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Sliding mode control of a class of uncertain nonlinear fractional-order time-varying delayed systems based on Razumikhin approach

Mohammad Ghamgosar, Seyed Mehdi Mirhosseini-Alizamini*, and Mahmood Dadkhah
Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Abstract

Within the current paper, we design a sliding-based control law to stabilize a set of systems that are nonlinear, fractional order involve delay, perturbation, and uncertainty. A control law-based sliding mode is considered in such a way that the variables of the closed loop system reach the sliding surface in a limited time and stay on it for later times. Then, using the Razomokhin stability theorem, the stability of the systems is proved and in the end, a calculation is found to search for useful methods.

Keywords. Delay systems, Fractional order systems, Sliding mode control, Norm-bound perturbation.
2010 Mathematics Subject Classification. 93D15, 93C10, 34K37.

1. Introduction

Fractional calculus features a history of 300 long times. A complete framework for fractional calculus was first presented by the Norwegian researcher Abel [36]. Fractional calculus work in control has been designed from Ostalop's work, which was driven to the plan of a controller called CRONE in 1991. In general, there are different modes for closed-loop control frameworks [6]. The system has Integer order and controller has Integer order, the system has Integer order and the controller has fractional order, the system has fractional order and controller has Integer, the system has fractional order and controller has fractional order. Different numerical strategies were utilized to unravel non-fractional optimal control and control issues [29–32]. A few strategies such as the Adomian decomposition method (ADM), and variational iteration method (VIM) have been utilized for the numerical arrangement of fractional issues [14, 34, 37, 43]. Stability could be an essential issue within the theory about the control system. In 1996, when Matigon introduced the stability of linear systems with fractional order, issues within the field of stability and stabilization of fractional systems have been of extraordinary intrigue to researchers [28]. If the system incorporates delay, planning the criteria that can guarantee system stability could be a challenging assignment [3, 41]. Regarding the stability of a delay system, we are interested in its asymptotic stability, however, other definitions have been proposed for the stability of such systems, including the stability of Mittag-Leffler and the stability called finite time stability (FTS) [2, 15, 23, 24, 27].

The Lyapunov-Kraskovsky second-order strategy is an efficient approach for assessing the stability criteria of systems that are nonlinear. The role of this approach in nonlinear systems is irrefutable but this approach isn't exceptionally efficient for delay fractional systems. In this strategy got to assess a functional called Lyapunov-Krasovsky and its derivative, which is exceptionally difficult to calculate for delay fractional systems [42]. Numerous efforts have been made to illuminate this issue and have been proceeded so distant. For a few of these things, they have been shown to be inaccurate or inadequate [18, 33, 45]. Lee Chen in a piece distributed in 2019 [7] gives a linear matrix inequality (LMI) for a delay fractional system. Utilizing an LMI a number of factors called choice factors, the negative definite of a certain matrix can be decided. In spite of the fact that an LMI incorporates an uncommon frame, it's pertinent in

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∗ Corresponding author. Email: m_mirhosseini@pnu.ac.ir.

numerous disparities of matrices and imperatives in control issues. Given the importance and challenge of the issue, inquire about proceeds.

So distant, no efficient and straightforward relationship has been found for the stability of delayed fractional systems, so theorems related to the stability criteria of delayed systems such as Razomykhin and Lyapunov-Kraskovsky theorems have been developed for certain classes of these cases [5]. In [16], a bilinear matrix inequality (BMI)-based approach the stability of delayed nonlinear fractional systems is outlined based on the state controller. Nearly all control issues confronted with unsettling influence and instability [20]. There are various strong controllers that are safe to indeterminacy in parameters and irritations [18]. A sliding mode controller (SMC) could be a capable and broadly utilized instrument among strong controllers to kill the efficiency of turbulence and instability in control hypothesis [12]. Within the realm of fractional systems, SMC has been utilized. We utilize, an integral type SMC including a fractional term for a deferred delayed fractional order system. Various methods such as back stepping-sliding mode, terminal sliding demonstrate and the adaptive sliding show has been created for fractional systems.

Within the current paper, a methodology based on, a new type of integral switching surface of the fractional-order is used to achieve sliding mode dynamics. The plan of the switching surface can warranty the asymptotic stability and desired performance [25]. In the first order SMC, the incidence of chattering is the main trouble. In order to fix this problem, a higher-order SMC is presented, which is an efficient one to dominate, the occurrence of chattering and switching control signals. Novel results on nonlinear fractional-order time-varying delayed systems with second-order sliding mode control are presented. SMC law is designed to create delay-independent stability on the sliding surface in finite time and it is used to reduce norm bounded uncertainties variation and bounded external disturbances to analyze the robust stability of the considered nonlinear fractional order delayed systems.

In recent years, there are various results are derived for nonlinear fractional order systems, for instance, the Output feedback finite-time dissipative control in [17], observer-based control results were developed in [35], mixed H∞ and passive control results presented in [19], and references therein. The problem of Finite-time H∞ control for neural networks has been reported in [40]. The problem of delay dependent and order dependent H∞ control with delays has been reported in [38]. Among these control methods, SMC has been well-definite as an impressive tool to prospect the transient response and also attain the robust performance of the systems. The main advantage of the SMC technique is its simplicity and inherent robustness towards the matched uncertainty. The SMC comprises discontinuous control input that drives the controlled system onto a specified sliding surface. Once the system is on the sliding surface, it becomes immune to matched uncertainties [11].

In [1], a fractional sliding mode is outlined for the fractional system with a delay in input. In [9], the terminal sliding mode controller with fractional order is utilized to control a specific category of control systems. For delayed fractional systems, this inquiry is progressing. In [4] by including a few limitations, a delay-independent sliding mode is displayed for the fractional system. Also, results on the second and higher-order SMC approach for uncertain nonlinear systems were developed in [21, 39]. In [13], an observer-based disturbance rejection control is designed for handling external disturbances. In [10], sliding mode observer for nonlinear fractional order systems and their advantages have been studied and finite-time SMC for higher-order systems was reported in [8]. From all considering existing researches, there has been little attention paid to the problem of second-order SMC design for delayed nonlinear fractional order.

Within the current article, fundamental preliminaries from fractional calculus are given in section 2. The issue beneath is considered and its characteristics are expressed in section 3. In section 4, we plan a suitable SMC conspire and after that, a sufficient condition to guarantee that system is stable over any relegated certain-time interim subject to time-delay and mismatched external disturbance influence and uncertainty. In section 5, some illustrations for various modes have been given to appear the sensibility of our hypothesis. At long last, we conclude our work with conclusions in section 6.

2. Preliminaries

In this section, two fundamental theorem and lemma are given.

Lemma 2.1. *Fractional time derivative of order α $(0 < \alpha < 1)$ of function $v(y,t) = y^T y$ can be calculated as:*

$$D_t^{\alpha} v(t) = \left(D_t^{\alpha} y\right)^T y + y^T \left(D_t^{\alpha} y\right) + 2\Psi, \tag{2.1}$$

where

$$\Psi = \sum_{k=1}^{\infty} \frac{\Gamma(1+\alpha)\left(D_t^{\alpha-k}y\right)\left(D_t^k y\right)^T}{\Gamma(1+k)\Gamma(1-k+\alpha)}. \tag{2.2}$$

$\Gamma(.)$ *is the gamma function and* Ψ *is bounded as follow:*

$$\exists\delta > 0, |\Psi| \leq \delta \|y(t)\|.$$

Proof. See proof in [44]. □

Theorem 2.2. *(Razumikhin stability for fractional order systems with delay) [26]*
For the delay fractional system as follows:

$${}_0^C D_t^\alpha y(t) = f\left(t, y_\theta\right), \tag{2.3}$$

where $y(t) \in \mathbb{R}^n$, $y_\theta(t) = y(t+\theta)$, $-\tau \leq \theta \leq 0$. *Suppose that* $v_1, v_2, v_3 : \mathbb{R}^+ \to \mathbb{R}^+$ *are scalar, continuous and nondecreasing functions, and* $v_1(0) = v_2(0) = v_3(0) = 0, v_2(,)$ *is strictly increasing, if there exists a continuous function* $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ *such that:*
(i) $v_1(\|y\|) \leq V(t,y) \leq v_2(\|y\|)$, $\forall t \geq 0$, $y \in \mathbb{R}^n$,
(ii) $D_t^\alpha V(t, y(t)) \leq -v_3(\|y\|)$, $0 < \alpha < 1$ *provided* $V(t+s, y(t+s)) \leq qV(t, y(t))$, *for* $q > 1$, $-\tau \leq s \leq 0$, *and* $t \geq 0$, *then system* (2.3) *is uniformly stable. Additionally,* $v_3(s) > 0$ *for* $s > 0$ *and there exists a continuous nondecreasing function* $\tilde{t}(s) : \mathbb{R}^+ \to \mathbb{R}^+$ *that* $\tilde{t}(s) > s$ *for* $s > 0$, *such that the condition (ii) is substituted by:*

$$D_t^\alpha V(t, y(t)) \leq -v_3(\|y\|), \quad 0 < \alpha < 1,$$

whenever $V(t+s, y(t+s)) \leq \tilde{t} V(t, y(t))$, $s \leq -\tau \leq 0$. *Then, system* (2.3) *is uniformly asymptotically stable.*

3. Problem statement

Consider a nonlinear fractional system with delay of the form:

$$\begin{array}{c} {}_{t_0}^C D_t^\alpha x(t) = Ax(t) + A_\tau x(t-\tau) + Bu(t) + Gd(t) + f\left(t, x(t), x(t-\tau)\right), \quad t \geq t_0, \\ x(t) = \Phi(t), \quad t_0 - \tau \leq t \leq t_0. \end{array} \tag{3.1}$$

where $0 < \alpha < 1$, and $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, are the state and control vectors respectively and A, A_τ, B, G are constant matrices with appropriate dimensions. $d(t)$ coordinated disturbance influence that's adequately smooth and bounded. Too, it is accepted that the upper bound of $d(t)$ is known and $|d(t)| \leq d_{\max}$; the vector f is a nonlinear and bounded function $\|f(.)\| \leq M$, that M is a known positive consistent; $\Phi(t) \in C\left(\left[t_0 - \tau, t_0\right], \mathbb{R}^n\right)$, is the persistent beginning initial function; $\tau > 0$ is the consistent time-delay of the system. In this paper, according to the sliding surface strategy, the control law is planned to fix the time-delay system (3.1) within the confront of external disturbance influence. Another, we consider the trace of uncertainty on the system (3.1). Uncertainty in a genuine system can come from many sources (3.1). Uncertainty in a genuine system can come from many sources. There are a few strategies to consider uncertainty in a nonlinear system. One of these strategies accepts it as a norm-bound and time-varying irritation [21, 22]. That is, we consider the terms including uncertainty in the following form:

$$f\left(t\ , x(t), x(t-\tau)\right) = \Delta A x(t) + \Delta A_\tau x(t-\tau), \tag{3.2}$$

where

$$\Delta A = E_1 F_1(t) H_1, \quad \Delta A_\tau = E_2 F_2(t) H_2, \tag{3.3}$$

also, $E_i, H_i (i = 1, 2)$ are known real constant matrices of appropriate dimensions, and $F_i (i = 1, 2)$ are unknown real time-varying matrices satisfying:

$$F_i(t) F_i^T(t) \leq I_n, \quad i = 1, 2, \quad t \geq t_0. \tag{3.4}$$

where I_n is identity matrix. Therefore, the system studied in this paper, consisting of disturbance and uncertainty, is as bellow:

$$ {}_0^C D_t^\alpha x(t) = (A + \Delta A)x(t) + (A_\tau + \Delta A_\tau)\, x(t-\tau) + Bu(t) + Gd(t), \quad t \geq 0. \tag{3.5}$$

Assumption 3.1. Rank and dimension of matrix B are equal.
Assumption 3.2. Controllability conditions are established for pair (A, B).

4. Main Results

Here our aim is to design sliding mode controller such that the trajectories of the SMC system are driven onto the predefined sliding surface in a finite instant and the closed loop system is asymptotically stable subject to all admissible parameter uncertainties and mismatched external disturbance. To do this, we will design the following integral type sliding surface for system (3.5):

$$\Lambda(t) = NI^{1-\alpha}x(t) - \int_{t_0}^{t} N\left[(A + \Delta A + BK)x(\xi) + \Delta A_\tau x(\xi - \tau)\right] d\xi, \tag{4.1}$$

and for system (3.1) we also define the sliding surface as follows:

$$\Lambda(t) = NI^{1-\alpha}x(t) - \int_{t_0}^{t} N(A + BK)x(\xi)d\xi. \tag{4.2}$$

In both cases m-dimensional vector $\Lambda(t) \in \mathbb{R}^m$ is the sliding surface, $K \in \mathbb{R}^{m\times n}$ is controller gain matrix to be determined later and $N \in \mathbb{R}^{m\times n}$ is a steady matrix, must be such that the $NB \in \mathbb{R}^{m\times m}$ is invertible. Integral I is the Riemann-Liouville fractional integral. In the following, we calculate the necessary parameters to design the sliding surface. Then, we use Theorem 2.2 to discover an adequate condition to guarantee that systems (3.1) and (3.5) are asymptotically stable over any allotted limited-time interval subject to nonentities time-delay and bungled external disturbance and uncertainty.

A. SMC Designing

In this section, we plan a reasonable SMC conspire to guarantee the directions of the SMC systems (3.1) and (3.5) reach the sliding surface inside a limited time.

Theorem 4.1. *Consider by system (3.5) involving time-varying perturbation. The following control law ensures the convergence of the state variables to the sliding surface inside a limited time.*

$$u(t) = Kx(t) - (NB)^{-1}NA_\tau x(t-\tau) - (NB)^{-1}\mu\Lambda(t) - (NB)^{-1}\rho\, \mathrm{sign}(\Lambda(t)), \tag{4.3}$$

with $\mu > 0$ and $\rho > d_{max}\,\|NG\|$ are designed parameters where d_{max} is the upper bound of the external disturbance. The function $\mathrm{sign}[.] : \mathbb{R}^m \to \mathbb{R}^m$ *is defined as* $\mathrm{sign}(\Lambda(t)) = \left[\mathrm{sign}\,(\Lambda_i(t)\right]_{m\times 1}$ *with*

$$\mathrm{sign}(\Lambda_i(t)) = \begin{cases} 1, & \Lambda_i(t) > 0, \\ 0, & \Lambda_i(t) = 0, \qquad i = 1, \cdots, m, \\ -1, & \Lambda_i(t) < 0. \end{cases} \tag{4.4}$$

Proof. After derivative from (4.1), we have

$$\dot{\Lambda}(t) = ND^\alpha x(t) - N\left[(A + \Delta A + BK)x(t) + \Delta A_\tau x(t-\tau)\right]. \tag{4.5}$$

Using (3.5), one has:

$$\begin{aligned} \dot{\Lambda}(t) &= N[(A + \Delta A)x(t) + (A_\tau + \Delta A_\tau)x(t-\tau) + Bu(t) + Gd(t)] - N[(A + \Delta A + BK)x(t) + \Delta A_\tau x(t-\tau)] \\ &= NA_\tau x(t-\tau) + NBu(t) + NGd(t) - NBKx(t). \end{aligned} \tag{4.6}$$

From(4.3) and above relation, it is straightforward to see that

$$\dot{\Lambda}(t) = -\mu\Lambda(t) - \rho sign(\Lambda(t)) + NGd(t). \tag{4.7}$$

Consider a non-negative quadratic function as bellow:

$$V(\Lambda(t)) = 0.5\Lambda^2(t). \tag{4.8}$$

Considering (4.7), $|d(t)| \leq d_{\max}$, and differentiation of (4.8) we have:

$$\begin{aligned}
\dot{V}(\Lambda(t)) &= \Lambda(t)\dot{\Lambda}(t) \\
&= \Lambda(t)\left[-\mu\Lambda(t) - \rho sign(\Lambda(t)) + NGd(t)\right] \\
&= -\mu\Lambda^2(t) - \rho \mid \Lambda(t) \mid + NGd(t)\Lambda(t) \\
&\leq -\mu\Lambda^2(t) - \rho \mid \Lambda(t) \mid + d_{max}\|NG\| \mid \Lambda(t) \mid \\
&\leq -\rho \mid \Lambda(t) \mid + d_{max}\|NG\| \mid \Lambda(t) \mid \\
&\leq -\underbrace{(\rho - d_{max}\|NG\|)}_{z} \mid \Lambda(t) \mid \\
&\leq -z \mid \Lambda(t) \mid .
\end{aligned}$$

Because $0 < \rho - d_{\max}\|NG\|$ we have $z > 0$ and

$$\dot{V}(\Lambda(t)) = \Lambda(t)\dot{\Lambda}(t) \leq -z \mid \Lambda(t) \mid < 0. \tag{4.9}$$

Hence the reaching law is satisfied. Using (4.9) and considering the two cases in which the positive or negative sign $\Lambda(t_0)$ is considered, it can be shown that the system variables reach the sliding surface at a finite time. In fact, by integrating (4.9), the time to reach the sliding level is calculated in this way $t_0 < T^\star \leq \left(\frac{|\Lambda(t_0)|}{\beta}\right) + t_0$. This completes the proof.

□

Theorem 4.2. *Consider by system (3.1) with sliding surface in (4.2). The following control law ensures the convergence of the state variables to the sliding surface inside a limited time.*

$$u(t) = Kx(t) - (NB)^{-1}NA_\tau x(t-\tau) - (NB)^{-1}\mu\Lambda(t) - (NB)^{-1}\rho\,\mathrm{sign}(\Lambda(t)), \tag{4.10}$$

where in $\mu > 0$ and $\rho > M\|N\| + d_{\max}\|NG\|$ are design parameters.

Proof. We go through a process like proving the Theorem 4.1

$$\dot{V}(\Lambda(t)) \leq -\mu\Lambda^2(t) - \rho|\Lambda(t)| + d_{\max}\|NG\|\Lambda(t)| + \|Nf\||\Lambda(t). \tag{4.11}$$

Because $\mu > 0$ and $\|f(.)\| \leq M$ so:

$$\begin{aligned}
\dot{V}(\Lambda(t)) &\leq -\rho|\Lambda(t)| + d_{max}\|NG\||\Lambda(t)| + M\|N\||\Lambda(t)| \\
&= -\left(\rho - d_{max}\|NG\| - M\|N\|\right)|\Lambda(t)| \\
&\leq -z|\Lambda(t)|,
\end{aligned}$$

where $z = (\rho - d_{max}\|NG\| - M\|N\|)$. According to the conditions of Theorem 4.2, we have $\rho > d_{max}\|NG\| + M\|N\|$ so $z = (\rho - d_{max}\|NG\| - M\|N\|) > 0$, hence, $\dot{V}(\Lambda(t)) < 0$.
This ensures the reaching of the state variables to the desired surface (4.2) for $\mu > 0$ and $\rho > M\|N\| + d_{\max}\|NG\|$. This completes the proof. □

B. Asymptotically Stable

In this area, a computational method for finding the matrix K is introduced using the stability criteria in Theorem 2.2.

Theorem 4.3. *Consider the system (3.5). Matrix K ensure that systems (3.5) is asymptotically stable subject to time-delay and disturbance and uncertainty,if all of $A + BK$ eigenvalues have a strictly negative real part and there is a unique solution for the Lyapanov equation:*

$$-Q = (A + BK)^T P + P(A + BK), \quad Q > 0, \tag{4.12}$$

such that

$$\lambda_{\min}(Q)r \geq 2rq\sigma\|P\| \left\|A_\tau - B(NB)^{-1}NA_\tau\right\| + 2\left(\|G\|d_{\max} + M + \delta\right)\|P\|, \tag{4.13}$$

where $d_{\max}$ is the upper bound of the external disturbance term and M is the upper bound of norm-bounded uncertainty function in (3.2). $\|x(t)\|$ is bounded in the set $\{x \in \mathbb{R}^n \mid \|x\| \leq r\}$ for a large positive number r, q is a positive number more than one, and δ is a positive constant and $\sigma = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}$.

Proof. By substituting (4.3) into (3.5) the state variables reach the sliding surface at the finite time according to Theorem 4.1, so we have $\Lambda(\mathrm{t}) = 0$ on the sliding surface for $t \geq T^\star$, and the closed loop system in the sliding mode is as bellow:

$${}_0^C D_t^\alpha x(t) = (A + BK + \Delta A)x(t) + (A_\tau - B(NB)^{-1}NA_\tau + \Delta A_\tau)x(t-\tau) + Gd(t), t \geq T^\star. \tag{4.14}$$

Since $\mathrm{Re}\left(\lambda_i(A + BK)\right) < 0$, $i = 1, 2 \ldots, n$, thus $A + BK$ is a Hurwitz matrix also $Q > 0$, so the matrix equation (4.12) has a unique solution P. Now, we can consider following Lyapunov function:

$$V(t) = x^T Px. \tag{4.15}$$

It is easy to verify that $V(t) = x^T Px$ is bounded by $\omega_1(x) = \lambda_{\min}(P)\|x\|^2$ and $\omega_2(x) = \lambda_{\max}(P)\|x\|^2$. So, the first condition of Theorem 2.2 are satisfied. Considering Eq. (4.14) and Lemma 2.1, the time derivative of Eq. (4.15) is obtained as:

$$D^\alpha V(t) = \left[A_{eq}x(t) + A_{eq\tau}x(t-\tau) + Gd + f\right]^T Px + x^T P\left[A_{eq}x(t) + A_{eq\tau}x(t-\tau) + Gd + f\right] + 2P\Psi.$$

where f is $f(t\ , x(t)\ , \ x(t-\tau)) = \Delta Ax(t) + \Delta A_\tau x(t-\tau)$, $A_{eq} = A + BK$ and $A_{eq\tau} = A_\tau - B(NB)^{-1}NA_\tau$, also we know that

$$\Delta A = E_1F_1(t)H_1, \quad \Delta A_\tau = E_2F_2(t)H_2.$$

and $\|f(.)\| \leq M$. Considering Eq. (4.14), the derivative of (4.15) is obtained as:

$$D^\alpha V(t) = x^T\left[A_{eq}^T P + PA_{eq}\right]x + \left[2x^T PA_{eq\tau}x(t-\tau)\right] + \left[2x^T PGd + 2x^T Pf(x)\right] + 2P\Psi.$$

As $-Q = A_{eq}^T P + PA_{eq}$, $Q > 0$, and $|d(t)| \leq d_{\max}$ and according to Lemma 2.6 we have

$$\begin{aligned}
D^\alpha V(t) &\leq -\lambda_{\min}(Q)\|x\|^2 + 2\|P\|\,\|A_{eq\tau}\|\,\|x\|\|x(t-\tau)\| + 2(\|Gd\| + \|f\|)\|P\|\|x\| + 2\delta\|x\|\|P\| \\
&\leq -\lambda_{\min}(Q)\|x\|^2 + 2\|P\|\,\|A_{eq\tau}\|\,\|x\|\|x(t-\tau)\| + 2(\|Gd\| + \|f\| + \delta)\|P\|\|x\| \\
&\leq -\lambda_{\min}(Q)\|x\|^2 + 2\|P\|\,\|A_{eq\tau}\|\,\|x\|\|x(t-\tau)\| + 2\left(\|G\|d_{\max} + M + \delta\right)\|P\|\|x\|.
\end{aligned}$$

Now, according to the Theorem 2.2, we consider the second condition of Razumikhin stability theorem. To do this, we consider $\tilde{t}(s) = q^2 s$, $q > 1$, the function $\tilde{t}$ satisfies the condition of the Razumikhin theorem. Now if the assumption $V(t+\theta, x(t+\theta)) \leq \tilde{t}V(t\ , x(t))$ holds, then we have to find a scalar and positive function v_3. From $V(t+\theta, x(t+\theta)) \leq \tilde{t}V(t\ , x(t))$, we have $x^T(t+\theta)Px(t+\theta) \leq q^2x^T(x)Px(t)$ hence $\lambda_{\min}(P)\|x(t+\theta)\|^2 \leq q^2\lambda_{\max}(P)\|x\|^2$ then

$$\|x(t+\theta)\|^2 \leq q^2\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|x\|^2.$$

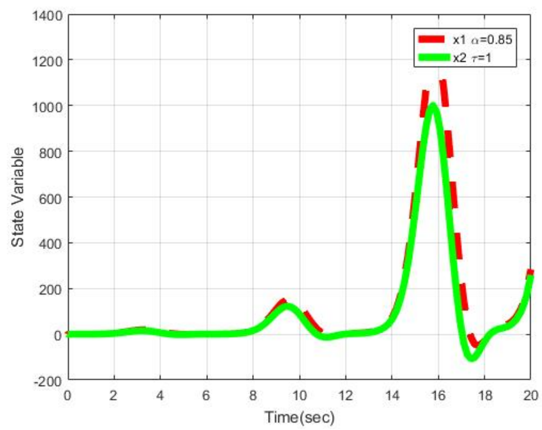


FIGURE 1. Trajectories of the open-loop system with $\alpha = 0.85$ for Example 5.1.

In the above relation, we consider $\theta = -\tau$ then $\|x(t-\tau)\|^2 \leq q^2 \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \|x\|^2$ hence $\|x(t-\tau)\| \leq q\sigma\|x\|$ where in $\sigma = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}$. To establish the second condition of the Theorem 2.2, it suffices to put $v_3(\|x\|) = \zeta\|x\|$, where $\zeta = \lambda_{min}(Q)r - 2r\sigma\|P\| \, \|A_{eq\tau}\| \, q - 2\left(\|G\|d_{max} + M + \delta\right)\|P\|$. According to, inequality (4.13), one has $\zeta > 0$, therefore

$$D^{\alpha}V(t) \leq -v_3(\|x\|)$$

So, if condition (4.13) is satisfied, according to the second condition of Theorem 2.2 , the closed-loop system (4.14) is stable. This completes the proof. □

Remark 4.4. According to the Assumption 2.2, the pair (A, B) is controllable, so we choose the matrix K in such a way that n eigenvalues of the matrix $A+BK$ have a strictly negative real part. If the relation (4.12) is not established, by repeating this process, we will examine another collection of eigenvalues. Using a transformation, inequality (4.12) can be written more easily. To do this, using the transformation $x(t) = Y\tilde{z}(t)$ system (4.14) can be rewritten as follows

$$D^{\alpha}\tilde{z}(t) = Y^{-1}A_{eq}Y\tilde{z}(t) + Y^{-1}A_{eq\tau}Y\tilde{z}(t-\tau) + Y^{-1}Gd(t)Y + Y^{-1}f(\tilde{z}(t)).$$

Now, by considering P equal to the identity matrix, we will have $1 = \sigma = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}$, and therefore we can reach the following relation:

$$\lambda_{\min}(Q)r \geq 2r\left\|Y^{-1}A_{eq\tau}Y\right\|q + 2\left(||Y^{-1}GT||d_{\max} + \widetilde{M} + \delta\right),$$

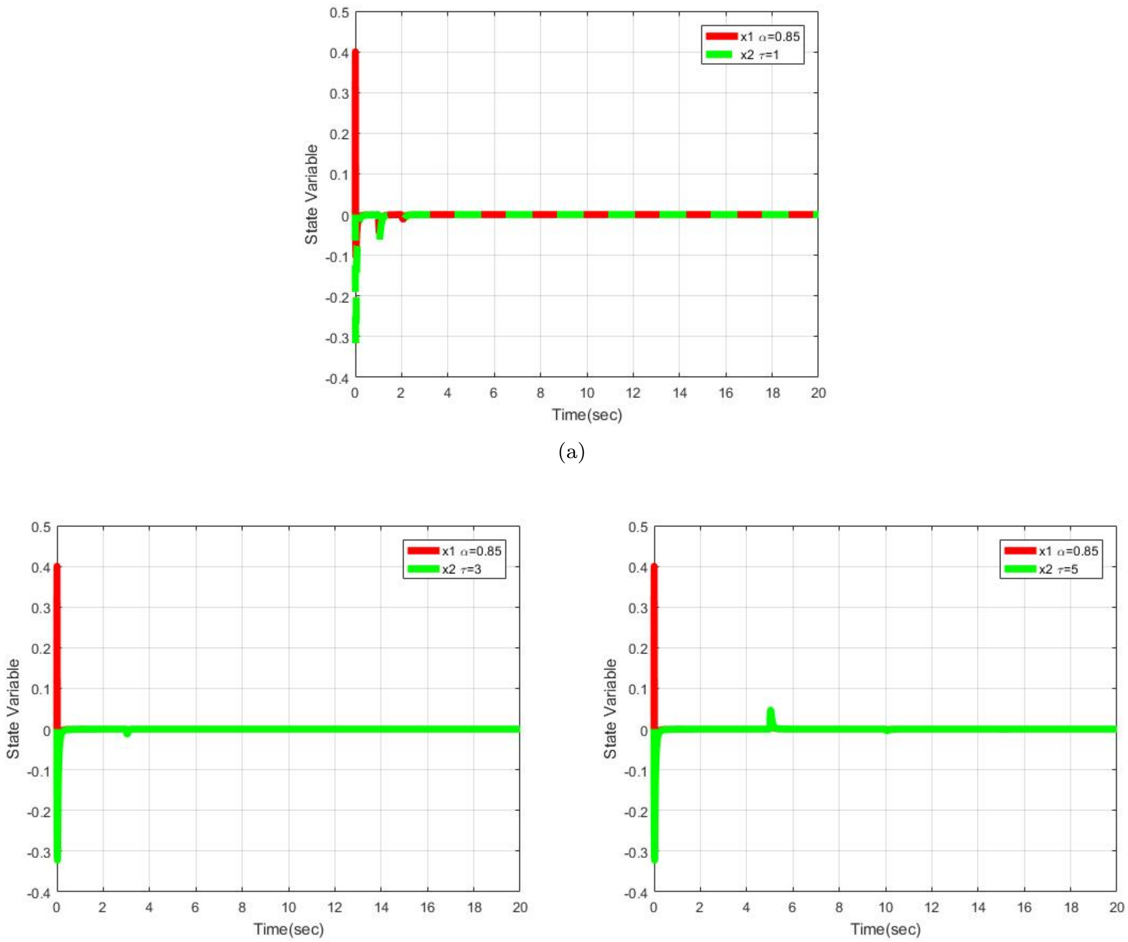
where $A^{\star}_{eq} = Y^{-1}A_{eq}Y\tilde{x}(t)$, $\widetilde{M} = M\|Y^{-1}\|\|Y\|$ and $Q = -((A^{\star}_{eq})^T + A^{\star}_{eq})$. The above relation can be used instead of inequality (4.12).

Corollary 4.5. *For system (3.1), stability on the sliding surface is determined if K is chosen in such a way that the conditions of Theorem 4.3 for the matrices A, B, Q are satisfied and*

$$\|f(.)\| \leq M.$$

5. Numerical Results

In this area, we provide three cases to demonstrate the effectiveness of our methods. Numerical simulation is created utilizing Ninteger toolbox in Matlab software.



(a) (b) (c)

FIGURE 2. (a) Time responses of states of the closed-loop system with $\alpha = 0.85$ and $\tau = 1$ for Example 5.1. (b) Time responses of states of the closed-loop system of the system with $\alpha = 0.85$ and $\tau = 3$ for Example 5.1. (c)Time responses of states of the closed-loop system with $\alpha = 0.85$ and $\tau = 5$ for Example 5.1.

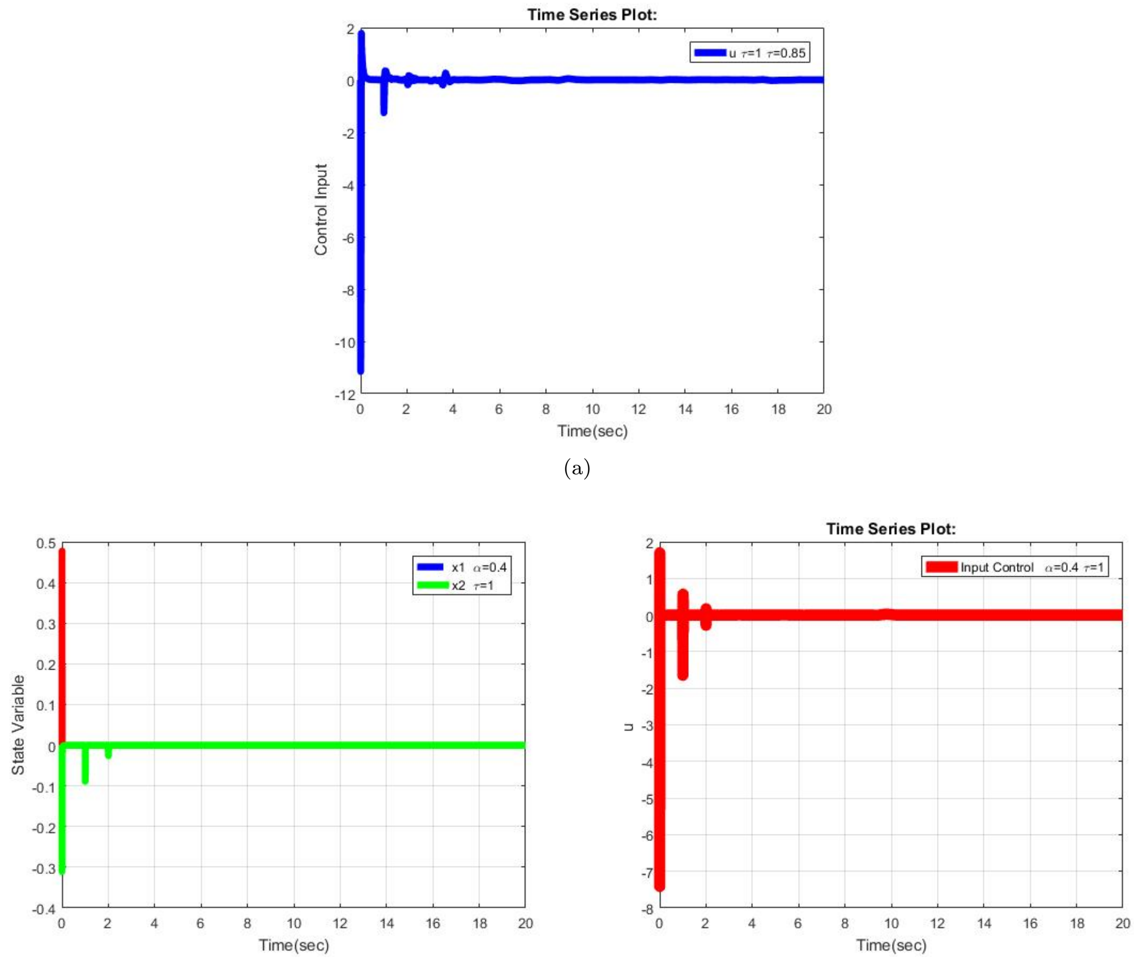
Example 5.1. Considering system (3.5) with $\alpha = 0.85$ and system matrices:

$$A = \begin{bmatrix} -4 & 5.5 \\ 8.5 & -10 \end{bmatrix}, \quad A_\tau = \begin{bmatrix} 0.1 & 0 \\ -0.1 & 0.2 \end{bmatrix}, \quad \mathrm{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The nonlinear function $\Delta A = E_1 F_1(t) H_1, \quad \Delta A_\tau = E_2 F_2(t) H_2$, satisfies condition (3.3) with

$$E1 = \begin{bmatrix} 1 \\ 2.5 \end{bmatrix}, \quad H1 = \begin{bmatrix} 1, 0 \end{bmatrix}, \quad F1 = F2 = \sin(t), \quad E2 = \begin{bmatrix} 0 \\ 2.3 \end{bmatrix}, \quad H2 = [0, 1].$$

Pair (A, B) is controllable. Time reaction of the system without state feedback controller appeared in Figure 1 so system is unstable for $u = 0$. In this case, we choose $\tau = 1, 3$ and 5 with $x(0) = [2, -1]^T$, at that point the controller gain matrix gotten as $K = [9.2500, -3.2500]$. The matrix N is chosen $N = [1, 0]$. By (4.1), the sliding surface function can be computed by movable parameters, $\rho = 8$ and $\mu = 2$. According to the conditions of Theorem 4.1 we have $\rho > d_{max}\|NG\|$ and $\mu > 0$. By applying control law in Theorem 4.1, we watch that the condition of Theorem 4.1



(a)

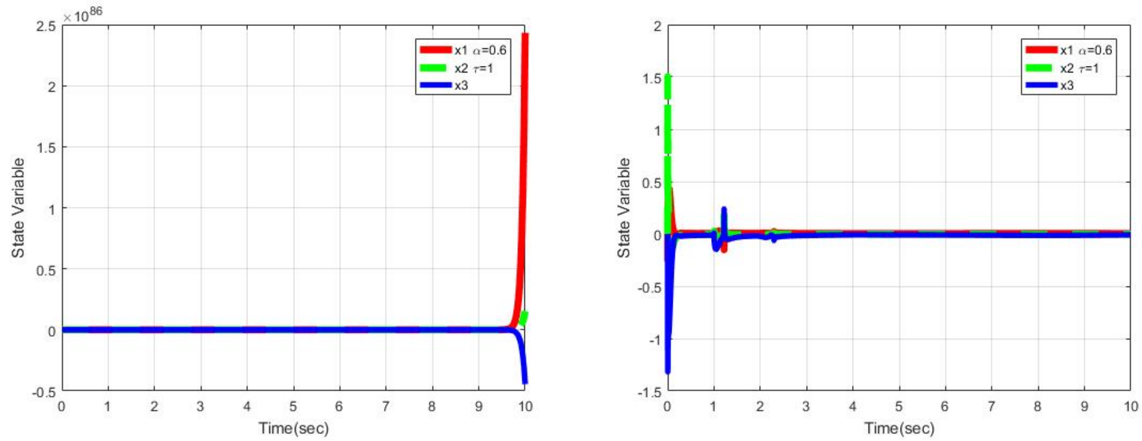
(b) (c)

FIGURE 3. (a) Feedback control law with $\alpha = 0.85$ and $\tau = 1$ for Example 5.1. (b) Trajectories of the closed-loop system of the system with $\alpha = 0.4$ and $\tau = 1$ for Example 5.1. (c) Feedback control law with $\alpha = 0.4$ and $\tau = 1$ for Example 5.1.

is satisfied and we show the simulation results of state $x(t)$ in Figures 2 where we present the trajectories of the closed-loop states with controller law (4.3). Figure 3(a) appears the reaction comes about of the feedback control law. Therefore, we show that system is stable. Also, Figures 3(b) and 3(c) present the trajectories of the closed-loop states and feedback control law for $\alpha = 0.4$. These figures show the efficiency of our method for selecting controller gain matrix K to guaranteed robustness against external disturbances and norm bounded uncertainty. It can also be concluded that the designed sliding mode controller, independent of delay, was able to stabilize the system.

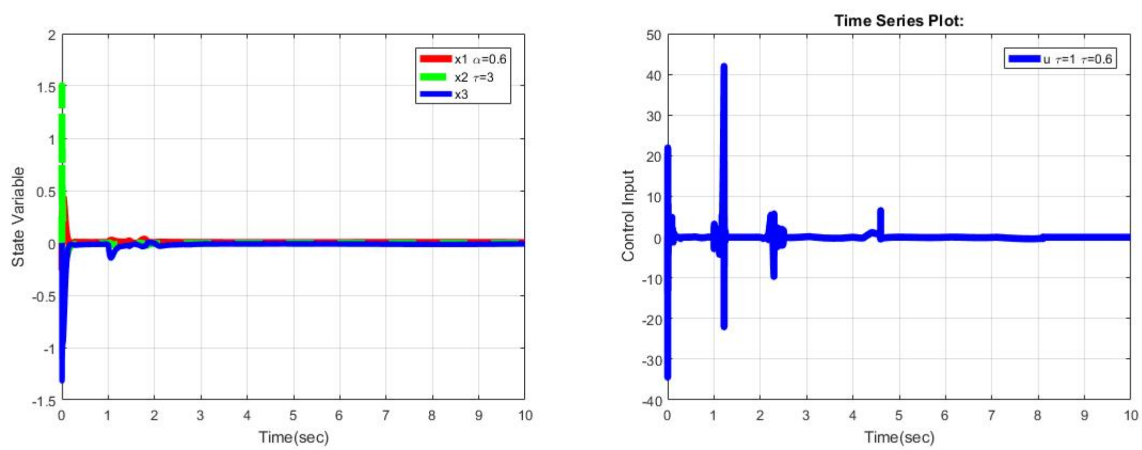
Example 5.2. Let us consider system (3.5) with $\alpha = 0.6$ and the following system characteristics:

$$A = \begin{bmatrix} 5 & 9 & 0 \\ 1 & -1 & 0 \\ 0 & -14.28 & 0 \end{bmatrix}, \quad A_\tau = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.1 & 0 & 0 \\ 0.2 & 0 & 0.1 \end{bmatrix}, \quad \mathrm{B} = \begin{pmatrix} 1 \\ 2.5 \\ 2 \end{pmatrix}, \quad G = \begin{pmatrix} 0.2 \\ 0.3 \\ 0.8 \end{pmatrix},$$



(a) (b)

FIGURE 4. (a) Time responses of states of the open-loop system with $\alpha = 0.6$ and $\tau = 1$ for Example 5.2. (b) Trajectories of the closed-loop system of the system with $\alpha = 0.6$ and $\tau = 1$ for Example 5.2.

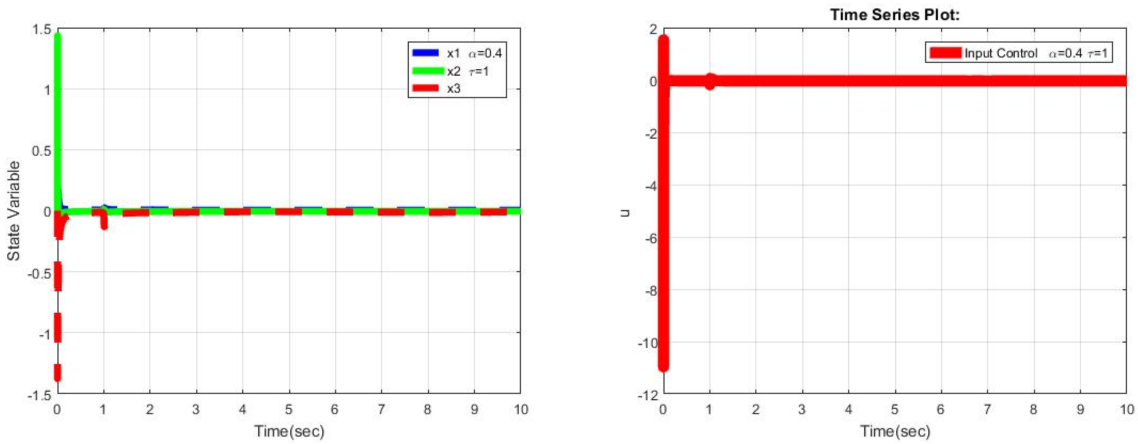


(a) (b)

FIGURE 5. (a) Trajectories of the closed-loop system with $\alpha = 0.6$ and $\tau = 3$ for Example 5.2. (b) Feedback control law with $\alpha = 0.6$ and $\tau = 1$ for Example 5.2.

$$\mathrm{d(t)} = 0.3\sin\left(\mathrm{t}e^{-2.3t}\right), \quad E1 = \begin{bmatrix} 0.5 \\ 0.8 \\ 0 \end{bmatrix}, \quad H1 = \begin{bmatrix} 1, 0, 0 \end{bmatrix}, \quad F1 = F2 = \sin(t), \quad E2 = \begin{bmatrix} 0 \\ 0.6 \\ 0.9 \end{bmatrix}, \quad H2 = [0, 1, 0].$$

Pair (A, B) is controllable and $d_{\max} = 0.3$, $||G|| = 0.8775$. This system is unstable for $u = 0$ as shown in Figure 4. In this example, we choose $\tau = 1$ and 3 with initial condition $x(0) = [1, 2, -2]^T$. The controller gain matrix gotten ass $K = [14.3428 \quad , 4.8925 \quad , 1.7130]$. Here, parameter $N = [1, 2.5, 2]$ so $||NG|| = 2.55$. By Eq. (4.1), the sliding surface function can be computed by adjustable parameters $\mu = 12$ and $\rho = 3$. By applying the control law in Theorem 4.1, we observe that the conditions of Theorem 4.1, $\rho > d_{\max}\|CG\|$ and $\mu > 0$, are satisfied. We appear, time responses of the closed-loop states in Figures 4(b), 5(a). Figure 5(b) appears the feedback comes about of the feedback control law.



(a) (b)

FIGURE 6. Trajectories of the closed-loop system of the system with $\alpha = 0.4$ and $\tau = 1$ for Example 5.2. (b) Feedback control law with $\alpha = 0.4$ and $\tau = 1$ for Example 5.2.

It is obvious from the figures that the closed-loop system is asymptotically stable. To anticipate the control signals from chattering, we supplant $\text{sign}(\Lambda(t))$ with $\frac{\Lambda(t)}{||\Lambda(t)||+0.04}$. This indicates that the sliding mode controller, independent of the delay, was able to stabilize the system. Finally, we repeated simulations for a smaller value of fractional order, alpha=0.4. Figures6(a), 6(b) show the result, stability and fast convergence is achieved.

Example 5.3. Considering system (3.1) with $\alpha = 0.7$ and the following system characteristics

$$A = \begin{bmatrix} -2.2 & 0.9 & -1.5 \\ 1.6 & 0.45 & 0.7 \\ -0.3 & -0.8 & -1 \end{bmatrix}, \quad A_\tau = \begin{bmatrix} 0 & 0 & 0.1 \\ -0.5 & 0 & 0 \\ 0.1 & 0.2 & 0 \end{bmatrix}, \tag{5.1}$$

$$B = \begin{bmatrix} 1 \\ 0.8 \\ -1.7 \end{bmatrix}, \quad f\left(t, x(t), x_\tau(t)\right) = \begin{bmatrix} 0.6\sin(x_1) \\ 0.2\sin(x_1) \\ 0.6\sin(x_1) \end{bmatrix}.$$

Pair (A, B) is controllable. It is clear from Figure 7(a) that the open-loop system is unstable. In this example, we choose $\tau = 1, 3$ and 5 with initial condition $x(0) = [1, -0.5, 0.5]^T$. The controller gain matrix gotten as $K = [17.7184, 120.7353, 53.5627]$. Here, parameter $N = [-0.1, 0.1, 0.1]$. By Eq. (4.2), the sliding surface function can be computed by adjustable parameters $\mu = 2$ and $\rho = 8$. With implementing the control law in Theorem 4.2, we observe that the conditions of Theorem 4.2, $\mu > 0$ and $\rho > M\|N\| + d_{\max}\|NG\|$, are satisfied. Figures 7(b), 7(c), and 8(a) present the trajectories of the closed-loop states with controller law (4.10). Figure 8(b) shows the simulation results of the feedback control law and sliding function. In this manner, the system states converged to the desired surface. Figures 9 show simulation result for $\alpha = 0.4$. Therefore, we can conclude the efficiency of our method in selecting the K controller gain matrix to achieve fast convergence and robustness against external selections and parameter uncertainty.

Example 5.4. In this example, we provide an example based on Chua's circuit model as in [15, 27], with disturbance and delay, which is in the form of nonlinear system (3.1) having the following system matrices:

$$A = \begin{bmatrix} -am_1 & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{bmatrix}, \quad A_\tau = \begin{bmatrix} -c & 0 & 0 \\ -c & 0 & 0 \\ 0 & -b & 0 \end{bmatrix}, \quad \text{B} = \begin{pmatrix} 1 \\ 2.5 \\ 2 \end{pmatrix}, \quad G = \begin{pmatrix} 0.2 \\ 0.3 \\ 0.8 \end{pmatrix},$$

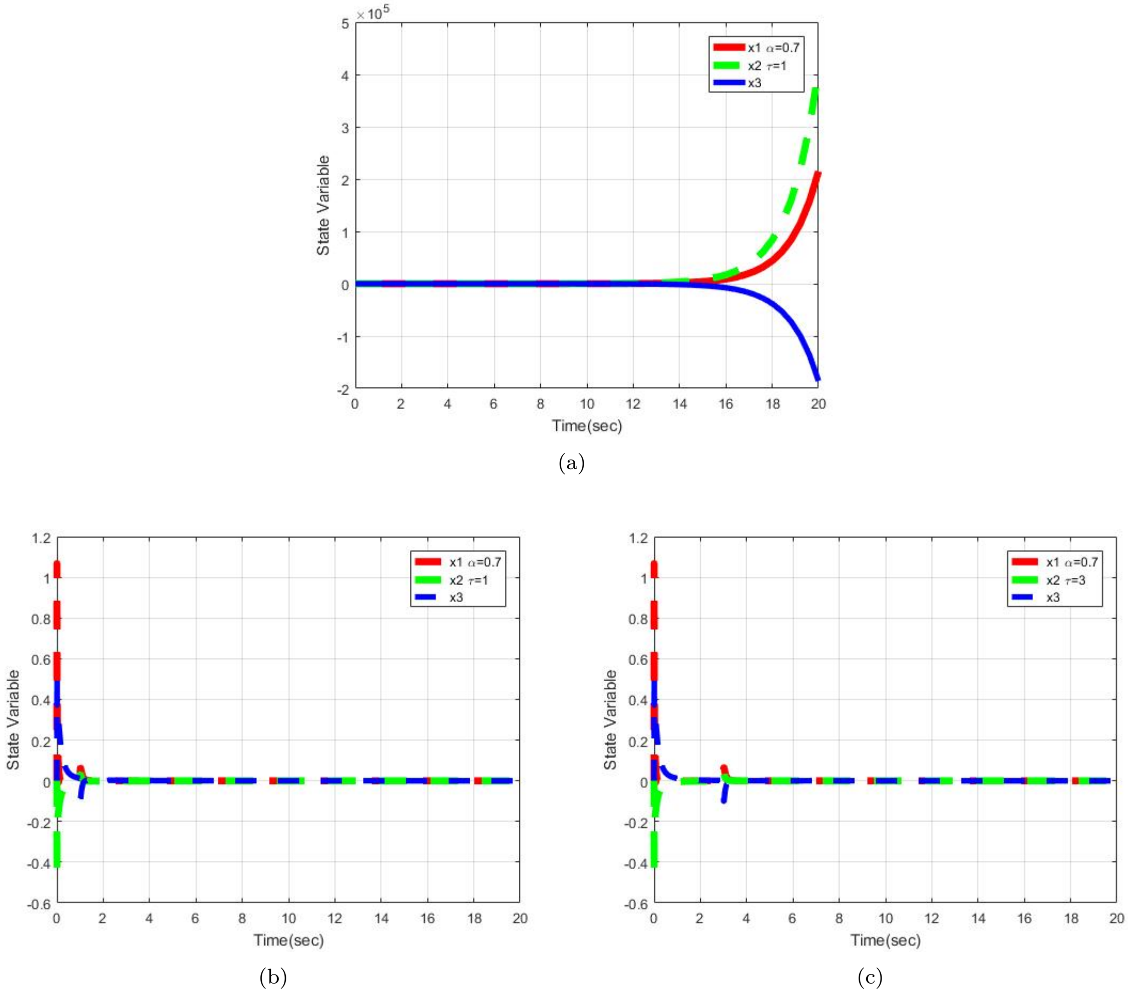
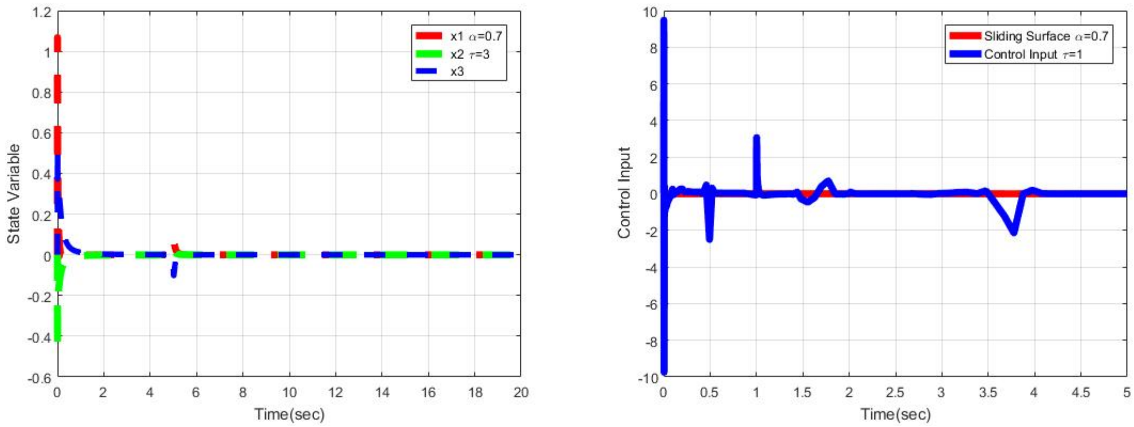


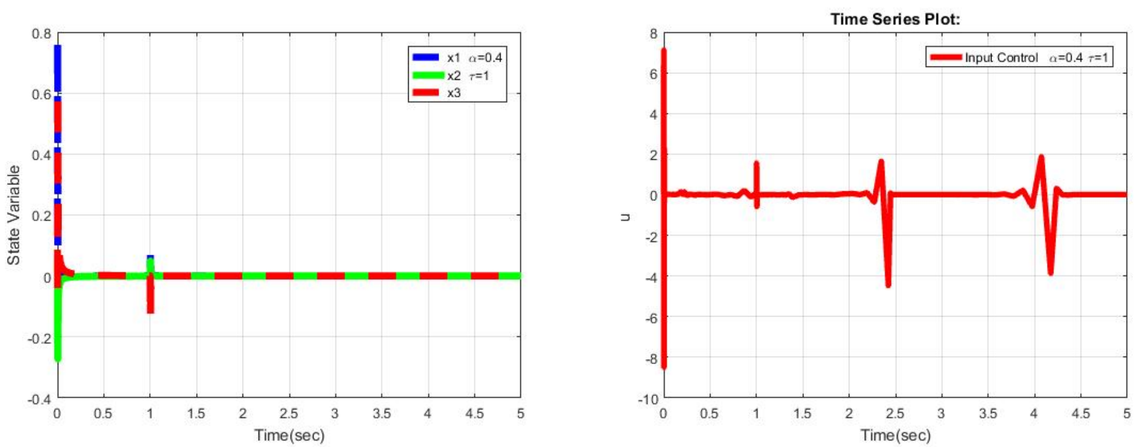
FIGURE 7. (a) Trajectories of the open-loop system with $\alpha = 0.7$ for Example 5.3. (b) Trajectories of the closed-loop system of the system with $\alpha = 0.7$ and $\tau = 1$ for Example 5.3. (c) Trajectories of the closed-loop system with $\alpha = 0.7$ and $\tau = 3$ for Example 5.3.

$\mathrm{d(t)} = 0.3\sin\left(\mathrm{t}e^{-2.3t}\right)$, $a = 9, b = 14.28, c = 0.1, b = 14.28, c = 0.1, m_1 = \frac{2}{7}$. Since a considerable number of hardware and software-based design and implementation approaches can be applied to Chua's circuits, these circuits also constitute excellent educative models that have pedagogical value in the study of nonlinear dynamics. In this example we choose $\tau = 1, 3$ and 5 with initial condition $x(0) = [0, -0.63, 0]^T$. The controller gain matrix gotten as $K = [-0.6252, 0.1199, 1.36063]$. Here, parameter $N = [1, 2.5, 2]$. By (4.10), the sliding surface function can be computed by adjustable parameters $\mu = 0.5$ and $\rho = 3$. With implementing the control law in Theorem 4.2, simulation results, shown in Figures 10, and 11(a). Figure 11(b) shows the feedback law control. It is clear from the figures that the closed-loop system is asymptotically stable. This proves the efficiency of the proposed method.



(a) (b)

FIGURE 8. (a) Trajectories of the closed-loop system of the system with $\alpha = 0.7$ and $\tau = 5$ for Example 5.3. (b) Feedback control law and sliding function with $\alpha = 0.7$ and $\tau = 1$ for Example 5.3.

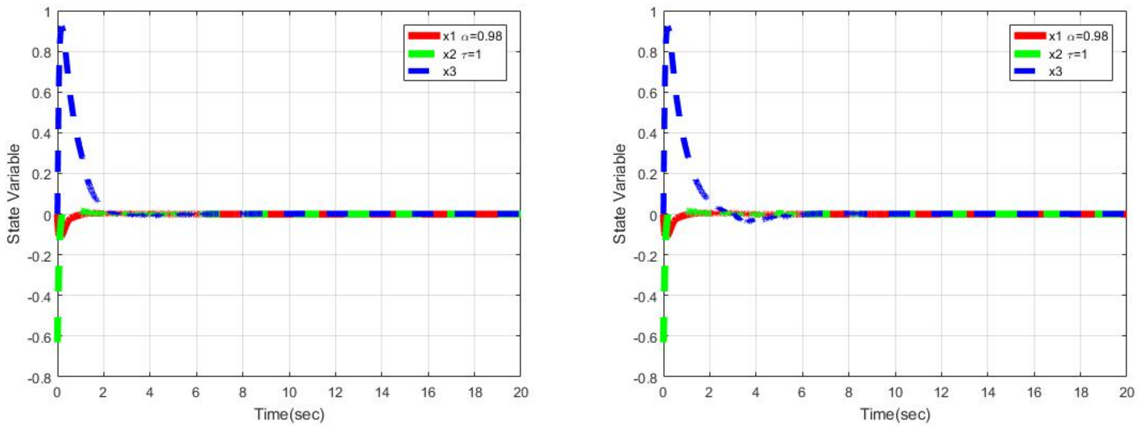


(a) (b)

FIGURE 9. (a) Trajectories of the closed-loop system of the system with $\alpha = 0.4$ and $\tau = 1$ for Example 5.3. (b) Feedback control law and sliding function with $\alpha = 0.4$ and $\tau = 1$ for Example 5.3.

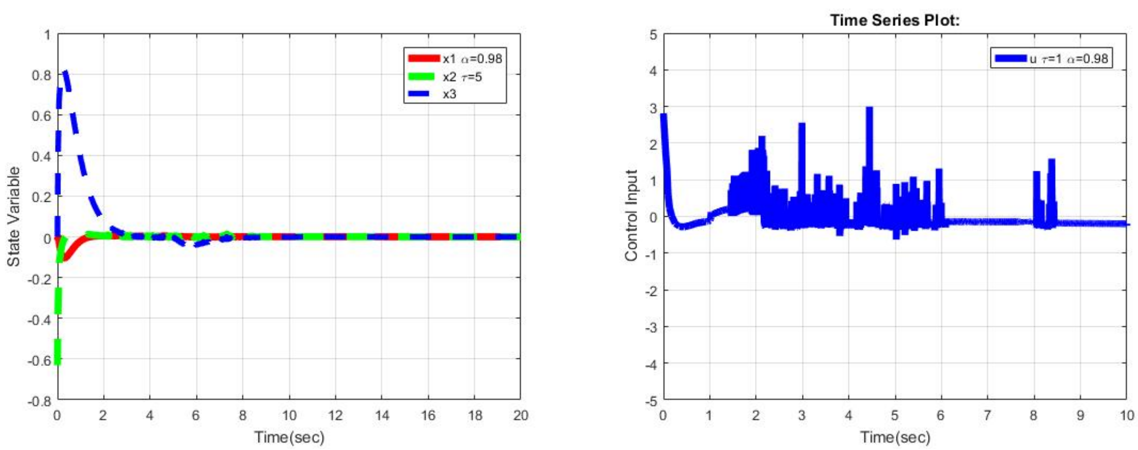
6. Conclusions

The stabilization of a class of nonlinear fractional-order time-varying delayed systems has been considered here, this system works as the Razumikhin approach subjected to nonlinear disturbance and parametric time-varying uncertainties. Theorems for each case are expressed separately. New and sufficient propositions have been expressed using relations based on inequalities. The performance of the main results is expressed using several examples in different dimensions. The studied system has Caputo fractional derivative; so, the studied results can be developed by considering other types of fractional derivatives. Some issues, such as LMI based SMC with LQR for a non-linear



(a) (b)

FIGURE 10. (a) Trajectories of the closed-loop system of the system with $\tau = 1$ for Example 5.4. (b) Trajectories of the closed-loop system of the system with $\tau = 3$ for Example 5.4.



(a) (b)

FIGURE 11. (a) Trajectories of the closed-loop system of the system with $\tau = 5$ for Example 5.4. (b) Feedback control law with $\tau = 1$ for Example 5.4.

fractional-order system, are still open problems. We will use the results obtained in this article to solve such a problem in a later work.

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