0$ and a local maximum at $u_2 = \frac{-2B-\sqrt{\Delta}}{6A} < 0$ with $u_1 < u_2$. Hence $f(u)$ has no positive root.
$II_5 : \Delta > 0$ and $B = 0$.
From these relations and by using the second derivative, we see that $f(u)$ has a local minimum at $u_1 = \frac{\sqrt{\Delta}}{6A} < 0$ and a local maximum at $u_2 = \frac{-\sqrt{\Delta}}{6A} > 0$. So that $f(u)$ has two positive roots, a unique positive root and has no positive root, when $f(u_2) > 0$, $f(u_2) = 0$ and $f(u_2) < 0$ respectively.
$II_6 : \Delta > 0$ and $C = 0$.
These relations imply $B \neq 0$. If $B > 0$, $f(u)$ has a local minimum at $u_1 = 0$ and a local maximum at $u_2 = \frac{-2}{3}\frac{B}{A} > 0$. Hence $f(u)$ has two positive roots, a unique positive root and has no positive root, when $f(u_2) > 0$, $f(u_2) = 0$ and $f(u_2) < 0$ respectively.
If $B < 0$ then $f(u)$ has a local minimum at $u_1 = \frac{-2}{3}\frac{B}{A} < 0$ and a local maximum at $u_2 = 0$.
$II_7 : \Delta = 0,\ B \neq 0$ and $C \neq 0$.
The relation $\Delta = 0$ implies $C = \frac{B^2}{3A} < 0$. Hence, $A < 0$, $C < 0$ and $D < 0$. When $B > 0$, $f(u)$ admits a tangent line which is horizontal at $u = \frac{-B}{3A} > 0$. When $B < 0$, $f(u)$ admits a tangent line which is horizontal at $u = \frac{-B}{3A} < 0$. Hence when $\epsilon > 0$ is sufficiently small, $f'(\frac{-B}{3A}+\epsilon) = \frac{-B^2}{3A}+3A\epsilon^2+C = 3A\epsilon^2 < 0$ and $f'(\frac{-B}{3A}-\epsilon) = \frac{-B^2}{3A}+3A\epsilon^2+C = 3A\epsilon^2 < 0$. This implies that $f(u)$ does not have any positive solution if $B > 0$. The case $B < 0$ is similar.
$II_8 : \Delta = 0$ *and* $B = 0$.
The unique root of $f(u)$ is $u^* = \sqrt[3]{\frac{-D}{A}} < 0$.

$II_9 : \Delta = 0$ *and* $C = 0$.
The unique root of $f(u)$ is $u^* = \sqrt[3]{\frac{-D}{A}} < 0$.

Case III: $R_0 = 1$. In this case $D = 0$ and $F(u) = 0$ reduces to,

$$g(u^*) = Au^{*2} + Bu^* + C = 0, \tag{3.3}$$

This equation does not have real roots when $\Delta^{'} = B^2 - 4AC < 0$. If $\Delta^{'} \geq 0$ and $C \geq 0$, it has one positive solution. If $\Delta^{'} \geq 0$ $C \leq 0$, when $B > 0$, g has maximum at $u^*_{max} = \frac{-B}{2A}$, with $g(u^*_{max}) = \frac{-\Delta^{'}}{-4A} \geq 0$.

4. Backward bifurcation

In most epidemic models, when $R_0 < 1$ and the initial values of the compartments of the model belong to the basin of attraction of the disease-free equilibrium point P_0, the disease dies out. At the same time, in some epidemiological models in the range $R_0 < 1$, there exist endemic equilibrium points which show that all initial states cannot be absorbed to P_0, i.e., the disease may become endemic. Backward bifurcation is the occurrence of this problem [24]. Now we prove the occurrence of backward bifurcation in the proposed model.

We use the Castillo-Chavez and Song theorem, i.e., the theorem 4.1 in [11].
Let $s = x_1$, $u = x_2$, $\tau = x_3$ and $q_1 = x_4$, then (2.3) becomes:

$$\begin{cases} \dfrac{dx_1}{dt} = \mu - \beta_1 x_1 x_2 - \mu x_1 + p_1 x_4 + p_2\left(1 - x_1 - x_2 - x_3 - x_4\right) = f_1, \\ \dfrac{dx_2}{dt} = \beta_1 x_1 x_2 + \beta_3 x_2 x_3 - (\mu + \beta_2 + p_3 + p_4)x_2 = f_2, \\ \dfrac{dx_3}{dt} = -\beta_3 x_2 x_3 + \beta_2 x_2 + p_5 x_4 + p_6\left(1 - x_1 - x_2 - x_3 - x_4\right) - \mu x_3 = f_3, \\ \dfrac{dx_4}{dt} = p_3 x_2 - (\mu + p_1 + p_5)x_4 = f_4. \end{cases} \tag{4.1}$$

If $R_0 = 1$ then $\beta_1 = \beta_1^* = \mu + \beta_2 + p_3 + p_4$. The Jacobian matrix $J(P_0, \beta_1^*)$ has three negative eigenvalues and the simple eigenvalue 0. Let $v = (0, 1, 0, 0)$ be the left eigenvector of $\mathcal{A}$ corresponding the zero eigenvalue, computed by $v\mathcal{A} = 0$. On the other hand, let $w = (w_1, w_2, w_3, w_4)^T$ be the right eigenvector of $\mathcal{A}$ associated with eigenvalue $\lambda_4 = 0$. We have,

$$\begin{aligned} w_1 &= \frac{-(\mu + p_1 + p_5)\Big[\beta_1(\mu + p_6) + p_2(\mu + \beta_2)\Big] + p_3\Big[p_1(\mu + p_6) - p_2(\mu + p_5)\Big]}{\mu(\mu + p_2 + p_6)}, \\ w_2 &= \frac{\mu + p_1 + p_5}{p_3} w_4 = \mu + p_1 + p_5, \\ w_3 &= \frac{(\mu + p_1 + p_5)\Big[p_2\beta_2 + p_6\beta_1 + \mu(\beta_2 - p_6)\Big] + \mu p_3(p_5 - p_6) + p_3(p_1 p_6 - p_2 p_5)}{\mu(\mu + p_2 + p_6)}, \\ w_4 &= p_3. \end{aligned}$$

By using these vectors, we compute constants **a**,**b** as in [11],

$$\begin{aligned} \mathbf{a} &= \sum_{k,i,j=1}^{n} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(P_0, \beta_1^*), = \sum_{i,j=1}^{4} w_i w_j \frac{\partial^2 f_2}{\partial x_i \partial x_j}(P_0, \beta_1^*) \\ &= 2w_2\Big(w_1\beta_1 + w_3\beta_3\Big), \end{aligned}$$

and

$$\mathbf{b} = \sum_{k,i=1}^{n} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \phi}(P_0, \beta_1^*) = \sum_{i=1}^{4} w_i \frac{\partial^2 f_2}{\partial x_i \partial \beta_1}(P_0, \beta_1^*)$$
$$= w_1 x_2 + w_2 x_1 = w_2 = \mu + p_1 + p_5.$$

As $\mathbf{b}$ is positive, by the sign of $\mathbf{a}$ we can determine the bifurcation of the system around DFE for $\beta_1 = \beta_1^*$. We consider

$$A_1 = p_1 p_3 \beta_1(\mu + p_6) + (\mu + p_1 + p_5)\beta_3 \Big[\beta_2(\mu + p_2) + p_6\beta_1\Big] + p_3 \quad \beta_3(p_5\mu + p_1 p_6),$$

and

$$A_2 = \beta_1 \Big[(\mu + p_1 + p_5)\Big(\beta_1(\mu + p_6) + p_2(\mu + \beta_2)\Big) + p_2 p_3(\mu + p_5)\Big] + \beta_3 \quad \Big[\mu p_6\Big((\mu + p_1 + p_5) + p_3\Big) + p_2 p_3 p_5\Big].$$

Now $A_1 > A_2$ if and only if $a > 0$, and part (4) in the theorem of Castillo-Chavez and Song imply the following result.

Theorem 4.1. *If $A_2 < A_1$, then backward bifurcation occurs when $R_0 = 1$. Furthermore, endemic equilibrium has asymptotic stability when $R_0 > 1$ and close to one.*

5. Global Stability of Equilibrium points

In this section, we study the global stability of equilibrium points of the system. To determine whether the addiction can invade the population, we study the global asymptotic stability of the DFE point P_0. We prove the global asymptotic stability of P_0 under certain conditions, which ensures that the addiction dies out for all initial values of the model components.

Lemma 5.1. *If $R_0 \leq \dfrac{\beta_1}{\beta_1 + \beta_3}$, DFE is GAS in Ω.*

Proof. We consider the function $V : \{(s, u, \tau, q_1) \in \Omega : s > 0, \tau > 0, q_1 > 0\} \to \mathbb{R}$ by $V(s, u, \tau, q_1) = u$, as a Lyaponuv function. Now,

$$\frac{dV}{dt} = \frac{du}{dt} \leq ((\beta_1(1 - \frac{1}{R_0}) + \beta_3)u.$$

Therefore, $\dfrac{dV}{dt} \leq 0$ when $R_0 \leq \dfrac{\beta_1}{\beta_1 + \beta_3}$. Furthermore, $\dfrac{dV}{dt} = 0$ if and only if $u = 0$. Hence by Lasalle invariance principle, P_0 is global asymptotic stable with respect to the invariant set Ω. See [21] for the proofs and applications of the notion of asymptotic stability with respect to invariant sets. □

In this part, we use the geometric stability method proved in [22, 23]. See [3, 9, 10, 18] for applications of this method.

Suppose that the system has a unique endemic equilibrium point. The Jacobian matrix of the system at the point (s, u, τ, q_1) is given by:

$$J = \frac{\partial f}{\partial x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_2 & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \tag{5.1}$$

$$\begin{aligned}
&a_{11} = -\mu - p_2 - -\beta_1 u, \quad a_{12} = -\beta_1 s - p_2, \quad a_{13} = -p_2, \quad a_{14} = p_1 - p_2, \\
&a_{21} = \beta_1 u, \quad a_{22} = \beta_1 s + \beta_3 \tau - (\mu + \beta_2 + p_3 + p_4), \quad a_{23} = \beta_3 u, \quad a_{24} = 0, \\
&a_{31} = -p_6, \quad a_{32} = -\beta_3 \tau + \beta_2 - p_6, \quad a_{33} = -\beta_3 u - p_6 - \mu, \quad a_{34} = p_5 - p_6, \\
&a_{41} = 0, \quad a_{42} = -p_3, \quad a_{43} = 0, \quad a_{44} = -(\mu + p_1 + p_5).
\end{aligned}$$

The second compound matrix, $J^{[2]}$ of $J = \frac{\partial f}{\partial x}$ is:

$$M = J^{[2]} = \begin{bmatrix} M_{11} & M_{12} & 0 & M_{14} & M_{15} & 0 \\ M_{21} & M_{22} & M_{23} & M_{24} & 0 & M_{26} \\ M_{31} & 0 & M_{33} & 0 & M_{35} & M_{36} \\ M_{41} & M_{42} & 0 & M_{44} & M_{45} & 0 \\ 0 & 0 & M_{53} & 0 & M_{55} & M_{56} \\ 0 & 0 & M_{63} & M_{64} & M_{65} & M_{66} \end{bmatrix}, \tag{5.2}$$

with the following components,
$M_{11} = -(2\mu + \beta_2 + p_2 + p_3 + p_4) + \beta_1(s-u) + \beta_3\tau$, $M_{12} = \beta_3 u$, $M_{14} = p_2$, $M_{15} = p_2 - p_1$,
$M_{21} = -\beta_3\tau - p_6 + \beta_2$, $M_{22} = -(2\mu + p_2 + p_6) - (\beta_1 + \beta_3)u$, $M_{23} = p_5 - p_6$, $M_{24} = -\beta_1 s - p_2$, $M_{26} = p_2 - p_1$,
$M_{31} = p_3$, $M_{33} = -\beta_1 u - (2\mu + p_1 + p_2 + p_5)$, $M_{35} = -\beta_1 s - p_2$, $M_{36} = -p_2$,
$M_{41} = p_6$, $M_{42} = \beta_1 u$, $M_{44} = -\beta_1 s + \beta_3\tau - \beta_3 u - (2\mu + \beta_2 + p_3 + p_4 + p_6)$, $M_{45} = p_5 - p_6$,
$M_{53} = \beta_1 u$, $M_{55} = \beta_1 s + \beta_3\tau - (2\mu + \beta_2 + p_1 + p_3 + p_4 + p_5)$, $M_{56} = \beta_3 u$,
$M_{63} = -p_6$, $M_{64} = -p_3$, $M_{65} = -\beta_3\tau + \beta_2 - p_6$, $M_{66} = -\beta_3 u - (2\mu + p_1 + p_5 + p_6)$.
We use the following matrix function,

$$P = \begin{bmatrix} \frac{1}{u} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{u} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{u} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Which yields the matrix $P_f P^{-1} = -diag(\frac{u'}{u}, 0, 0, \frac{u'}{u}, \frac{u'}{u}, 0)$ and,

$$Q = P_f P^{-1} + PMP^{-1} = \begin{bmatrix} A_{11} & A_{12} & 0 & A_{14} & A_{15} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} & 0 & A_{26} \\ A_{31} & 0 & A_{33} & 0 & A_{35} & A_{36} \\ A_{41} & A_{42} & 0 & A_{44} & A_{45} & 0 \\ 0 & 0 & A_{53} & 0 & A_{55} & A_{56} \\ 0 & 0 & A_{63} & A_{64} & A_{65} & A_{66} \end{bmatrix}, \tag{5.3}$$

in which,
$A_{11} = -\mu - \beta_1 u - p_2$, $A_{12} = \beta_3$, $A_{14} = p_2$, $A_{15} = p_2 - p_1$,
$A_{21} = -(\beta_3\tau + p_6 - \beta_2)u$, $A_{22} = -(2\mu + p_2 + p_6) - (\beta_1 + \beta_3)u$, $A_{23} = p_5 - p_6$, $A_{24} = -(\beta_1 s + p_2)u$, $A_{26} = p_2 - p_1$,
$A_{31} = p_3 u$, $A_{33} = -\beta_1 u - (2\mu + p_1 + p_2 + p_5)$, $A_{35} = -(\beta_1 s + p_2)u$, $M_{36} = -p_2$,
$A_{41} = p_6$, $A_{42} = \beta_1$, $A_{44} = -\beta_3 u - p_6$, $A_{45} = p_5 - p_6$,
$A_{53} = \beta_1$, $A_{55} = -\mu - p_1 - p_5$, $A_{56} = \beta_3$,
$A_{63} = -p_6$, $A_{64} = -p_3 u$, $M_{65} = -\beta_3\tau + \beta_2 - p_6$, $A_{66} = -\beta_3 u - (2\mu + p_1 + p_5 + p_6)$.
We use the norm introduced in [18]. At first, we prove the following lemma.

Lemma 5.2. *There exists $\chi > 0$, with $D_+ \parallel z \parallel \leq -\chi \parallel z \parallel$ in which $z \in \mathbb{R}^6$ is the solution of,*

$$\frac{dz}{dt} = Q(\phi_t(l))z \tag{5.4}$$

provided that, $p_2 < p_1$, $p_5 + \beta_3 < \beta_2 + p_3$, $p_3 + \beta_3 < \mu$, $p_3 + \beta_3 + p_2 + \beta_1 < \mu + p_1$ and $p_1 + p_2 + 2\beta_1 < 2\mu + p_6$.

Proof. To demonstrate the existence of such $\chi > 0$, we need the sixteen separate cases, see [9]. We demonstrate two cases with complete details.
Case 1: $U_1 > U_2$; z_1, z_2, $z_3 > 0$ and $|z_1| > |z_2|$. In this case, $\| z \| = |z_1| + |z_3|$ and

$$D_+ \| z \| = (A_{11}z_1 + A_{12}z_2 + A_{14}z_4 + A_{15}z_5) + (A_{31}z_1 + A_{33}z_3 + A_{35}z_5 + A_{36}z_6)$$

We have,

$$(A_{11} + A_{31})z_1 + A_{33}z_3 + A_{12}z_2 \leq \max\{A_{11} + A_{31} + \beta_3, A_{33} + \beta_3\} \| z \|,$$

and

$$\begin{array}{ll} [*] & A_{14}z_4 = p_2 z_4 \\ [**] & A_{15}z_5 + A_{35}z_5 = [p_2 - p_1 - (\beta_1 s + p_2)u]z_5 \\ [***] & A_{36}z_6 = -p_2 z_6 \end{array}$$

The following situations occur :
(1): If $z_4, z_5, z_6 > 0$, we delete $[***]$ and by the condition $p_2 - p_1 + (-\beta_1 s - p_2)u < 0$, $[**]$ is also deleted. Furthermore,

$$A_{14}z_4 \quad \leq \quad p_2 \| z \| .$$

We consider the condition:

$$\max\{A_{11} + A_{31} + \beta_3, A_{33} + \beta_3\} + \max\{p_2, p_1 - p_2 + (\beta_1 s + p_2)u\} < 0 \tag{5.5}$$

(2): If $z_4, z_5, z_6 < 0$, then we delete $[*]$ and,

$$\begin{aligned} A_{15}z_5 + A_{35}z_5 + A_{36}z_6 &\leq p_2|z_6| + \big(p_1 - p_2 + (\beta_1 s + p_2)u\big)|z_5| \\ &\leq \max\{p_2,\ p_1 - p_2 + (\beta_1 s + p_2)u\} \| z \| . \end{aligned}$$

In this case, we use (5.5).
(3): If $z_4, z_5 > 0$ and $z_6 < 0$, then $[**]$ is deleted and,

$$A_{14}z_4 + A_{36}z_6 = p_2(|z_4| + |z_6|) \leq p_2 \| z \|$$

in this case, we use (5.5).
(4): If $z_4, z_5 < 0$ and $z_6 > 0$, then $[*]$ and $[***]$ are deleted and,

$$A_{15}z_5 + A_{35}z_5 \quad \leq \quad (p_1 - p_2 + (\beta_1 s + p_2)u) \| z \|,$$

we use (5.5).
(5): If $z_4, z_6 > 0$ and $z_5 < 0$, then $[***]$ is deleted,

$$A_{14}z_4 + A_{15}z_5 + A_{35}z_5 \quad \leq \quad \max\{p_2,\ p_1 - p_2 + (\beta_1 s + p_2)u\} \| z \|,$$

and we use (5.5).
(6): If $z_4, z_6 < 0$ and $z_5 > 0$, then $[*]$ and $[**]$ are deleted,

$$A_{36}z_6 \leq p_2 \| z \|$$

and we use (5.5).
(7): If $z_5, z_6 > 0$ and $z_4 < 0$ then $[*]$, $[**]$ and $[***]$ are deleted due to their negativity.
(8): If $z_5, z_6 < 0$ and $z_4 > 0$ hence,

$$A_{14}z_4 + A_{15}z_5 + A_{35}z_5 + A_{36}z_6 \leq \max\{p_2, p_1 - p_2 + (\beta_1 s + p_2)u\} \| z \|$$

and we use (5.5).
The following conditions are sufficient for this case:

$$\begin{cases} p_2 - p_1 - (\beta_1 s + p_2)u < 0 \\ \max\{A_{11} + A_{31} + \beta_3, A_{33} + \beta_3\} + \max\{p_2, p_1 - p_2 + (\beta_1 s + p_2)u\} < 0 \end{cases}$$

Case 2: $U_1 > U_2$, z_2, $z_3 > 0 >$ z_1 and $|z_1| < |z_2|$. In this case, $\| z \| = |z_2| + |z_3|$ and

$$\begin{aligned} D_+ \| z \| &= z_2^{'} + z_3^{'} \\ &= (A_{21}z_1 + A_{22}z_2 + A_{23}z_3 + A_{24}z_4 + A_{26}z_6) + (A_{31}z_1 + A_{33}z_3 + A_{35}z_5 + A_{36}z_6). \end{aligned} \tag{5.6}$$

We have,

$$\begin{aligned} A_{22}|z_2| + (A_{23} + A_{33})|z_3| &\leq \max\{A_{22}, A_{23} + A_{33}\}(|z_2| + |z_3|) \\ &= \max\{A_{22}, A_{23} + A_{33}\} \| z \| . \end{aligned}$$

We suppose the inequality $A_{21}+A_{31} > 0$, and then delete the term $(A_{21}+A_{31})z_1$ because of its negativity. Furthermore we have the following terms:

$$\begin{aligned} [*] \quad & A_{24}z_4 = (-\beta_1 s - p_2)u\, z_4, \\ [**] \quad & A_{35}z_5 = (-\beta_1 s - p_2)u\, z_5, \\ [***] \quad & A_{26}z_6 + A_{36}z_6 = (p_2 - p_1 - p_2)z_6 = -p_1 z_6. \end{aligned}$$

The following situations occur :
(1): If $z_4, z_5, z_6 > 0$, we delete all terms in $[*], [**]$ and $[***]$ due to negativity.
(2): If $z_4, z_5, z_6 < 0$. We have,

$$A_{24}z_4 + A_{35}z_5 + A_{26}z_6 + A_{36}z_6 \leq (p_1 + (2\beta_1 s + 2p_2)u) \| z \| .$$

Thus

$$D_+ \| z \| \leq \Big(\max\{A_{22}, A_{23} + A_{33}\} + (p_1 + (2\beta_1 s + 2p_2)u) \Big) \| z \| .$$

Hence, we suppose the following condition:

$$\max\{A_{22}, A_{23} + A_{33}\} + p_1 + (2\beta_1 s + 2p_2)u < 0. \tag{5.7}$$

(3): If $z_4, z_5 > 0$ and $z_6 < 0$, then $[*]$ and $[**]$ are deleted and,

$$A_{26}z_6 + A_{36}z_6 \leq p_1|z_6| \leq p_1 \| z \|,$$

hence we use (5.7).
(4): If $z_4, z_5 < 0$ and $z_6 > 0$, then $[***]$ is deleted and,

$$A_{24}z_4 + A_{35}z_5 \leq (\beta_1 s + p_2)u \| z \|,$$

hence we use (5.7).
(5): If $z_4, z_6 > 0$ and $z_5 < 0$, then $[*]$ and $[***]$ are deleted and,

$$A_{35}z_5 \leq (\beta_1 s + p_2)u \| z \|,$$

hence we use (5.7).
(6): If $z_4, z_6 < 0$ and $z_5 > 0$, then $[**]$ is deleted and,

$$A_{24}z_4 + A_{26}z_6 + A_{36}z_6 \leq (p_1 + (\beta_1 s + p_2)u) \| z \|,$$

hence we use (5.7).
(7): If $z_5, z_6 < 0$ and $z_4 > 0$, then $[**]$ and $[***]$ are deleted and,

$$A_{24}z_4 \leq (\beta_1 s + p_2)u \| z \|,$$

hence we use (5.7).
(8): If $z_5, z_6 < 0$ and $z_4 > 0$, then $[*]$ is deleted and,

$$A_{35}z_5 + A_{26}z_6 + A_{36}z_6 \leq (p_1 + (\beta_1 s + p_2)u)\|z\|,$$

hence we use (5.7).

The following conditions are sufficient for this case:

$$\begin{cases} A_{21} + A_{31} > 0 \\ \max\{A_{22}, A_{23} + A_{33}\} + p_1 + (2\beta_1 s + 2p_2)u < 0 \end{cases}$$

The inequalities of this lemma imply the negativity of the coefficient of $\parallel z \parallel$ in all cases. □

As it is mentioned in [3], when backward bifurcation occurs, for example our system, we need to prove the following lemma which is an analogous to proposition 5.2. of [3] for our model.

Lemma 5.3. *Let ψ be a simple closed curve in Ω, then there exist $\xi > 0$ and surfaces φ^k minimizing $\mathcal{S}$ with respect to $\sum(\psi, \Omega)$ and for all $k = 2, 3, ,...$ and $t \in [0, \epsilon]$, $\varphi_t^k \subseteq \Omega$.*

Proof. Let $\xi = \frac{1}{2} min\{u : (s, u, \tau, q_1) \in \psi\}$. Obviously $\xi > 0$. The inequality $\dfrac{du}{dt} \geq -(\mu+\beta_2+p_3+p_4)u$ implies the existence of $\epsilon > 0$ for which the trajectories remain in Ω, for $t \in [0, \epsilon]$, if $u(0) \geq \xi$. Now we prove that there exists $\{\varphi^k\}$ minimizing $\mathcal{S}$ in the set $\sum(\psi, \widetilde{\Omega})$ in which $\widetilde{\Omega} = \{(s, u, \tau, q_1) \in \Omega : u \geq \xi\}$. For $\varphi(l) = (s(l), u(l), \tau(l), q_1(l)) \in \sum(\psi, \Omega)$ define the surface, $\widetilde{\varphi}(l) = (\widetilde{s}(l), \widetilde{u}(l), \widetilde{\tau}(l), \widetilde{q_1}(l))$ as follows:

$$\begin{cases} \varphi(l) & if \quad u(l) \geq \xi, \\ (s, \xi, \tau, q_1) & if \quad u(l) < \xi,\ s + \xi + \tau + q_1 \leq 1, \\ A & if \quad u(l) < \xi,\ s + \xi + \tau + q_1 > 1, \end{cases}$$

in which

$$A = (\frac{s}{\sqrt{3}(s + \tau + q_1)}(1 - \xi), \xi, \frac{\tau}{\sqrt{3}(s + \tau + q_1)}(1 - \xi), \frac{q_1}{\sqrt{3}(s + \tau + q_1)}(1 - \xi)).$$

Now $\widetilde{\varphi}(l) \in \sum(\psi, \widetilde{\Omega})$. We will prove $\mathcal{S}\widetilde{\varphi} \leq \mathcal{S}\phi$.
We denote

$$\frac{\partial \widetilde{\varphi}}{\partial l_1} \wedge \frac{\partial \widetilde{\varphi}}{\partial l_2} = (\widetilde{x_1}, \widetilde{x_2}, \widetilde{x_3}, \widetilde{x_4}, \widetilde{x_5}, \widetilde{x_6})^T.$$

and

$$\frac{\partial \varphi}{\partial l_1} \wedge \frac{\partial \varphi}{\partial l_2} = (x_1, x_1, x_2, x_3, x_4, x_5, x_6)^T.$$

We prove,

$$\parallel \frac{\partial \widetilde{\varphi}}{\partial l_1} \wedge \frac{\partial \widetilde{\varphi}}{\partial l_2} \parallel \leq \parallel \frac{\partial \varphi}{\partial l_1} \wedge \frac{\partial \varphi}{\partial l_2} \parallel .$$

Three cases can occur:

1. If $u(l) \geq \xi$, then $\widetilde{\varphi} = \varphi$ and therefore, $| \widetilde{x_i} | = | x_i |$ $(i = 1, 2, ..., 6)$, hence,

$$\parallel \frac{\partial \widetilde{\varphi}}{\partial l_1} \wedge \frac{\partial \widetilde{\varphi}}{\partial l_2} \parallel = \parallel \frac{\partial \varphi}{\partial l_1} \wedge \frac{\partial \varphi}{\partial l_2} \parallel .$$

2. If $u(l) < \xi$ and $s(l) + \xi + \tau(l) + q_1(l) \leq 1$, then $\widetilde{\varphi}(l) = (s(l), \xi, \tau(l), q_1(l))$. Therefore it follows $\widetilde{x_i} = x_i$ $(i = 1, 3, 5)$

and $\widetilde{x_i} = 0$ $(i = 2, 4, 6)$. Thus $| \widetilde{x_i} | \leq | x_i |$ $(i = 1, 2, ..., 6)$, which imply

$$\parallel \frac{\partial \widetilde{\varphi}}{\partial l_1} \wedge \frac{\partial \widetilde{\varphi}}{\partial l_2} \parallel \leq \parallel \frac{\partial \varphi}{\partial l_1} \wedge \frac{\partial \varphi}{\partial l_2} \parallel .$$

3. If $u(l) < \xi$ and $s(l) + \xi + \tau(l) + q_1(l) > 1$, then

$$\widetilde{\varphi}(l) = (\frac{s}{\sqrt{3}(s+\tau+q_1)}(1-\xi), \xi, \frac{\tau}{\sqrt{3}(s+\tau+q_1)}(1-\xi), \frac{q_1}{\sqrt{3}(s+\tau+q_1)}(1-\xi)).$$

In this case, using $\frac{\partial \widetilde{s}}{\partial l_j} + \frac{\partial \widetilde{\tau}}{\partial l_j} + \frac{\partial \widetilde{q_1}}{\partial l_j} = 0$ we obtain,

$$\frac{\partial \widetilde{\varphi}}{\partial l_1} = z_1(l_1)f_1 + z_2(l_1)f_2 \qquad \textit{and} \qquad \frac{\partial \widetilde{\varphi}}{\partial l_2} = z_1(l_2)f_1 + z_2(l_2)f_2$$

in which,

$$f_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

and

$$z_1(l_j) = (1-\xi)\,\frac{(\tau+q_1)\frac{\partial s}{\partial l_j} - s(\frac{\partial \tau}{\partial l_j} + \frac{\partial q_1}{\partial l_j})}{\sqrt{3}(s+\tau+q_1)^2}$$

$$z_2(l_j) = (1-\xi)\,\frac{(s+q_1)\frac{\partial \tau}{\partial l_j} - \tau(\frac{\partial s}{\partial l_j} + \frac{\partial q_1}{\partial l_j})}{\sqrt{3}(s+\tau+q_1)^2}$$

for $j = 1, 2$. Therefore,

$$\frac{\partial \widetilde{\varphi}}{\partial l_1} \wedge \frac{\partial \widetilde{\varphi}}{\partial l_2} = (z_1(l_1)z_2(l_2) - z_2(l_1)z_1(l_2))f_1 \wedge f_2 = \frac{(1-\xi)^2}{3(s+\tau+q_1)^4} K \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

in which,

$$K = q_1(s+\tau+q_1)x_2 - \tau(s+\tau+q_1)x_3 + s(s+\tau+q_1)x_6.$$

This yields,

$$\| \frac{\partial \widetilde{\varphi}}{\partial l_1} \wedge \frac{\partial \widetilde{\varphi}}{\partial l_2} \| \leq | x_2 | + | x_3 | + | x_6 | \leq \| \frac{\partial \varphi}{\partial l_1} \wedge \frac{\partial \varphi}{\partial l_2} \| .$$

Furthermore $\widetilde{u}(l) = max\{u(l), \xi\}$, hence $\frac{1}{\widetilde{u}} \leq \frac{1}{u}$. Therefore,

$$\mathcal{S}\widetilde{\phi} = \int_{\mathcal{B}} \| \widetilde{P} \,.\, (\frac{\partial \widetilde{\phi}}{\partial l_1} \wedge \frac{\partial \widetilde{\phi}}{\partial l_2}) \| \; dl \leq \int_{\mathcal{B}} \| P \,.\, (\frac{\partial \phi}{\partial l_1} \wedge \frac{\partial \phi}{\partial l_2}) \| \; dl = \mathcal{S}\phi.$$

Let $\delta = \inf\{\mathcal{S}\phi : \phi \in \sum(\psi, \Omega)\}$ and the sequence $\{\phi^k\}$ minimizes $\mathcal{S}$ in the set $\sum(\psi, \Omega)$, then $\lim_{k\to\infty} \mathcal{S}\phi^k = \delta$. Now consider the sequence $\{\widetilde{\phi}^k\} \subset \sum(\psi, \Omega)$ as in the above definition, from the boundedness of $\{\mathcal{S}\widetilde{\phi}^k\}$ and $\mathcal{S}\widetilde{\phi}^k \leq \mathcal{S}\phi^k$, we have $\lim_{k\to\infty} \mathcal{S}\widetilde{\phi}^k \leq \delta$. Furthermore $\widetilde{\phi}^k \in \sum(\psi, \Omega)$ hence $\mathcal{S}\widetilde{\phi}^k \geq \delta$ and $\lim_{k\to\infty} \mathcal{S}\widetilde{\phi}^k \geq \delta$ which imply $\lim_{k\to\infty} \mathcal{S}\widetilde{\phi}^k = \delta$. Now,

$$\inf\{\mathcal{S}\widetilde{\phi} \,:\, \widetilde{\phi} \in \sum(\psi, \widetilde{\Omega})\} \leq \inf\{\mathcal{S}\phi \,:\, \phi \in \sum(\psi, \Omega)\} = \delta.$$

The relation $\widetilde{\phi} \in \sum(\psi, \Omega)$ implies that $\inf\{\mathcal{S}\widetilde{\phi} \,:\, \widetilde{\phi} \in \sum(\psi, \widetilde{\Omega})\} \geq \delta$, hence $\inf\{\mathcal{S}\widetilde{\phi} \,:\, \widetilde{\phi} \in \sum(\psi, \widetilde{\Omega})\} = \delta$. At the final, we can show that $\lim_{k\to\infty} \mathcal{S}\widetilde{\phi}^k = \delta = \inf\{\mathcal{S}\widetilde{\phi} \,:\, \widetilde{\phi} \in \sum(\psi, \widetilde{\Omega})\}$, i.e. $\{\widetilde{\phi}^k\}$ minimizes $\mathcal{S}$ with respect to $\sum(\psi, \widetilde{\Omega})$. $\square$

From Lemma 5.2 and 5.3, we have the following result.

Theorem 5.4. *All ω-limit points of (2.1) in Ω° are equilibrium points and therefore each positive semi-trajectory tends to a steady state.*

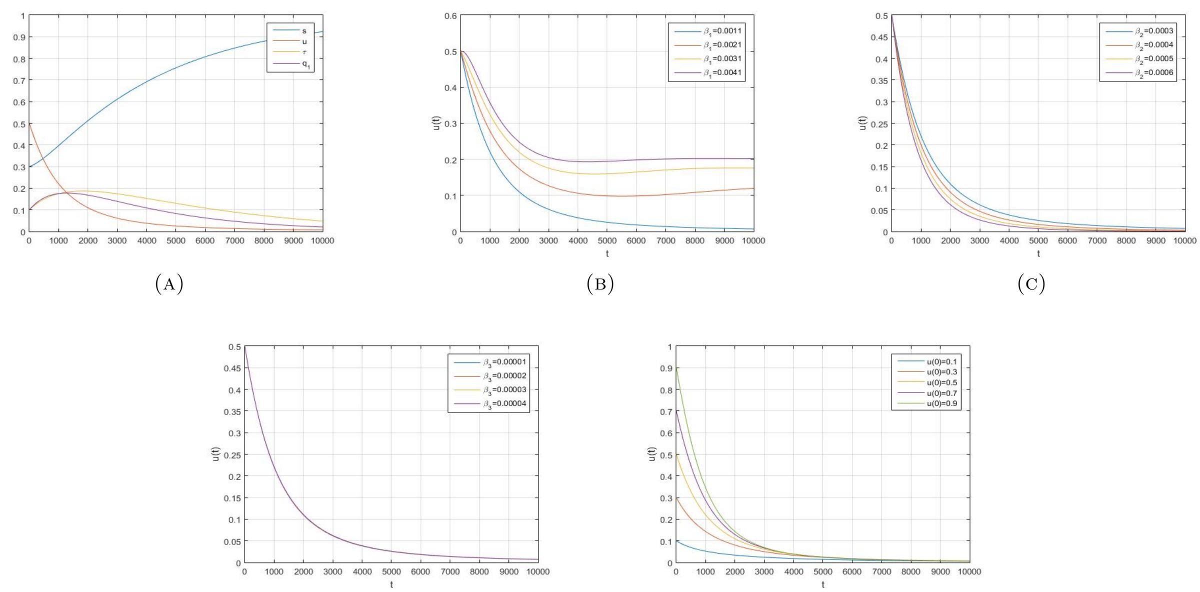
And finally, we have the following useful result.

Theorem 5.5. *Let the conditions in Lemma 5.2 hold,*
(1) If the system has the unique steady state P_0, i.e., the drug-free equilibrium, all trajectories converge to P_0.
(2) If the system has a unique endemic steady state, all trajectories converge to it.

6. Numerical Simulation

In this section, we will simulate the system using MATLAB software, so that the obtained analytical results can be seen numerically. We present three cases.

Case 1. $R_0 < \frac{\beta_1}{\beta_1+\beta_3} < 1$.
We choose $\mu = 3 \times 10^{-4}$, $\beta_1 = 11 \times 10^{-4}, \beta_2 = 3 \times 10^{-4}$, $\beta_3 = 10^{-5}$, $p_1 = 10^{-6}$, $p_2 = 10^{-3}$, $p_3 = 4 \times 10^{-4}$, $p_4 = 2 \times 10^{-4}$, $p_5 = 10^{-4}$ and $p_6 = 10^{-4}$. In this case, $R_0 \simeq 0.9166$ and $\frac{\beta_1}{\beta_1+\beta_3} \simeq 0.99099$. See Figure 2.

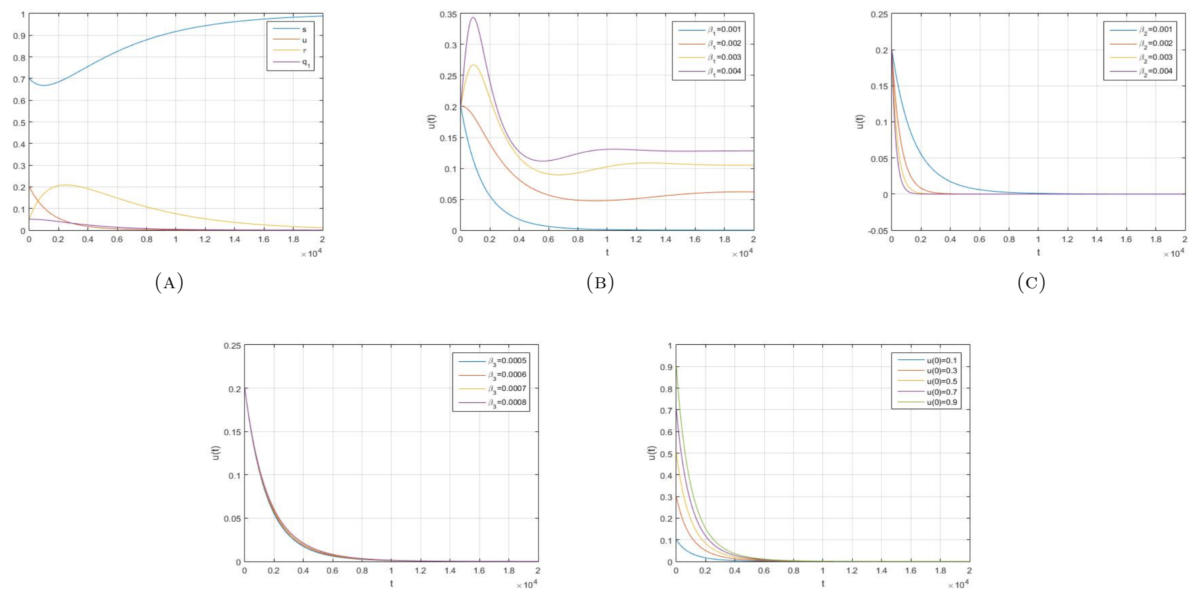


(A) (B) (C) (D) (E)

Figure 2. (a) shows the plot of the solution of the system. (b), (c) and (d) show the sensitivity of $u(t)$ with respect to β_1, β_2, and β_3 respectively. (e) shows the convergence of $u(t)$, (infectious compartment) of five solution curves of the system to the DFE. In this case, Lemma (5.1) shows the global stability of DFE.

Case 2. $\frac{\beta_1}{\beta_1+\beta_3} < R_0 < 1$.
We choose $\mu = 2 \times 10^{-4}$, $\beta_1 = 10^{-3}, \beta_2 = 10^{-3}$, $\beta_3 = 5 \times 10^{-4}$, $p_1 = 5 \times 10^{-5}$, $p_2 = 5 \times 10^{-5}$, $p_3 = 10^{-4}$, $p_4 = 10^{-4}$,

$p_5 = 10^{-4}$ and $p_6 = 10^{-4}$. In this case, $R_0 \simeq 0.7142$ and $\frac{\beta_1}{\beta_1+\beta_3} \simeq 0.6666$. See Figure 3.



(A) (B) (C)

(D) (E)

FIGURE 3. (a) shows the plot of the solution of the system. (b), (c) and (d) show the sensitivity of $u(t)$ with respect to β_1, β_2 and β_3 respectively. (e) shows the convergence of $u(t)$, (infectious compartment) of five solution curves of the system to the DFE.

Case 3. $R_0 > 1$ and parameters satisfy the relations in lemma 5.2.
We choose $\mu = 10^{-3}$, $\beta_1 = 3 \times 10^{-3}, \beta_2 = 10^{-3}$, $\beta_3 = 3 \times 10^{-4}$, $p_1 = 3 \times 10^{-3}$, $p_2 = 2 \times 10^{-3}$, $p_3 = 2 \times 10^{-4}$, $p_4 = 10^{-4}$, $p_5 = 10^{-4}$and $p_6 = 10^{-2}$. In this case $R_0 \simeq 1.304347$. See Figure 4.

7. Conclusion

In this article, the model of White and Comiskey has been modified. We added two compartments for drug users under consultation rehabilitation and incarcerated drug users. We studied the steady states of the model, its existence, and local and global stability. We showed that DFE is locally and globally stable under appropriate conditions. With the aid of the geometric stability method, we studied the global stability of the endemic steady states. Furthermore, we proved that backward bifurcation can occur. The occurrence of this bifurcation showed that the reduction of the basic reproduction number to $R_0 < 1$ is not enough for the control of the epidemic.

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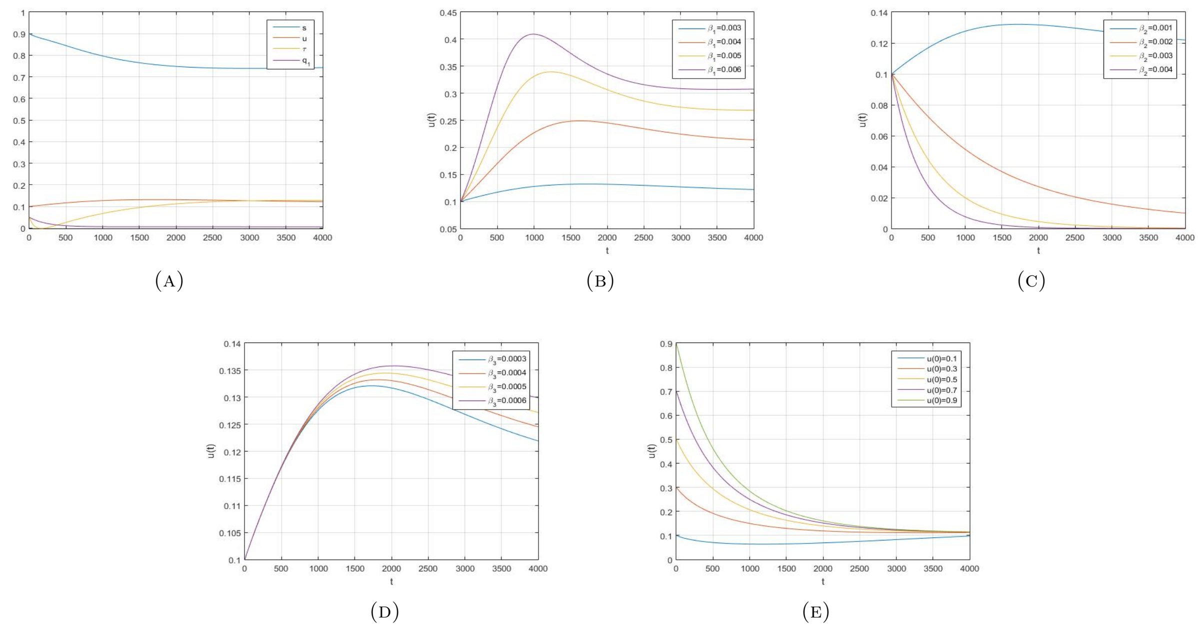

FIGURE 4. (a) shows the plot of the solution of the system. (b), (c) and (d) show the sensitivity of $u(t)$ with respect to β_1, β_2, and β_3 respectively. (e) shows the convergence of $u(t)$, (infectious compartment) of five solution curves of the system to the endemic equilibrium.

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